Coefficient Estimates of Toeplitz Determinant for a Certain Class of Close-to-Convex Functions

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Abstract Let $S$ denote the class of analytic and univalent functions in $D$, where $D$ is defined as the unit disk, $D := \{ z \in \mathbb{C} : |z| < 1 \}$ and having the Taylor representation form of $S$. We will determine the estimation for the Toeplitz determinants where the elements are the Taylor coefficients of the class close-to-convex functions in $S$.

Keywords: Univalent functions, Analytic functions, Toeplitz determinants, Close-to-convex functions

Introduction and Preliminaries

Let $A$ denote the class of analytic functions normalized by $f(0) = 0 = f'(0) - 1$ in $D$, where $D$ is defined as the unit disk, $D := \{ z \in \mathbb{C} : |z| < 1 \}$. If $f \in A$ then $f(z)$ has the following representation.

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$  \hspace{1cm} (1)

We denote $S$ as the class of univalent functions in $A$. There are three major subclasses in $S$ that include starlike functions, convex functions, and close-to-convex functions. Let $S^*$ represent the class of starlike functions in $S$. A function $f \in A$ becomes a starlike function, if it satisfies the following condition,

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0$$  \hspace{1cm} (2)

for $z \in D$. In the starlike functions, it has an important member as well, as the class $S$ is the Koebe function that is defined as follows,

$$k(z) = \frac{z}{(1-z)^2}$$  \hspace{1cm} (3)

In most issues for the $S^*$ and $S$ classes, the Koebe function play the role of extreme function. A function $f \in A$ is supposed to be a convex function, if it meets the following condition,

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0$$  \hspace{1cm} (4)

for $z \in D$. We denote $CV$ for the class of convex functions in $S$. In 1952, Kaplan had stated that if function $f \in S$ and if there exists a real number, $\alpha$, where $|\alpha| < \pi/2$ and a function $g(z)$ is convex which satisfy these conditions,
Research paper due to the results of the Toeplitz determinants function, where along with the diagonal. We refer closely connected to the Hankel determinants are those of functions. For example, the results of the estimates of Hankel determinants for various subclasses of univalent functions. Each of them will have to deal with finding the upper bound for the estimate of Hankel determinants for many classes of analytic functions. There are several results of early investigation of Hankel determinants for many classes of analytic functions started in the 1960s. In many branches of mathematics, the Hankel matrices and determinants play such a significant role and have several applications. There are several results of the Hankel determinants in the literature. The early investigation of Hankel determinants for many classes of analytic functions started in the 1960s. In 1966 and 1967, Pommerenke had studied Hankel’s determinants for the class of univalent functions given by [6] (cited by [16]). After that, many recent papers have been concerned to the problem of finding the estimate of Hankel determinants for various subclasses of univalent functions. Each of them will have to deal with finding the upper bound for \( |a_4a_3 - a_3^2| \) of their own classes. Some of the results, will also find the more general functional \( |a_4a_3 - \mu a_3^2| \) with the real \( \mu \) for various classes of functions. For example, the results of [2], [3], [9], [12], [13], [17] and [18].

Closely connected to the Hankel determinants are the Toeplitz determinants. Along with the reverse diagonal, the Hankel matrices and determinants play constant entries, while the Toeplitz matrices have constant entries along with the diagonal. We referred [22] for a description of the application of the Toeplitz matrices to a wide variety of areas especially in pure and applied mathematics. The symmetric Toeplitz determinant \( T_q(n) \) has been recently introduced by [7] for the analytic functions \( f \) of the form (1), defined as follows

\[
T_q(n) := \begin{bmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_n & \cdots & a_{n+q-2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q-2} & \cdots & a_n
\end{bmatrix}
\]

where \( n, q = 1, 2, 3, \cdots \) and \( a_4 = 1 \). In particular,

\[
T_2(2) := \begin{bmatrix}
a_2 & a_3 \\
a_3 & a_2
\end{bmatrix}, \quad T_2(3) := \begin{bmatrix}
a_3 & a_4 \\
a_4 & a_3
\end{bmatrix}, \quad T_3(1) := \begin{bmatrix}
1 & a_2 & a_3 \\
a_2 & 1 & a_2 \\
a_3 & a_2 & 1
\end{bmatrix}, \quad T_3(2) := \begin{bmatrix}
a_2 & a_3 & a_4 \\
a_3 & a_2 & a_3 \\
a_4 & a_3 & a_2
\end{bmatrix}.
\]

The estimates of the Toeplitz determinant \( |T_q(n)| \) for functions in the starlike and the close-to-convex functions were studied by [7] for small values of \( n \) and \( p \). Similarly, for the authors in [19] and [20], they had studied the estimates of the Toeplitz determinants \( |T_q(n)| \) for functions belonged to a certain conic domain and bounded rotation, \( R \) respectively. There seems to be little in the literature concerning the estimates of the Toeplitz determinants, aside from the result of [7]. The authors in [1] pointed out that the result obtained by [7] is not valid because of the \( c_1 > 0 \) is not justified, since the functional \( |T_q(n)| \) for \( n, q = 1, 2, 3, \cdots \) is not rotationally invariant. Consequently, the authors in [7] agreed to retract their research paper due to the results were not fully proved.
In the following year, the authors in [1] came out with the sharp estimate for the Toeplitz determinants in which the elements were the Taylor coefficients of functions in $S$ and some of its classes. In this purpose paper to give the estimates for Toeplitz determinants $T_n(n)$ for the class $K_{α,δ}$ which is the class of close-to-convex functions. Let $P$ represent the class of analytic functions $p$ in the unit disk $D$, which has the following form

$$p(z) = 1 + \sum_{n=1}^{∞} C_n z^n$$  \hspace{1cm} (8)$$
such that $Re \ p(z) > 0$ in the unit disk, $D$. These functions are sometimes called the Carathéodory functions. Before we continue to prove the main result, we need some preliminary results for the functions in $P$.

**Lemma 2.** [11, page 41]. For a function $p \in P$ of the form (8), the sharp inequality $|c_n| \leq 2$ holds for each $n \geq 1$. Equality holds for the function $p(z) = (1 + z)/(1 - z)$.

**Lemma 3.** [8, Theorem 1]. Let $p \in P$ be the form (8) and $μ \in \mathbb{C}$. Then

$$|c_n - μc_{n-1}| \leq 2 \max\{1, |2μ - 1|\} \text{ for } 1 \leq k \leq n - 1.$$  \hspace{1cm} (9)$$
If $|2μ - 1| \geq 1$ then the inequality is sharp for the function $p(z) = (1 + z)/(1 - z)$ or its rotations. If $|2μ - 1| \leq 1$ then the inequality is sharp for $p(z) = (1 + z^2)/(1 - z^2)$ or its rotations.

**Lemma 4.** [6, Theorem 1]. Let $g \in S^*$ be in the form

$$g(z) = z + \sum_{n=2}^{∞} b_n z^n$$  \hspace{1cm} (10)$$
Then, for any $λ \in \mathbb{C}$,

$$|b_1 - 2b_2^λ| \leq \max\{1, |3 - 4λ|\}$$

The inequality is sharp for Koebe function in the form (3) and its rotations if $|3 - 4λ| \geq 1$, and for $(k(z^2))^2$ and its rotations if $|3 - 4λ| < 1$.

**Lemma 5.** [15, Theorem 2.2]. Let $g \in S^*$ be in the form of (9). Then

$$|2b_n b_m - b_{n+m-1}| \leq λ_1 n m - (n + m - 1) \text{ for } λ_1 \geq \frac{2(n + m - 1)}{nm},$$

where $n, m = 2,3,\ldots$. the inequality is sharp for the Koebe function in the form (3) and its rotations.

**Lemma 6.** [2, Theorem 3.1]. Let $g \in S^*$ be in the form of (9). Then

$$|b_2 b_4 - b_3^λ| \leq 1,$$

and the inequality is sharp for the Koebe function, in the form (3) and its rotations.

**Lemma 7.** Let $f \in K_{α,δ}$ be the form (1). Then

$$|a_2 a_4 - 2α^2| \leq \frac{1}{72} \left[340 + 148(A_{α,δ})^2 + 120 A_{α,δ} + 36 A_{α,δ} \sqrt{\frac{1369}{81} + \frac{2116}{81} \delta^2}\right].$$

**Proof:** Noted that,

$$f(z) = z + \sum_{n=2}^{∞} a_n z^n = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$  \hspace{1cm} (10)$$
and

$$f'(z) = 1 + \sum_{n=2}^{∞} n a_n z^{n-1} = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots$$  \hspace{1cm} (11)$$
\[ zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n = z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + \cdots \]  
(12)

\[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots \]  
(13)

and

\[ p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \]  
(14)

By referring to the definition 1, we have

\[ e^{ia} \frac{zf'(z)}{g(z)} > \delta, \text{ where } \cos(\alpha) > \delta, |\alpha| < \frac{\pi}{2}. \]

A Carathéodory function exists, as \( p \in P \) of the form (14) as such that,

\[ e^{ia} \frac{zf'(z)}{g(z)} - \delta = p(z). \]  
(15)

Firstly, we need to find the representation theorem of class \( K_{a\delta} \). Let \( f \in S \) be in the form of (1), then from Left-Hand Side (LHS) of the equation (15), it can be written in the form,

\[ e^{ia} \frac{zf'(z)}{g(z)} - \delta = e^{ia} \left( 1 + \sum_{n=1}^{\infty} c_n z^n \right) - \delta, \]

and

\[ e^{ia} \frac{zf'(z)}{g(z)} - \delta - i \sin(\alpha) = \cos(\alpha) - \delta + \sum_{n=1}^{\infty} e^{ia} c_n z^n, \]

which gives

\[ e^{ia} \frac{zf'(z)}{g(z)} - \delta - i \sin(\alpha) \cos(\alpha) - \delta = 1 + \sum_{n=1}^{\infty} \left( \frac{e^{ia} c_n}{\cos(\alpha) - \delta} \right) z^n, \]

and

\[ e^{ia} \frac{zf'(z)}{g(z)} - \delta - i \sin(\alpha) \cos(\alpha) - \delta = 1 + \sum_{n=1}^{\infty} q_n z^n, \]  
(16)

where \( q_n = \left( \frac{e^{ia} c_n}{\cos(\alpha) - \delta} \right) \). Then, we can relate the equation (16) to the function in \( P \) with

\[ e^{ia} \frac{zf'(z)}{g(z)} - \delta - i \sin(\alpha) \cos(\alpha) - \delta = p(z), \]

where \( p(z) \) has the form of (8). Take note that, \( \cos(\alpha) - \delta \) must always be positive. Next, we find the coefficients of \( z \), by using the following equation,

\[ e^{ia} \frac{zf'(z)}{g(z)} - \delta - i \sin(\alpha) = p(z) \cos(\alpha) - \delta \]

which implies,

\[ zf'(z) = e^{-ia} \left[ g(z) \right] \left[ p(z) \cos(\alpha) - \delta + \delta + i \sin(\alpha) \right] \]  
(17)

Take note that, \( \Lambda_{a\delta} = \cos(\alpha) - \delta \) and from the Right-Hand Side (RHS) of the equation (17), we have
\[
\left( e^{-ia} \right) \left( \sum_{n=2}^{\infty} b_n z^n \right) \left( \sum_{n=1}^{\infty} c_n z^n \right) (\cos(\alpha) - \delta + \delta i \sin(\alpha)) \\
= \left( e^{-ia} \right) \left( \sum_{n=2}^{\infty} b_n z^n \right) \left( \cos(\alpha) - \delta + \sum_{n=1}^{\infty} (\cos(\alpha) - \delta) c_n z^n \right) + \delta + i \sin(\alpha) \\
= \left( e^{-ia} \right) \left( \sum_{n=2}^{\infty} b_n z^n \right) \left( e^{ia} + \sum_{n=1}^{\infty} A_{a \delta} c_n z^n \right)
\]

Also, from the Left-Hand Side (LHS) of the equation (17), we have
\[
z f(z) = z + \sum_{n=2}^{\infty} n a_n z^n = z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + \cdots
\]

By comparing the coefficients of \( z \) in both LHS and RHS of the equation (17), we will get
\[
2a_2 = A_{a \delta} e^{-ia} c_1 + b_2 \tag{18}
\]
\[
3a_3 = A_{a \delta} e^{-ia} c_2 + A_{a \delta} e^{-ia} b_2 c_1 + b_3 \tag{19}
\]
\[
4a_4 = A_{a \delta} e^{-ia} c_3 + A_{a \delta} e^{-ia} b_2 c_2 + A_{a \delta} e^{-ia} b_3 c_1 + b_4 \tag{20}
\]

Now, we must find the equation \( a_2 a_4 = 2a_2^2 \) by using the coefficients that we had obtained in (18), (19) and (20). We let \( A = a_2 a_4 \),
\[
A = \frac{1}{2} \left( A_{a \delta} e^{-ia} c_1 + b_2 \right) \left( A_{a \delta} e^{-ia} c_3 + A_{a \delta} e^{-ia} b_2 c_1 + A_{a \delta} e^{-ia} b_3 + b_4 \right)
\]
\[
= \frac{1}{8} \left( \left( A_{a \delta} \right)^2 e^{-2ia} c_1 c_3 + b_2 c_2 c_1 + b_3 c_3 \right) + A_{a \delta} e^{-ia} \left( b_2 c_1 + b_2 c_3 + b_3 c_2 + b_3 b_2 c_1 + b_4 b_2 \right)
\]

Also, by letting \( B = 2a_3^2 \), we will have
\[
B = \frac{2}{9} \left( A_{a \delta} e^{-ia} c_2 + A_{a \delta} e^{-ia} b_2 c_1 + b_3 \right)^2
\]
\[
= \frac{2}{9} \left( \left( A_{a \delta} \right)^2 e^{-2ia} c_2^2 + 2b_2 c_2 c_1 + b_3 c_2^2 \right) + A_{a \delta} e^{-ia} \left( 2b_3 b_2 c_1 + 2b_3 c_2 + b_3 b_3 c_1 \right)
\]

Then, we will subtract the equations of \( A \) and \( B \) that we have obtained so that it becomes \( A - B = a_2 a_4 - 2a_3^2 \), and it yields
\[
A - B = \left( A_{a \delta} \right)^2 e^{-2ia} \left( \frac{c_1 c_3}{8} + \frac{b_2 c_2 c_1}{8} + \frac{b_3 c_3}{8} + \frac{2c_2^2}{9} - \frac{4b_2 c_2 c_1}{9} - \frac{2b_3^2}{9} \right)
\]
\[
+ A_{a \delta} e^{-ia} \left( \frac{b_2 c_1}{8} + \frac{b_2 c_3}{8} + \frac{b_3^2 c_2}{8} + \frac{b_3 b_2 c_1}{8} + \frac{b_3 b_3 c_1}{9} - \frac{4b_3 c_2^2}{9} + b_2 b_3 \right)
\]

and when simplify it further, we get
\[
A - B = \frac{1}{72} \left[ \left( 9b_3 b_2 - 16b_2^2 \right) + A_{a \delta} e^{-ia} \left( 9b_4 - 23b_3 b_2 \right) \right]
\]
\[
+ \left( A_{a \delta} \right)^2 e^{-2ia} \left( 9b_3^2 - 16b_2^2 \right) + A_{a \delta} e^{-ia} \left( 9b_4 - 32b_3 \right) + A_{a \delta} e^{-ia} \left( 9b_3 - 23A_{a \delta} e^{-ia} c_2 c_1 \right) + \left( A_{a \delta} \right)^2 e^{-2ia} \left( 9c_1 c_3 - 16c_2^2 \right)
\]

By applying the triangle inequality for the equation (21), we have
\[
72 \left| a_2 a_4 - 2a_3^2 \right| \leq \left| 9b_4 b_2 - 16b_2^2 + A_{a \delta} c_1 \right| \left| 9b_4 - 23b_3 b_2 \right| \left| \frac{c_1}{8} \right| \left| 9b_1 - 16b_2^2 \right| + \left| A_{a \delta} \right| \left| c_2 \right| \left| 9b_3^2 - 32b_3 \right| + A_{a \delta} \left| b_2 \right| \left| 9c_3 - 23A_{a \delta} e^{-ia} c_2 c_1 \right| + \left( A_{a \delta} \right)^2 \left| c_1 c_3 - 16c_2^2 \right|
\]

By letting \( Y_1 = |9b_4 b_2 - 16b_2^2|, Y_2 = |9b_4 - 23b_3 b_2|, Y_3 = |9b_3 - 16b_2^2|, Y_4 = |9b_3^2 - 32b_3|, \)
\[ Y_5 = |9c_3 - 23A_{a\delta}e^{-i\alpha}c_2c_1|, \text{ and } Y_6 = |9c_4c_3 - 16c_2^2|. \text{ Then, by applying the Lemma 4, 5, and 6, we will get} \\
\begin{align*}
Y_1 & \leq 9 |b_4b_2 - b_3^2| + 7 |b_3| \\
& \leq 9 + 63 \\
& \leq 72 \\
Y_2 & \leq 9 \left| b_4 + \frac{23b_3b_2}{9} \right| \\
& \leq 9 \left( \frac{46}{3} - 4 \right) \\
& \leq 102 \\
Y_3 & \leq 9 \left| b_3 - \frac{16b_2^2}{9} \right| \\
& \leq 9 \left( \frac{64}{9} - 3 \right) \\
& \leq 37 \\
Y_4 & \leq 32 \left| b_1 - \frac{9b_2}{32} \right| \\
& \leq 32 \left( \frac{3}{8} - \frac{9}{8} \right) \\
& \leq 60 \\
\end{align*}
\]

For \( Y_5 = 9|c_3 - \mu c_2c_1| \leq 18 \max\{1,|2\mu - 1|\}, \) where \( \mu = \frac{23}{9} A_{a\delta} e^{-i\alpha}. \) Take note that,

\[ (2\mu - 1)^2 = \left[ \left( \frac{23}{9} A_{a\delta} e^{-i\alpha} \right) - 1 \right]^2 \\
= \left[ \frac{14}{9} + \frac{23}{9} \cos(2\alpha) - \frac{46}{9} \delta \cos(\alpha) \right]^2 + \left[ \frac{23}{9} \sin(2\alpha) - \frac{46}{9} \delta \sin(\alpha) \right]^2. \tag{27} \]

Firstly, we need to expand the equation (27) by letting

\[ Q_1 = \left[ \frac{14}{9} + \frac{23}{9} \cos(2\alpha) - \frac{46}{9} \delta \cos(\alpha) \right]^2 \text{ and } Q_2 = \left[ \frac{23}{9} \sin(2\alpha) - \frac{46}{9} \delta \sin(\alpha) \right]^2. \]

From \( Q_1, \) we will have,

\[ Q_1 = \frac{196}{81} + \frac{644}{81} \cos(2\alpha) + \frac{529}{81} \cos(2\alpha)^2 + \frac{2116}{81} \delta^2 (\cos(\alpha))^2. \]

Then from \( Q_2 \) we will have,

\[ Q_2 = \frac{529}{81} (\sin(2\alpha))^2 - \frac{2116}{81} \delta \sin(2\alpha) \sin(\alpha) + \frac{2116}{81} \delta^2 (\sin(\alpha))^2. \]

Hence, we will add both equations, \( Q_1 \) and \( Q_2, \) so we will get

\[ Q_1 + Q_2 = \frac{725}{81} + \frac{644}{81} \cos(2\alpha) + \frac{2116}{81} \delta^2 - \frac{2116}{81} \delta \sin(2\alpha) \sin(\alpha). \]

By letting \( f(\alpha) = Q_1 + Q_2, \) then we need to find the critical point of the equation by differentiating the function \( f(\alpha) \) with respect to \( \alpha, \) so we will have

\[ \frac{df(\alpha)}{d\alpha} = \frac{1288}{81} \sin(2\alpha) - \frac{2116}{81} \delta (\sin(2\alpha) \cos(\alpha) + 2 \sin(\alpha) \cos(2\alpha)) \\
= \frac{1}{81} \left[ \sin(2\alpha) \left( 1288 - 2116 \delta (\cos(\alpha) + 2 \sin(\alpha) \cot(2\alpha)) \right) \right]. \tag{28} \]

After differentiating the function \( f(\alpha), \) we will let the differential equation (28) to zero, and we will get
\[
\frac{1}{81}\left[\sin(2\alpha)(1288 - 2116\delta\cos(\alpha) + 2\sin(\alpha)\cos(2\alpha))\right] = 0.
\]

Since the equation \(\sin(2\alpha)\) can only be equivalent to zero and we had obtained the following critical points,

\[\alpha = \frac{\pi}{2}\text{ and }\alpha = 0.\]

From the critical points that we had acquired, we know that,

\[1 + \frac{2116\delta^2}{81} \leq |2\mu - 1|^2 \leq \frac{1369}{81} + \frac{2116}{81}\delta^2\]

and

\[1 + \frac{2116\delta^2}{81} \leq |2\mu - 1| \leq \sqrt{\frac{1369}{81} + \frac{2116}{81}\delta^2}\]

Therefore,

\[Y_5 = \left|9c_1 - 23 A_{\alpha\delta} e^{-i\alpha} c_2 c_3\right| \leq 18\left(\frac{1369}{81} + \frac{2116}{81}\delta^2\right)\]

(29)

Again, by using Lemma 2 and 3 for the equation \(Y_6\), we will have

\[\left|9c_1 c_3 - 16c_2^2\right| \leq 9|c_1 c_3 - c_4| + 9\left|c_4 - \frac{16}{9}c_2^2\right| \leq 18 + 46 \leq 64.\]

Hence, by using the relations of the equations (23), (24), (25), (26), (29) and (30), the inequality (22), we will get

\[|a_2 a_4 - 2a_3^2| \leq \frac{1}{72}\left[72 + \left(A_{\alpha\delta}\right)^2(102) + \left(37 + A_{\alpha\delta}\right)^2(37) + A_{\alpha\delta}(2)(60)\right] + 120A_{\alpha\delta} + 36A_{\alpha\delta}\left(\frac{1369}{81} + \frac{2116}{81}\delta^2\right)\]

This concludes the proof.

**Theorem 2.9.** Let \(f \in K_{\alpha\delta}\) be in the form of (1). Then

\[|T_3(2)| \leq 44 + 4\left[\frac{340}{72} + \frac{148}{72}\left(A_{\alpha\delta}\right)^2 + \frac{120}{72}A_{\alpha\delta} + \frac{36}{72}A_{\alpha\delta}\left(\frac{1369}{81} + \frac{2116}{81}\delta^2\right)\right].\]

**Proof.** Let \(f \in K_{\alpha\delta}\) be in the form of (1). Then by Lemma 7,

\[|T_3(2)| = \left|a_2^3 - 2a_2a_3^2 - 2a_2a_3^2 + 2a_2^3a_4\right| \leq |a_2|^3 + 2|a_2||a_3|^2 + |a_2a_4 - 2a_3^2| \leq 44 + 4\left[\frac{340}{72} + \frac{148}{72}\left(A_{\alpha\delta}\right)^2 + \frac{120}{72}A_{\alpha\delta} + \frac{36}{72}A_{\alpha\delta}\left(\frac{1369}{81} + \frac{2116}{81}\delta^2\right)\right].\]

This concludes the proof. By letting \(\delta = 0\) and \(\alpha = 0\), the results that were obtained can be reduced to the result of [1].

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