Ordered discrete and continuous Z-numbers

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INTRODUCTION

Real-world information is flawed, and natural language is often used to represent this feature. Such information is often characterized by fuzziness, which implies that soft constraints are imposed on the values of variables of interest. Furthermore, reliability is another essential property of information. Any estimation of values of interest, be it precise or soft, are subject to the confidence with regards to sources of information. Thus, fuzziness from the one side and partial reliability form the other side are strongly associated to each other [1]. To discuss this concept, Zadeh in [2] introduced the concept of Z-number as a formal description of such information. Basically, the concept of Z-number relates to the issue of reliability of information. A Z-number is an ordered pair of fuzzy numbers (A, B). It is associated with a real valued uncertain variable X, with the first component, A, playing the role of a fuzzy restriction (R(X)) on the values which X can take, written as X is A such that A is a fuzzy. The second component B is a measure of reliability (certainty) of the first component [2].

B. Kang et al. [3] proposed an approach of dealing with Z-numbers which naturally arises in the areas of decision making, control, regression analysis and others. The approach is based on transforming a Z-number into fuzzy number on the basis of fuzzy Zadeh expectation of the fuzziness. The advantage of this approach is its low analytical and computational complexity, which allows for a wide spectrum of its applications. Unfortunately, converting Z-number to fuzzy leads to significant loss of original information and reducing the benefit of using Z-number-based information in the first place.

The authors in [4] developed some basics for direct computation with Z-number, by suggesting a general and computationally effective approach to deal with discrete Z-number. The authors provided motivation to use discrete Z-numbers as an alternative to the continuous one, based on the fact that natural language-based information has a discrete framework and it is not required to decide upon a reasonable assumption to use some type of probability distributions. Furthermore, it has lower computational complexity than that with continuous Z-numbers. Some basic theoretical aspects of arithmetic operations over discrete Z-numbers such as addition, subtraction, multiplication, division, square root of a Z-number, and other operations are proposed as well as a series of numerical examples are provided by them to illustrate the validity of the suggested approach.

A mathematical property called ordered, is required for constructing temporal discrete Z-numbers. Consider the set of complex numbers, C. It is not ordered naturally but when the relation | |: C → R is employed on C such that |C| = |a + ib| = √a² + b² ∈ R, then the ordered property is deduced indirectly. Fig. 1 shows the coordinates of complex number.
Similarly, discrete Z-number is an ordered pair of discrete fuzzy numbers, however, this does not guarantee that discrete Z-number is an ordered set too. This paper proves that both discrete and continuous Z-numbers can be ordered by applying a linear ordering relation \( < \) between set of discrete or continuous Z-numbers and any arbitrary ordered subset of \( \mathbb{R} \). The rest of the paper is organized as follows: Section 2 contains some basic definitions related to this work; the concepts of ordered discrete and continuous Z-number are revealed in Section 3; a sample of the implementation is presented in Section 4; and finally, the conclusion is drawn in Section 5.

**PRELIMINARIES**

Here are some important definitions which are essential in this work.

**Definition 1.1** [7] The relation \( \prec \) on \( X \times X \) is a partial ordering on \( X \) if it satisfies the following properties:

1. (Reflexivity) \( x \prec x \) for every \( x \in X \).
2. (antisymmetry) If \( x_1 \prec x_2 \) and \( x_2 \prec x_1 \), then \( x_1 = x_2 \).
3. (transitivity) If \( x_1 \prec x_2 \) and \( x_2 \prec x_3 \), then \( x_1 \prec x_3 \).

A pair \((X, \prec)\) is called a partially ordered set. A partially ordered set \((X, \leq)\) is said to be totally ordered (also called linearly ordered), provided that for every \( x_1, x_2 \in X \) and \( x_1 \neq x_2 \), either \( x_1 \leq x_2 \) or \( x_2 \leq x_1 \). A partial order \( \leq \) is then said to be a linearly ordered.

**Definition 2.2** [7] A partially ordered set in which every pair of element has the greatest lower bound and the least upper bound is called a lattice.

**Definition 2.2** [7] A lattice \((Z, \lor, \land)\) is a distributive lattice if the following additional identity holds for all \( a, b, c \in Z \), \( a \land (b \lor c) = (a \land b) \lor (a \land c) \).

**Definition 4.2** [8] A fuzzy number \( A \) of the real line \( R \) with membership function \( \mu_A : \mathbb{R} \rightarrow [0, 1] \) is a discrete fuzzy number if its support is finite, i.e., there exist \( \{x_1, \ldots, x_n\} \in R \) with \( x_1 < x_2 < \cdots < x_n \), such that \( \text{supp}(A) = \{x_1, \ldots, x_n\} \) and there exist natural numbers \( s, t \) with \( 1 \leq s \leq t \leq n \) satisfying the following conditions:

1. \( \mu_A(x_i) = 1 \) for any natural number \( i \) with \( s \leq i \leq t \)
2. \( \mu_A(x_i) \leq \mu_A(x_j) \) for each natural number \( i, j \) with \( 1 \leq i \leq j \leq s \)
3. \( \mu_A(x_i) \geq \mu_A(x_j) \) for each natural number \( i, j \) with \( 1 \leq i \leq j \leq n \).

**Definition 3.5** [5] A continuous fuzzy number is a fuzzy subset \( A \) of the real line \( \mathbb{R} \) with membership function \( \mu_A : \mathbb{R} \rightarrow [0, 1] \) which possesses the following properties:

1. \( A \) is a normal fuzzy set.
2. \( A \) is a convex fuzzy set.
3. \( \alpha \)-cut \( A^\alpha \) is a closed interval for every \( \alpha \in (0, 1] \).
4. The support of \( A \), \( \text{supp}(A) \), is bounded.

A continuous fuzzy number \( A \) with the membership function defined as

\[
\mu_A = \begin{cases} 
\frac{x - a}{b - a} & \text{if } a \leq x < b \\
1 & \text{if } b \leq x \leq c \\
\frac{d - x}{d - c} & \text{if } c < x \leq d \\
0 & \text{otherwise}
\end{cases}
\]

\[\mu_A = \left\{ \begin{array}{ll}
\frac{x - a}{b - a} & \text{if } a \leq x < b \\
1 & \text{if } b \leq x \leq c \\
\frac{d - x}{d - c} & \text{if } c < x \leq d \\
0 & \text{otherwise}
\end{array} \right. \]

and denoted as \((a, b, c, d)\).

**Fig. 2 Trapezoidal fuzzy number.**

**Definition 2.3** [4] A discrete Z-number is an ordered pair \( Z = (A, B) \) where \( A \) is a discrete fuzzy number playing a role as a fuzzy constraint on values that a random variable \( X \) may take:

\[ X \text{ is } A \]

and \( B \) is a discrete fuzzy number with a membership function \( \mu_B : \{b_1, \ldots, b_n\} \rightarrow [0, 1] \), \( \{b_1, \ldots, b_n\} \subseteq [0, 1] \), playing a role of a fuzzy constraint on the probability measure of \( A \):

\[ P(A) \text{ is } B \]

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**ORDERED Z-NUMBER**

In [6], the concept of minimum and maximum of both discrete and continuous Z-number was introduced and denoted as MIN and MAX, respectively. They showed that for the discrete Z-number, the triple \((Z_D, \text{MIN}, \text{MAX})\) is a distributive lattice, where \( Z_D \) represents the set of discrete Z-numbers whose support is a sequence of consecutive natural numbers. The term MIN and MAX serve as meet and joint of \( Z_D \) which implies immediately that discrete Z-number is partially ordered. Similarly, the triple \((Z_C, \text{MIN}, \text{MAX})\) is also a distributive lattice, where \( Z_C \) represents the set of continuous Z-numbers support, which is a bounded set of natural numbers. Since set of natural numbers is well-ordered and has a least element, hence, continuous Z-number is partially ordered. However, [9] did not show explicitly that \((Z_D, \text{MIN}, \text{MAX})\) is a distributive lattice. Therefore, in this paper the relation \( \prec \) on Z-number (discrete or continuous) is shown to be partially ordered in Theorem 3.1.

A discrete or continuous Z-number can be ordered using two different methods. The first one is by using the method proposed by Kang B in [3], which is, converting discrete or continuous Z-number to a discrete or continuous generalized fuzzy number. However, this method may lead to sufficient loss of original information. The second method, which is the most preferable, is by creating a relation between
set of $Z$-number (discrete or continuous) and any arbitrary ordered subset in $\mathbb{R}$ as follows.

**Definition 3.1** Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be two $Z$-numbers (discrete or continuous). Then $Z_1 \preceq Z_2$ if and only if $A_1 = A_2$ and $B_1 = B_2$, namely, $\mu_a(x) = \mu_b(x)$ and $\mu_b(x) = \mu_b(x)$, respectively.

**Theorem 3.1** The relation $(\mathcal{Z}_D, \preceq)$ is well-defined.

**Proof** Consider two sets $(\mathcal{Z}_D, \preceq)$ and $(\mathcal{G}, \leq)$ with relation $(\mathcal{Z}_D, \preceq) \Rightarrow (\mathcal{G}, \leq)$ for $G \subseteq \mathbb{R}$ and $(\mathcal{Z}_D, \preceq)$ means a set of $Z$-numbers (discrete or continuous) with binary operation $\prec$. The relation $\Rightarrow$ is well defined due to its tautology as shown in Table 2, where $T = \text{True}$ and $F = \text{False}$.

<table>
<thead>
<tr>
<th>$(\mathcal{Z}_D, \prec)$</th>
<th>$(\mathcal{G}, \leq)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

Both sides must be exactly the same in order the relation $\Rightarrow$ to be true.

**Theorem 3.2.** The set $([-1, 0], \leq)$ is partially ordered.

**Proof.** Consider $([-1, 0], \leq)$. It is reflexive since $x \leq x, \forall x \in [-1, 0]$. It is antisymmetric since $x \leq y$ and $y \leq x \Rightarrow x = y$ for $x, y, z \in [-1, 0]$. It is transitive since $x \leq y$ and $y \leq z \Rightarrow x \leq z$ for $x, y, z \in [-1, 0]$. Hence, $([-1, 0], \leq)$ is a partially ordered set.

**Corollary 3.1** The relation $(\mathcal{Z}_D, \prec) \Rightarrow ([0, 1], \leq)$ is well defined.

**Proof** By replacing $G = [-1, 0]$ in Theorem 3.1, the proposed relation $(\mathcal{Z}_D, \prec) \Rightarrow ([0, 1], \leq)$ is well defined.

**Theorem 3.3** The set $(\mathcal{Z}_D, \prec)$ with relation defined as $(\mathcal{Z}_D, \prec)$ is a partially ordered set.

**Proof** The relation $(\mathcal{Z}_D, \prec)$ is well defined by Corollary 3.1. Furthermore, $([-1, 0], \leq)$ is partially ordered by Theorem 3.2. The reflexive property holds for $(\mathcal{Z}_D, \prec)$ by invoking the reflexivity of $([0, 1], \leq)$. In other words, $Z_1 \preceq Z_1, \forall Z_1 \in (\mathcal{Z}_D, \prec)$. Similarly, for antisymmetry and transitive properties for $(\mathcal{Z}_D, \prec)$, by invoking the antisymmetry and transitive properties of $([-1, 0], \leq)$.

Hence, $(\mathcal{Z}_D, \prec)$ with relation defined as $(\mathcal{Z}_D, \prec) \Rightarrow ([0, 1], \leq)$ is a partially ordered set.

**Theorem 3.4** Let $\mathcal{Z}_D$ be a set of discrete $Z$-numbers and $\prec$ be a linear ordering relation. The set $(\mathcal{Z}_D, \prec)$ is said to be totally ordered, by creating a relation between $\mathcal{Z}_D$ and any arbitrary ordered set in $\mathbb{R}$.

**Proof** Let $H$ be any arbitrary ordered set in $\mathbb{R}$, namely, $(H, <)$ in $\mathbb{R}$, as proven in Theorem 3.1. Obviously, similar relation is well defined when it is replaced by $(\mathcal{Z}_D \times \mathcal{G}, \prec)$ such that $\prec: (\mathcal{Z}_D \times \mathcal{G}, \prec) \rightarrow \mathcal{G} \ni (Z_1, g_1) \prec (Z_2, g_2) \Rightarrow g_1 < g_2$.

**Theorem 4.1** Let $(F, d_F)$ and $(T, d_T)$ be metric spaces, where $(T, <)$ is a linearly ordered set with minimal element $t_0 \in T$. Let $S_t \in F \times T$ be an augmented trajectory of a dynamic motion $g \in F^T$ defined for all $t \in T$. The relation $\prec$ on $S_t \times S_{t'}$, generated by $g(\cdot)$, is called a temporal ordering on $S_t$, and is defined as $\forall (Z_1, t_1), (Z_2, t_2) \in S_t (Z_1, t_1) \prec (Z_2, t_2) \Rightarrow t_1 < t_2$, where $Z_1 \preceq Z_2$.
and \( Z \) are ordered discrete Z-numbers. For any set \( K \subseteq S \), a pair \((K, \prec')\) is said to be a temporal set on \( S \).

**Definition 4.2** Let \( S \) be an augmented dynamic trajectory with appropriate temporal ordering \( \prec' \). Let \((K, \prec')\) be a temporal set on \( S \). A discrete Z-number in the universe \( K \) is called a temporal discrete Z-number which is denoted as \( Z = (A, B) \).

Fig. 3 illustrates the relationship between the augmented trajectory \( S \), temporal set \( K \) and the temporal discrete Z-number \( Z \).

![Fig. 3 Relation between \( S \), \( K \) and \( Z \).](image)

The following Lemma, theorem and corollary lead to temporal discrete Z-numbers as a class of ordered discrete Z-numbers.

**Lemma 4.1.** Let \( S \) be an augmented trajectory, then every temporal ordering \( \prec' \) on \( S \) is a partial ordering on \( S \).

**Proof.** By Lemma 4.1 the temporal ordering \( \prec' \) on \( S \) is a partial ordering. For any distinct elements of \( S \) i.e. \((Z_1, t_1) \neq (Z_2, t_2)\), then there exist \( g(t_1) = Z_1 \) and \( g(t_2) = Z_2 \) when \( t_1 \neq t_2 \). Since \((T, \prec)\) is linearly ordered, then \( t_1 < t_2 \) or \( t_2 < t_1 \). This implies that \((Z_1, t_1) \prec (Z_2, t_2)\) i.e. \((Z_1, t_1) \prec' (Z_2, t_2)\). Hence the temporal ordering \( \prec' \) on \( S \) is a partial ordering on \( S \).

**Theorem 4.2.** Let \( S \) be an augmented trajectory, then every temporal ordering \( \prec' \) on \( S \) is linearly ordered.

**Proof.** By Definition 4.1 of temporal ordering \( \prec' \) is defined as \((Z_1, t_1) \prec' (Z_2, t_2)\). Hence, we can simply say that by Lemma 4.1, Theorem 4.2, Definition 4.1, and 4.2, every temporal discrete Z-number is an ordered discrete Z-number.

The detailed derivation of temporal discrete Z-number and its implementation procedure are presented in [11], whereby some of the data used are obtained from [12] to illustrate the procedure for analyzing EEG signal of an epileptic seizure.

**Numerical example:**

Some of the data used are taken from [12]. Let consider an EEG data set of an epileptic seizure which is given in Table 2. By applying Z-number clustering algorithm one can partition the data set into clusters which are represented by membership function of temporal discrete Z-number.

<table>
<thead>
<tr>
<th>( x_{t,1} )</th>
<th>( x_{t,2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.05</td>
<td>0.774906</td>
</tr>
<tr>
<td>1.10</td>
<td>0.822311</td>
</tr>
<tr>
<td>1.15</td>
<td>0.874949</td>
</tr>
<tr>
<td>1.20</td>
<td>0.933029</td>
</tr>
<tr>
<td>1.25</td>
<td>0.996711</td>
</tr>
<tr>
<td>1.30</td>
<td>1.066098</td>
</tr>
<tr>
<td>1.35</td>
<td>1.141221</td>
</tr>
<tr>
<td>1.40</td>
<td>1.22203</td>
</tr>
<tr>
<td>1.45</td>
<td>1.308371</td>
</tr>
<tr>
<td>1.50</td>
<td>1.399882</td>
</tr>
<tr>
<td>1.55</td>
<td>1.496474</td>
</tr>
<tr>
<td>1.60</td>
<td>1.586384</td>
</tr>
</tbody>
</table>

Firstly, in order to obtain a type-2 temporal fuzzy set cluster, fuzzy fuzzifier is used as shown in Fig. 5.
Supposed the membership functions of $A^t$ and $B^t$ for $x$ dimension are represented as follows

\[ A^t = 0/0 + 0.3/1.5 + 1/2.2 + 0.1/3 + 0/0 \]

and

\[ B^t = 0.8/0.77 + 1/0.79 + 0.9/0.8 + 0.4/0.9 + 0/1 \]

Therefore, the membership functions are used to determine the measure of uncertainty for $Z^t$ in $x$ dimension with respect to the time of occurrence.

The numerical example illustrates the implementation procedure of applying temporal discrete Z-number to analyze EEG signal data of epileptic seizure and finally to determine the measure of uncertainty with respect to time of occurrence.

**CONCLUSION**

Even though both discrete and continuous Z-numbers are pairs of discrete and continuous fuzzy numbers, however they not simply imply discrete and continuous Z-numbers are ordered immediately as fuzzy numbers with respect to their membership values. A complex number is an example such case. In other words, both discrete and continuous Z-numbers cannot be ordered on their own. This paper proposed the idea of ordered discrete and continuous Z-number by creating a relation between set of discrete or continuous Z-numbers and any arbitrary ordered subset of it.

The proposed structure is successfully used to construct temporal discrete Z-number with the purpose to analyze electroencephalogram signal of an epileptic seizure.

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