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RESEARCH ARTICLE

Backstepping in infinite dimensional for the time fractional order partial differential equations

Ibtisam Kamil Hanan [a, b], Muhammad Zaini Ahmad [a], Fadhel Subhi Fadhel [b], Ghazwan Raheem Mohammed [c,*]

[a] *Institute of Engineering Mathematics, Universiti Malaysia Perlis, Pauh Putra Main Campus, 02600 Arau, Perlis, Malaysia*
[b] *Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, P. O. Box 47077, Baghdad, Iraq*
[c] *Mayoralty of Baghdad, Rasheed Municipality, Khilany squarel Al-Kholfaa street, Baghdad, Iraq*

* *Corresponding author: ghazwanraheem1980@gmail.com*

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Abstract

This paper focuses on the application of backstepping control scheme for the time fractional order partial differential equation (FPDE). The fractional derivative is presented by using Caputo fractional derivative. The design technique here can exhaust systems with an arbitrary finite number of open loop unstable eigenvalues and is not limited to a certain kind of boundary actuation. We show how the FPDE is converted into a Mittag-Leffler stability by designing invertible coordinate transformation. Numerical simulation is given to demonstrate the effectiveness of the proposed control scheme.

Keywords: Boundary control, backstepping, stabilisation, coordinate transformation, fractional order PDE

INTRODUCTION

We can define the boundary control as a distributed parameter control that has been widely studied and developed in the control theory. It is not very recent when the researchers started to investigate the general parabolic equations and the boundary feedback stabilisation of this kind of equations. Triggiani [1] and Lasiecka [2] used the semi group theory in order to evaluate a general form to obtain the eigenvalues for parabolic problems. Then, an auxiliary functional observers has been developed by Nambu [3], to stabilise diffusion equations by the use of boundary observation and feedback. Moreover Bensoussan et al. [4] discussed the stabilisation of the optimal control setting by the boundary control.

There is a huge amount of attention these days on the boundary control of K-S equation [5,6], Burgers equation [7], KdV-Burgers equation, C-H equation [8,9], heat equation [10–13]. The boundary control of the K-S equation is studied in [5] with an external excitation, specifically by using the semi group theory and Banach contraction fixed point to satisfy the existence and uniqueness of the solution. It has been presented in [7] the existence of an optimal controller and suitable index of performance $J(y;\, u)$ regarding to Galerkin method. While in [14], a new simple controller was proposed for Chen's chaotic system.

The heat equation is a typical parabolic equation, which has rich physics background. Recently, many researchers have been focusing on the heat equation with backstepping control law [15–19], which is still a boundary control. Nevertheless, according to our knowledge, there are only few attempts on the method of boundary feedback stabilisation to deal with the unstable FPDE. The boundary stabilisation of fractional wave equation based on numerical solution technique has given a boundary control of a Caputo fractional wave equation through a fractional order boundary controller was studied in [20,21].

One important feature of the fractional order models over the integer order ones is that many real life applications can be described by utilising notation of fractional order [22,23].

Based on what was listed above, this paper focuses on the following FPDE

$$ {}_0^cD_t^q s(x,t) = \xi s_{xx}(x,t) + \gamma(x)s(x,t) \quad \text{in } (0,1)\times(0,\infty), \tag{1}$$

where the boundary conditions are

$$ s(0,t) = 0, \;\; t>0, \tag{2}$$

or

$$ s_x(0,t) = 0, \;\; t>0, \tag{3}$$

$$ s(1,t) = \int_0^1 k_1(\zeta)s(\zeta,t)\,d\zeta, \tag{4}$$

where, $0<q\le 1, \xi>0, \gamma(x)\in L_\infty(0,1), {}_0^cD_t^q$ represents the Caputo fractional order derivative and ${}_0I_t^q$ is the representation of the Riemann-Liouville fractional order integral, and are defined as [24].

$$ D_t^q s(x,t) = \begin{cases} \dfrac{\partial}{\partial t}s(x,t), & q=1, \\ {}_0I_t^{1-q}\dfrac{\partial}{\partial t}s(x,t), & 0<q<1, \end{cases}$$

and

$$ {}_0I_t^q s(s,t) = \frac{1}{\Gamma(q)}\int_0^t (t-r)^{q-1}s(x,r)\,dr, \;\; q>0.$$

Equation (1) could be applied in many real life applications. It is introduced in [25] to better characterise reaction diffusion processes in inhomogeneous environment in space. For instance, the dispersive transport media that performs reaction diffusion process [26], following through sourced porous media [27], and many other dynamical diffusion processes in disordered media [28]. In our work, equation (1) is considered to model a thin rod with heat loss to the surrounding environment. Also, it deals with the heat generation in the thin rod due to space in an inhomogeneous environment.

When $q=1$, Eq. (1), is reduced to the classical integer order unstable heat equation. The boundary feedback control law with Dirichlet boundary conditions by using discretise backstepping method was studied in [29].

Recently we proposed the backstepping method for stabilising nonlinear FPDE with constant coefficients. The semi-discretised fractional order backstepping approach introduced to find the boundary controller function which stabilises nonlinear FPDE with Dirichlet boundary conditions by transformation it into an equivalent stable closed loop [30-32]. In this paper, we devote to discuss the boundary control for Eq. (1). We use the backstepping method for the semi discretisation of the Eq. (1) to derive a Dirichlet and Neumann boundary feedback control law that makes the closed loop system stable. We show that the integral kernel in the control law is bounded.

The rest of this paper is set up as follows: our problem is discussed and discretised with Dirichlet condition in Section 2 and with homogeneous Neumann boundary condition is presented in Section 3. Finally, Section 4 provides a numerical simulation to illustrate the effectiveness of proposed control scheme. The conclusions are devoted in the last section.

Dirichlet boundary conditions

Consider the following system,

$$\begin{aligned} &{}_0^cD_t^q s(x,t)=\xi s_{xx}(x,t)+\gamma(x)s(x,t) && \text{in } (0,1)\times(0,\infty),\\ &s(0,t)=s(1,t)=0 && \text{in } (0,\infty). \end{aligned} \tag{5}$$

For any positive and large value of γ the system (5) will be unstable [33]. Take the coordinate transformation bellow,

$$r(x,t)=s(x,t)-\int_0^x k(x,\zeta)s(\zeta,t)d\zeta, \quad x\in(0,1), t\in(0,\infty) \tag{6}$$

together with the Dirichlet boundary feedback control,

$$s(1,t)=\int_0^1 k_1(\zeta)s(\zeta,t)\,d\zeta. \tag{7}$$

The system,

$$\begin{aligned} &{}_0^cD_t^q s(x,t)=\xi s_{xx}(x,t)+\gamma(x)s(x,t) && \text{in } (0,1)\times(0,\infty),\\ &s(0,t)=0 && \text{in } (0,\infty),\\ &s(x,0)=s_0(x) && \text{in } (0,1), \end{aligned} \tag{8}$$

one can transform the above system into the following system,

$$\begin{aligned} &{}_0^cD_t^q r(x,t)=\xi r_{xx}(x,t)-a\,r(x,t) && \text{in } (0,1)\times(0,\infty),\\ &r(0,t)=r(1,t)=0 && \text{in } (0,\infty),\\ &r(x,0)=r_0(x) && \text{in } (0,1). \end{aligned} \tag{9}$$

The q -order derivative of the transformation (6) due to time is,

$$\begin{aligned} {}_0^cD_t^q r(x,t) &= \xi s_{xx}(x,t)+\gamma(x)s(x,t)-\xi k(x,x)s_x(x,t)+\xi k(x,0)s_x(0,t)+\\ &\quad \xi k_\zeta(x,x)s(x,t)-\xi k_\zeta(x,0)s(0,t)-\\ &\quad \int_0^x\left(\xi k_{\zeta\zeta}(x,\zeta)s(\zeta,t)+k(x,\zeta)\gamma(\zeta)s(\zeta,t)\right)d\zeta. \end{aligned}$$

Next, we differentiate (6) with respect to x obtained,

$$\begin{aligned} r_{xx}(x,t) &= s_{xx}(x,t)-\frac{d}{dx}k(x,x)s(x,t)-k(x,x)s_x(x,t)-k_x(x,x)s(x,t)\\ &\quad -\int_0^x k_{xx}(x,\zeta)s(\zeta,t)d\zeta. \end{aligned}$$

Then, we get that ${}_0^cD_t^q r(x,t)-\xi r_{xx}(x,t)+a r(x,t)=0$

$$\begin{aligned} 0 &= \left(\xi k_\zeta(x,x)+\xi\frac{d}{dx}k(x,x)+\xi k_x(x,x)+\gamma(x)+a\right)s(x,t)+\xi k(x,0)\times\\ &\quad s_x(0,t)-\int_0^x\left(\xi k_{\zeta\zeta}(x,\zeta)-\xi k_{xx}(x,\zeta)+k(x,\zeta)(\gamma(\zeta)+a)\right)s(\zeta,t)d\zeta. \end{aligned}$$

The first step of obtaining Eq. (6) in a constructive form is to discretise (1), (2) and (4). The following step is to develop a transformation with strange coordinate for the discretised system. Finally, examine the convergence of the infinite dimensional transformation. We define,

$s_i=s(il,t),\ \gamma_i=\gamma(il)$ for $i=0,1,\ldots,m+1$ where $m\in\mathbb{N}$, $l=1/(m+1)$.

The discrete coordinate transformation for system (6) can be formulated as,

$$r_i=s_i-\psi_{i-1}(s_1,s_2,\ldots,s_{i-1}),\quad i=1,\ldots,m. \tag{10}$$

A discretised of System (1), (2) and (4) due to the space x, using finite differences is,

$$\begin{aligned} &s_0=0,\\ &{}_0^cD_t^q s_i=\xi\frac{s_{i+1}-2s_i+s_{i-1}}{l^2}+\gamma_i s_i \qquad i=1,\ldots,m,\\ &s_{m+1}=\psi_m(s_1,s_2,\ldots,s_m). \end{aligned} \tag{11}$$

The target transformed System (9) has the discretised form,

$$\begin{aligned} &r_0=0,\\ &{}_0^cD_t^q r_i=\xi\frac{r_{i+1}-2r_i+r_{i-1}}{l^2}-ar_i \qquad i=1,\ldots,m,\\ &r_{m+1}=0. \end{aligned} \tag{12}$$

The finite dimensional transformation (10) convergence to the infinite dimensional one (6) has been proven to have a boundedness kernel [29,34].

The Gain Kernel

In this subsection, the derivation of the kernel's recursiver elationship is presented as follows:

The first step is to differentiate the transformation (10) with respect to time in q-order to get:

$${}_0^cD_t^q r_i={}_0^cD_t^q s_i-\sum_{j=1}^{i-1}\Gamma(2-q)s_j^{q-1}\frac{\partial^q\psi_{i-1}}{\partial s_j^q}\frac{\partial^q s_j}{\partial t^q}. \tag{13}$$

Substituting Eqs. (11) and (12) in (13) gives,

$$\begin{aligned} \xi\frac{r_{i+1}-2r_i+r_{i-1}}{l^2}-ar_i &= \xi\frac{s_{i+1}-2s_i+s_{i-1}}{l^2}+\gamma_i s_i-\\ &\quad \sum_{j=1}^{i-1}\Gamma(2-q)s_j^{q-1}\frac{\partial^q\psi_{i-1}}{\partial s_j^q}\frac{\partial^q s_j}{\partial t^q}. \end{aligned} \tag{14}$$

Solving the obtained equation for ψ_i's , we get the recursive formula,

$$\psi_i=\left(2+\frac{a l^2}{\xi}\right)\psi_{i-1}-\psi_{i-2}-\frac{l^2}{\xi}(\gamma_i+a)s_i+ \sum_{j=1}^{i-1}\Gamma(2-q)s_j^{q-1}\frac{\partial^q\psi_{i-1}}{\partial s_j^q}\left(s_{j+1}-2s_j+s_{j-1}+\frac{l^2}{\xi}\gamma_j s_j\right), \tag{15}$$

for $i=1,\mathrm{K}\,,m$ with initial values $\psi_0=\psi_{-1}=0$. Writing the ψ_i's in the linear form gives,

$$\psi_i=\sum_{j=1}^{i}k_{ij}s_j,\quad i=1,\mathrm{K}\,,m, \tag{16}$$

and performing simple calculations we obtain the general recursive relationship,

$$k_{i,1}=\frac{l^2}{\xi}(a+\gamma_1)k_{i-1,1}-k_{i-2,1}+k_{i-1,2}, \tag{17}$$

$$k_{i,j}=\frac{l^2}{\xi}(a+\gamma_j)k_{i-1,j}+k_{i-1,j-1}+k_{i-1,j+1}-k_{i-2,j},\ j=2,\mathrm{K}\,,i-2, \tag{18}$$

$$k_{i,i-1}=\frac{l^2}{\xi}(a+\gamma_{i-1})k_{i-1,i-1}+k_{i-1,i-2}, \tag{19}$$

$$k_{i,i}=k_{i-1,i-1}-\frac{l^2}{\xi}(a+\gamma_i), \tag{20}$$

for $i=4,\mathrm{K}\,,m$ with initial conditions,

$$k_{1,1}=\frac{-l^2}{\xi}(a+\gamma_1), \tag{21}$$

$$k_{2,1}=\frac{-l^4}{\xi^2}(a+\gamma_1)^2, \tag{22}$$

$$k_{2,2}=\frac{-l^2}{\xi}(a+\gamma_1)-\frac{l^2}{\xi}(a+\gamma_2), \tag{23}$$

$$k_{3,1}=\frac{-l^6}{\xi^3}(a+\gamma_1)^3-\frac{l^2}{\xi}(a+\gamma_2), \tag{24}$$

$$k_{3,2}=\frac{l^2}{\xi}(a+\gamma_2)\left(\frac{-l^2}{\xi}(a+\gamma_1)-\frac{l^2}{\xi}(a+\gamma_2)\right)-\frac{l^4}{\xi^2}(a+\gamma_1)^2, \tag{25}$$

$$k_{3,3}=\frac{-l^2}{\xi}(a+\gamma_1)-\frac{l^2}{\xi}(a+\gamma_2)-\frac{l^2}{\xi}(a+\gamma_3). \tag{26}$$

In fact, these gains are bounded. Besides, based on the [29], we see that:

Proposition 1 [29]*:* The sequence $\{k_{i,j}\}$ defined in (17)-(26) has the elements that should satisfy,

$$\left|k_{i,i-j}\right|\le\binom{i}{j+1}\left(\frac{l^2}{\xi}(a+\gamma)\right)^{j+1}+(i-j)\times \sum_{h=1}^{[j/2]}\frac{1}{h}\binom{j-h}{h-1}\binom{i-h}{j-2h}\left(\frac{l^2}{\xi}(a+\gamma)\right)^{j-2h+1}, \tag{27}$$

where $\gamma=\max_{x\in[0,1]}\left|\gamma(x)\right|$

Proposition 2 [29]: Suppose that two functions $r(x)\in L_\infty(0,1)$ and $s(x)\in L_\infty(0,1)$ satisfy the relationship,

$$r(x)=s(x)-\int_0^x\tilde{k}(x,\zeta)s(\zeta)\,d\zeta,\quad\forall x\in[0,1], \tag{28}$$

where $\tilde{k}\in C_r([0,1];L_\infty(0,1))$. Then, n and N are positive constants with size dependence only on $\tilde{k}$, such that $n\|r\|_\infty\le\|s\|_\infty\le N\|r\|_\infty$, $n\|r\|\le\|s\|\le N\|r\|$.

Proposition 3 [35]: Assume that x is a continuous and differentiable function defined on $x:[0,\infty)\to\Re$. Then for any given $t\ge0$, one can get

$$\frac{1}{2}\,{}_0^cD_t^q x^2(t)\le x(t)\,{}_0^cD_t^q x(t),\quad\forall q\in(0,1]. \tag{29}$$

Theorem 1. Suppose that $\gamma(x)\in L_\infty(0,1)$ and $\xi,a>0$, $r(.,t)$ is a continuous and differentiable function on the interval $[0,\infty)$ and the Laplace transform of $r^2(.,t)$ exist. For any $s_0\in L_\infty(0,1)$, system (1), with (2) and (4) has a unique solution satisfy,

$$\|s(x,t)\|^2\le n_1\|s_0\|^2E_q(-2at^q),\ n_1>0, \tag{30}$$

where $E_q(t)=\sum_{i=0}^{\infty}\frac{t^i}{\Gamma(qi+1)},\ t\in\Re.$

Proof. By Proposition 2, there exist $v>0$ such that,

$$\|s(.,t)\|\le v\|r(.,t)\|,\ \|r_0\|\le v\|s_0\|. \tag{31}$$

Let,

$$R(t)=(1/2)\int_0^1 r^2(.,t)\,dx. \tag{32}$$

By Proposition 3, we have ${}_0^cD_t^qR(t)=(1/2)\int_0^1{}_0^cD_t^qr^2(.,t)\,dx\le-2a\,R(t).$

Let,

$$W(t)=-2a\,R(t)-{}_0^cD_t^qR(t). \tag{33}$$

The Laplace transform of (33) is $\hat{R}(s)=\frac{s^{q-1}R(0)-\hat{W}(s)}{s^q+2a}$, where

$$R(0)=(1/2)\int_0^1 r_0^2(x)\,dx.$$

By the use of the uniqueness and existence theorem [36], inverting the transform of (33) leads to the following unique solution,

$$R(t)=E_q(-2at^q)R(0)-W(t)*[t^{q-1}E_q(-2at^q)],\ t\ge0. \tag{34}$$

In addition, it is clear that t^{q-1} and $E_q(-2at^q)$ are both nonnegative functions, then

$$R(t)\le E_q(-2at^q)R(0),\ t\ge0. \tag{35}$$

Implies that $\|s(.,t)\|^2\le v^2\|s_0\|^2E_q(-2at^q),\ 0\le t<\infty.$

For the convergence of the finite difference approximations obtained from (11), and (12), to the solutions of (1)-(4), and (9), respectively see [37].

Neumann boundary conditions

We continue the discussion of the proposed control scheme. In this section, we consider the case when this scheme has a Neumann boundary condition. In this case, one can use the following technique, starting with a finite-dimensional coordinate transformation of a backstepping-style,

$$\begin{aligned} &r_0 = s_0, \\ &r_1 = s_1, \\ &r_i = s_i - \psi_{i-1}(s_1,\ldots,s_{i-1}), \quad i = 2,\ldots,m, \\ &r_{m+1} = 0, \end{aligned} \tag{36}$$

Now, the original system has been transformed to the semi-discretise infinite-dimensional system,

$$\begin{aligned} &{}_0^cD_t^q r(x,t) = \xi r_{xx}(x,t) - a\,r(x,t) && \text{in } (0,1)\times(0,\infty), \\ &r_x(0,t) = 0 && \text{in } (0,\infty), \\ &r(1,t) = 0 && \text{in } (0,\infty), \\ &r(x,0) = r_0(x) && \text{in } (0,1). \end{aligned} \tag{37}$$

by using the same technique as in Dirichlet case, one can obtain,

$$\begin{aligned} \psi_i = {} & \left(2 + \frac{a l^2}{\xi}\right)\psi_{i-1} - \psi_{i-2} - \frac{l^2}{\xi}(\gamma_i + a)s_i + \Gamma(2-q)s_1^{q-1} \times \\ & \frac{\partial^q \psi_{i-1}}{\partial s_1^q}\left(s_2 - \left(1 - \frac{l^2}{\xi}\gamma_1\right)s_1\right) + \sum_{j=2}^{i-1}\Gamma(2-q)s_j^{q-1}\frac{\partial^q \psi_{i-1}}{\partial s_j^q} \times \\ & \left(s_{j+1} - \left(2 - \frac{l^2}{\xi}\gamma_j\right)s_j + s_{j-1}\right). \end{aligned} \tag{38}$$

Writing the ψ_i's in the linear form (16) we have,

$$k_{i,1} = \left(\frac{l^2}{\xi}(a + \gamma_1) + 1\right)k_{i-1,1} - k_{i-2,1} + k_{i-1,2}, \tag{39}$$

where $k_{i,j}, k_{i,i-1}$, and $k_{i,i}$ are defined in (18)-(20). The initial conditions for the recursion can be written as,

$$k_{2,1} = \frac{-l^4}{\xi^2}(a+\gamma_1)^2 - \frac{l^2}{\xi}(a+\gamma_1), \tag{40}$$

$$k_{3,1} = -\left(\frac{l^2}{\xi}(a+\gamma_1)+1\right)^2\frac{l^2}{\xi}(a+\gamma_1) - \frac{l^2}{\xi}(a+\gamma_2), \tag{41}$$

$$k_{3,2} = \frac{-l^2}{\xi}(a+\gamma_2)\left(\frac{l^2}{\xi}(a+\gamma_1) + \frac{l^2}{\xi}(a+\gamma_2)\right) - \left(\frac{l^2}{\xi}(a+\gamma_1)+1\right)\frac{l^2}{\xi}(a+\gamma_1), \tag{42}$$

$k_{1,1}, k_{2,2}$, and $k_{3,3}$ are defined as the same case in Dirichlet.

Same as for the Dirichlet case, it is shown by the numerical solution that the coefficients $\{(m+1)k_{m,j}\}_{j=1}^{m}$ are still bounded. The numerical solution also shows the coefficients oscillation, and this oscillation is increased if *m* is increasing, see Fig. 1 and Fig. 2.

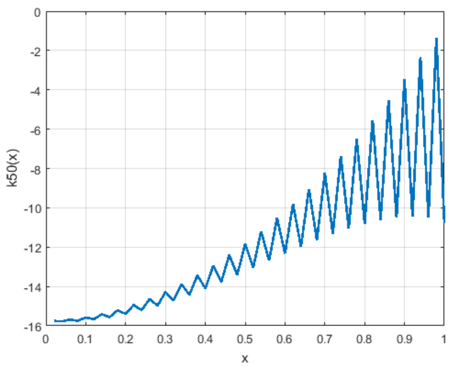

Fig. 1 Oscillation of the approximating kernel for $m = 50, \gamma = 10, \xi = 1, a = 1$

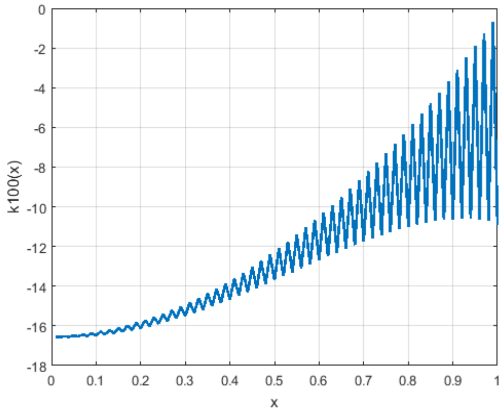

Fig. 2 Oscillation of the approximating kernel for $m = 100, \gamma = 10, \xi = 1, a = 1$

Lemma 1: The sequence $\{k_{i,j}\}$ defined in (39)-(42) has elements that should satisfy the following

$$\begin{aligned} \left|k_{i,i-j}\right| \le {} & \binom{i}{j+1}\left(\frac{l^2}{\xi}(a+\gamma)\right)^{j+1} + (i-j)\times \\ & \sum_{h=1}^{[j/2]}\frac{1}{h}\binom{j-h}{h-1}\binom{i-h}{j-2h}\left(\frac{l^2}{\xi}(a+\gamma)\right)^{j-2h+1} + \\ & 2\sum_{h=0}^{[(j-1)/2]}\sum_{z=0}^{j-2h-1}\binom{h+z}{h}\binom{i-h-1}{z}\left(\frac{l^2}{\xi}(a+\gamma)\right)^{z+1}, \end{aligned} \tag{43}$$

where $\gamma = \max_{x\in[0,1]}|\gamma(x)|$

Proof. First, estimate the initial values of *k*'s as follows

$$\left|k_{1,1}\right| \le \frac{l^2}{\xi}(a+\gamma), \tag{44}$$

$$\left|k_{2,1}\right| \le \left(\frac{l^2}{\xi}(a+\gamma)+1\right)\frac{l^2}{\xi}(a+\gamma) = \frac{l^4}{\xi^2}(a+\gamma)^2 + \frac{l^2}{\xi}(a+\gamma), \tag{45}$$

$$\left|k_{2,2}\right| \le \frac{2l^2}{\xi}(a+\gamma), \tag{46}$$

$$\left|k_{3,1}\right| \le \frac{l^6}{\xi^3}(a+\gamma)^3 + \frac{2l^4}{\xi^2}(a+\gamma)^2 + \frac{2l^2}{\xi}(a+\gamma), \tag{47}$$

$$\left|k_{3,2}\right| \le \frac{3l^4}{\xi^2}(a+\gamma)^2 + \frac{l^2}{\xi}(a+\gamma), \tag{48}$$

$$\left|k_{3,3}\right| \le \frac{3l^2}{\xi}(a+\gamma), \tag{49}$$

and $k_{i,i}$ and $k_{i,i-1}$ as,

$$\left|k_{i,i}\right| \le \frac{il^2}{\xi}(a+\gamma), \tag{50}$$

$$\left|k_{i,i-1}\right| \le \frac{i(i-1)}{2}\frac{l^4}{\xi^2}(a+\gamma)^2 + \frac{l^2}{\xi}(a+\gamma). \tag{51}$$

By the use of the general identity of $k_{i,j}$ and mathematical induction one can obtain the inequality (43) of Lemma 1.

Lemma 2. The sequence $\{(m+1)k_{m,j}\}_{j=1,\ldots,m,\,m\ge 1}$ remains uniformly bounded in *m* and *j* as $m \to \infty$.

Proof. We can write

$$
\begin{aligned}
(m+1)\left|k_{m,m-bm}\right| &\le (m+1)\binom{m}{bm+1}\left(\frac{u}{(m+1)^2}\right)^{bm+1} + \\
&(m+1)(m-bm)\sum_{h=1}^{[bm/2]}\frac{1}{h}\binom{bm-h}{h-1}\binom{m-h}{bm-2h}\left(\frac{u}{(m+1)^2}\right)^{bm-2h+1} \\
&+2(m+1)\sum_{h=0}^{[(bm-1)/2]}\sum_{z=0}^{bm-2h-1}\binom{h+z}{h}\binom{m-h-1}{z}\left(\frac{u}{(m+1)^2}\right)^{z+1}.
\end{aligned} \quad (52)
$$

The first two terms appeared in expression (27), therefore the third term could be estimated from (52) by using the inequality

$$\binom{h+z}{h}=\binom{h+z}{z}\le (h+z)^z, \quad (53)$$

$$\binom{m-h-1}{z}\le\binom{m}{z}, \quad (54)$$

and $h+z\le h+z_{\max}=h+bm-2h-1=bm-h-1\le bm<m+1,$ we obtain

$$
\begin{aligned}
&(m+1)\sum_{h=0}^{[(bm-1)/2]}\sum_{z=0}^{bm-2h-1}\binom{h+z}{h}\binom{m-h-1}{z}\left(\frac{u}{(m+1)^2}\right)^{z+1}, \\
&\le \frac{u}{m+1}\sum_{h=0}^{[(bm-1)/2]}\sum_{z=0}^{bm-2h-1}\left(\frac{h+z}{m+1}\right)^z\binom{m}{z}\left(\frac{u}{m+1}\right)^z, \\
&\le \frac{u}{m+1}\sum_{h=0}^{[(bm-1)/2]}\sum_{w=0}^{bm}\binom{m}{w}\left(\frac{u}{m}\right)^w 1^{m-w}, \\
&\le \frac{u}{m+1}\left(1+\frac{u}{m}\right)^m\sum_{h=0}^{[(bm-1)/2]}1, \\
&\le ue^u.
\end{aligned}
$$

Theorem 2. Suppose that $\gamma(x)\in L_\infty(0,1)$ and $\xi,a>0,\ r(.,t)$ is a continuous and differentiable function on the interval $[0,\infty)$ and the Laplace transform of $r^2(.,t)$ exist. For any $s_0\in L_\infty(0,1)$, system (1), with (3) and (4) has a unique solution satisfy

$$\|s(x,t)\|^2\le n_2\|s_0\|^2 E_q(-2at^q),\ \ n_2>0. \quad (55)$$

Proof. The proof is similar to Theorem 1.

Neumman type of actuation

This subsection presents the extension of control law from the Dirichlet into the Neumann type, which can be obtained by setting $r(1,t)=0,$ so that

$$r(1,t)=s(1,t)-\int_0^x \tilde{k}(x,\zeta)s(\zeta)\,d\zeta,\quad 0\le x\le 1. \quad (56)$$

The boundary condition of the target system using $s_x(1,t)$ at $x=1$ is presented by

$$r_x(1,t)=c_1 r(x,t). \quad (57)$$

Derive Eq. (56), with respect to x

$$r_x(1,t)=s_x(1,t)-\tilde{k}(1,1)s(1,t)-\int_0^x \tilde{k}_x(x,\zeta)s(\zeta)\,d\zeta. \quad (58)$$

Substitute Eqs. (56), and (57), in (58)

$$s_x(1,t)=c_1 s(1,t)+\tilde{k}(1,1)s(1,t)+\int_0^1 \tilde{k}_x(1,\zeta)s(\zeta)\,d\zeta-c_1\int_0^1 \tilde{k}(1,\zeta)s(\zeta)\,d\zeta. \quad (59)$$

The discrete version of the original system, the target systems and the coordinate transformation lead to the formula

$$r(1,t)=s(1,t)-\psi_m. \quad (60)$$

Differentiating Eq. (60), in space x and solve for $s_x(1,t)$ we obtain

$$s_x^{\text{dis}}(1,t)=c_1 s_m+\frac{k_{m,m}}{l}s_m+\sum_{j=1}^{m-1}\frac{k_{m,j}-k_{m-1,j}}{l}s_j-c_1\sum_{j=1}^{m}k_{m,j}s_j. \quad (61)$$

Comparing Eqs. (59), and. (61), it is guaranteed that

$$s_x^{\text{dis}}(1,t)\xrightarrow{m\to\infty}s_x(1,t),\quad \forall t>0. \quad (62)$$

Since it is uniform boundedness of $k_{i,j}/l$

Numerical simulation

The main target of this section is to test the effectiveness of the presented theoretical results, in Eq. (1), by taking a simulate example with $\gamma(x)=\gamma=17,\ \xi=0.3.$ Let the initial condition be

$$s_0(x)=-0.01e^{6.7x}\sin(8\pi x) \quad (63)$$

Using the technique presented in Proposition 1, consider $a=1,$ $m=200$ with the kernel $k(x)\approx k_m(x)$ see Fig. 3. While Fig. 4 shows oscillation after enlarging some part of $k_m(x)$. This implies to a non-continuous limiting kernel function $k(x)$.

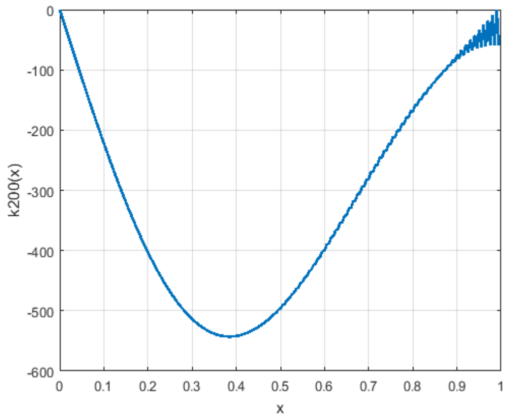

Fig. 3 Kernel function $k_m(x)$ for m=200.

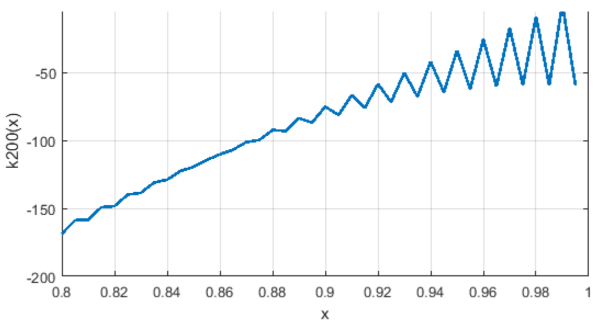

Fig. 4 Oscillation of the approximating kernel function.

The results of the simulations of the open loop system $s(1,t)=0$ are presented in Fig. 5 and Fig. 7 with fractional order $q=0.7,$ and integer order $q = 1$, respectively. While the results of the closed-loop systems are presented in Fig. 6 and Fig. 8. The system was discretised using a BTCS finite difference method with 200 steps and Dirichlet controller.

One can see that the state of the uncontrolled fractional system quickly grows and the initial condition is rapidly smoothed even though the fractional system is unstable. While for the controlled fractional system, the instability is quickly suppressed and the state converges to the zero equilibrium.

The results show the effectiveness of the proposed controller to stabilise the FPDE system and for integer order $q = 1$, we obtained the same results that obtained by [29].

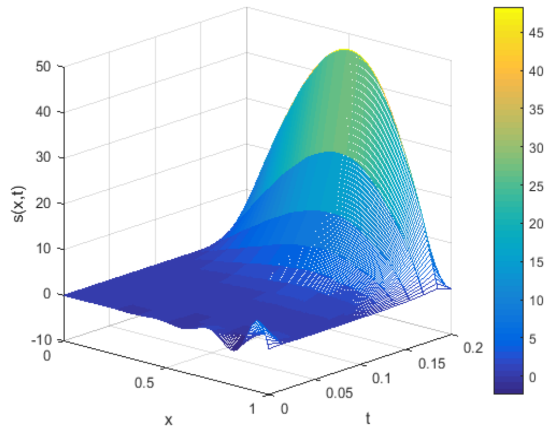

Fig. 5 Solution of the system without control when q = 0.7.

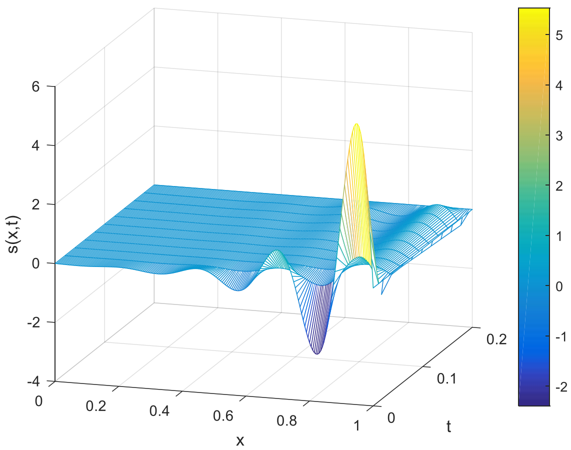

Fig. 6 Approximation of controlled system when q=0.7

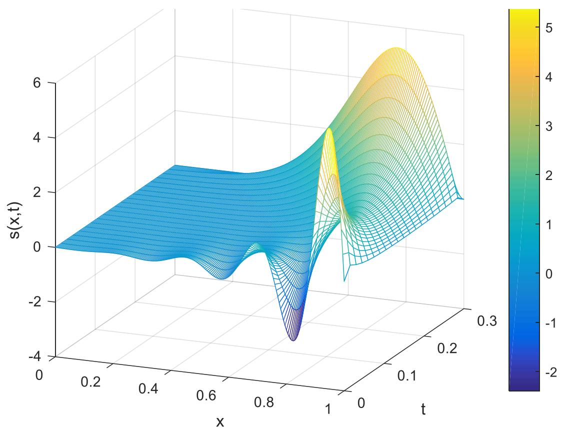

Fig. 7 Solution of the system without control when q = 1.

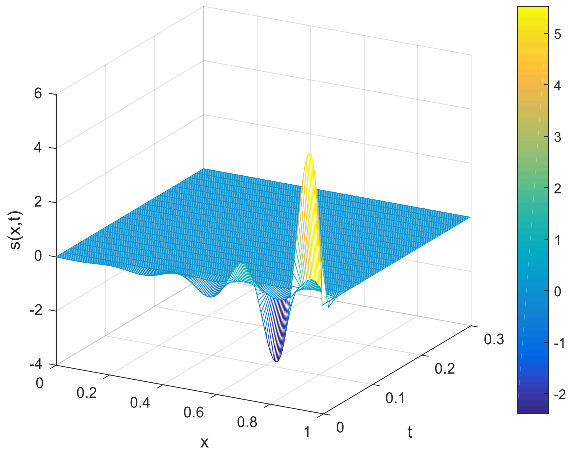

Fig. 8 Approximation of controlled system when q=1

CONCLUSION

In this paper we proposed the discretised technique for designing boundary feedback controller for the time FPDE with two different cases (Dirichlet and Neumann) of boundary conditions at x = 0. Numerical simulation shows that our results are in satisfactory agreement in dealing with unstable FPDE. We hope that the result have could be provided some insights into the qualitative analysis of the design of fractional order controller.

For future work, one can assume more applications of the proposed procedure for a board class of FPDE,

$$ {}_0^cD_t^q s(x,t) = \xi s_{xx}(x,t) + \gamma(x)s(x,t) + \lambda(x)s(0,t) + \int_0^x f(x,y)s(y,t)\,dy,\ 0 < x < 1, $$

with boundary conditions

$$ s(0,t) = 0,\ \text{or}\ s_x(0,t) = \alpha s(0,t),\ t \ge 0, $$

$$ s(1,t) = \int_0^1 k(1,y)s(y,t)\,dy,\ t \ge 0. $$

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