MJFAS MALAYSIAN JOURNAL OF FUNDAMENTAL AND APPLIED SCIENCES
PRINT ISSN: 2289-5981 | ONLINE ISSN: 2289-599X

RESEARCH ARTICLE

Polycyclic transformations of crystallographic groups with quaternion point group of order eight

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Article history
Received 21 September 2017
Accepted 22 October 2017

Abstract

Exploration of a group's properties is vital for better understanding about the group. Amongst other properties, the homological invariants including the nonabelian tensor square of a group can be explicated by showing that the group is polycyclic. In this paper, the polycyclic presentations of certain crystallographic groups with quaternion point group of order eight are shown to be consistent; which implies that these groups are polycyclic.

***Keywords*: Crystallographic groups, polycyclic presentations, quaternion, point group**

INTRODUCTION

In mathematical view, a crystallographic group is the description on the symmetrical pattern of a crystal. It is a symmetry group which has configuration in space. It is an extension of a free abelian group of finite rank by a finite point group. Research on homological invariants of a group has been increasing in number since it is related to the study of the properties of the group using mathematical approach. It includes the nonabelian tensor square ($G \otimes G$), the exterior square ($G \wedge G$), and the Schur multiplier ($M(G)$) of a group. The nonabelian tensor square is requisite in determining the other properties of the group. The nonabelian tensor squares of all groups up to order 30 were computed (Brown *et al.*, 1987). In 1999, the nonabelian tensor squares of 2-generator 2-groups of class 2 were explicated (Kappe *et al.*, 1999) while, in 2008, the homological invariants of all infinite two-generator groups of nilpotency class two were found (Mohd Ali *et al.*, 1998). Meanwhile, the homological invariants of the symmetric group of order six, S_3 were constructed (Ramachandran *et al.*, 2008). The nonabelian tensor square of groups of orders $8q$ where q is an odd prime had been computed (Rashid *et al.*, 2013). Moreover, Zainal *et al*. focused on the nonabelian tensor square and the Schur multiplier of some groups of odd prime power order (Zainal *et al.*, 2013). The crystallographic groups with cyclic point group of order two and its nonabelian tensor square was first explored by Masri (Masri, 2009) while later in 2014 it has been extended on finding other homological invariants of these groups (Mat Hassim *et al.*, 2014). Besides, the homological invariants of crystallographic groups with nonabelian point group, particularly dihedral group of order eight, have been determined (Mohd Idrus *et al.*, 2015; Wan Mohd Fauzi *et al.*, 2015). Furthermore, the homological invariants of crystallographic groups with symmetric point group of order six are found (Tan *et al.*, 2016).

The groups being considered are taken from Crystallographic, Algorithms and Table (CARAT) package (CARAT, 2014). By using the technique on computing the nonabelian tensor square of polycyclic groups (Blyth and Morse, 2009), these groups are transformed from matrix representation to polycyclic presentation before their homological invariants can be computed (Mohammad *et al.*, 2016). It is crucial to perform the consistency check for those polycyclic presentations so that we can proceed to find the homological invariants of the groups. Recently, the polycyclic presentations of the first crystallographic group with quaternion extension was verified to satisfy its consistency relations (Mohammad *et al.*, 2015). Therefore, in this research, the polycyclic presentations of second, third and fourth of torsion free crystallographic groups of dimension six with quaternion point group of order eight will be proved to be consistent.

SOME PRELIMINARIES

To find its homological invariants, we use the technique in (Blyth and Morse, 2009). The polycyclic presentations of these crystallographic groups are shown to be consistent. The following definitions are used throughout this research.

Definition 1: (Eick and Nickel, 2008) **Polycyclic Presentation**

Let F_n be a free group on generators ($g_1, g_2, \ldots\ldots, g_n$) and R be a set of relations of a group G. The relations of a polycyclic presentation have the form:

$$
\begin{aligned}
g_i^{e_i} &= g_{i+1}^{x_{i,i+1}} \ldots g_n^{x_{i,n}} && \text{for } i \in I, \\
g_j^{-1} g_i g_j &= g_{j+1}^{y_{i,j,j+1}} \ldots g_n^{y_{i,j,n}} && \text{for } j < i, \\
g_j g_i g_j^{-1} &= g_{j+1}^{z_{i,j,j+1}} \ldots g_n^{z_{i,j,n}} && \text{for } j < i \text{ and } j \notin I
\end{aligned}
$$

for some $I \subseteq \{1, \ldots, n\}, e_i \in \square$ for $i \in I$ and $x_{i,j}, y_{i,j,k}, z_{i,j,k} \in \square$ for all i, j and k.

Definition 2: (Eick and Nickel, 2008) **Consistent Polycyclic Presentation**

Let G be a group generated by $g_1, g_2, ..., g_n$ and $e_i, e_j \in$ □. The consistency of the relations in G can be determined using the following consistency relations.

$$g_k(g_j g_i) = (g_k g_j)g_i \qquad \text{for } k > j > i,$$
$$(g_j^{e_j})g_i = g_j^{e_j - 1}(g_j g_i) \qquad \text{for } j > i, j \in I,$$
$$g_j(g_i^{e_i}) = (g_j g_i)g_i^{e_i - 1} \qquad \text{for } j > i, i \in I,$$
$$(g_i^{e_i})g_i = g_i(g_i^{e_i}) \qquad \text{for } i \in I,$$
$$g_j = (g_j g_i^{-1})g_i \qquad \text{for } j > i, i \notin I.$$

RESULTS AND DISCUSSION

Based on Definition 1 and Definition 2, the second, third and fourth polycyclic presentations for the crystallographic groups of dimension six with quaternion point group of order eight will be explicated. All of the groups are generated by $l_1, l_2, l_3, l_4, l_5, l_6$ but different a, b where

$$l_1 = \begin{bmatrix} 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&1\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$l_3 = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&1\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$l_5 = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad l_6 = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1 \end{bmatrix}.$$

and for a, b

$Q_2(6)$;

$$a = \begin{bmatrix} 0&-1&1&-1&0&0&-\frac{1}{2}\\ 0&0&0&1&0&0&0\\ -1&-1&0&1&0&0&\frac{1}{2}\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&1&0&\frac{-1}{4}\\ 0&0&0&0&0&1&\frac{1}{4}\\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad b = \begin{bmatrix} -1&-1&0&1&0&0&0\\ 1&1&-1&0&0&0&0\\ 0&1&-1&1&0&0&0\\ -1&0&-1&1&0&0&0\\ 0&0&0&0&0&1&\frac{-1}{2}\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix}$$

$Q_3(6)$;

$$a = \begin{bmatrix} 0&0&0&1&0&-1&-\frac{1}{4}\\ 0&0&1&0&1&0&\frac{1}{2}\\ 0&-1&0&0&0&1&\frac{1}{4}\\ -1&0&0&0&1&0&0\\ 0&0&0&0&1&0&\frac{1}{4}\\ 0&0&0&0&0&1&\frac{1}{4}\\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad b = \begin{bmatrix} 0&0&1&0&1&-1&0\\ 0&0&0&-1&0&0&0\\ -1&0&0&0&0&0&0\\ 0&1&0&0&-1&-1&0\\ 0&0&0&0&0&-1&\frac{-1}{2}\\ 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix}$$

$Q_4(6)$;

$$a = \begin{bmatrix} 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ -1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&1&0&\frac{-1}{4}\\ 0&0&0&0&0&1&\frac{1}{4}\\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad b = \begin{bmatrix} 0&0&0&1&0&0&0\\ 0&0&-1&0&0&0&0\\ 0&1&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&0&0&0&1&\frac{-1}{2}\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix}$$

Therefore, Theorem 1 until Theorem 3 are developed.

Theorem 1
Let $Q_2(6)$ be the second crystallographic group of dimension six with quaternion point group of order eight and its polycyclic presentation is given as in the following:

$$\begin{aligned} Q_2(6) = \langle & a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \mid a^2 = cl_2l_3l_4l_6,\ b^2 = cl_5l_6^{-1}, \\ & b^a = bcl_1^{-1}l_2l_5^{-2}l_6^2,\ c^2 = l_5l_6^{-1},\ c^a = cl_2l_4l_5^{-1}l_6, \\ & c^b = c,\ l_1^a = l_1^{-1}l_2l_4^{-1},\ l_1^b = l_3^{-1}, l_1^c = l_1^{-1}, \\ & l_2^a = l_1^{-1}l_2l_3,\ l_2^b = l_1^{-1}l_3^{-1}l_4^{-1},\ l_2^c = l_2^{-1}, \\ & l_3^a = l_2^{-1}l_3^{-1}l_4^{-1},\ l_3^b = l_1,\ l_3^c = l_3^{-1},\ l_4^a = l_1l_3l_4, \\ & l_4^b = l_1^{-1}l_2l_3, l_4^c = l_4^{-1},\ l_5^a = l_6, l_5^b = l_5, \\ & l_5^c = l_5, l_6^a = l_5,\ l_6^b = l_6,\ l_6^c = l_6, \\ & l_j^{l_i} = l_j,\ l_j^{l_i^{-1}} = l_j \text{ for } j > i,\ 1 \le i, j \le 6 \rangle. \end{aligned} \tag{1}$$

Then, the polycyclic presentation is consistent.

Proof: By Definition 1, $Q_2(6)$ is generated by $a,b,c,l_1,l_2,l_3,l_4,l_5$ and l_6. Let $g_1 = a,\ g_2 = b,\ g_3 = c,\ g_4 = l_1,\ g_5 = l_2,\ g_6 = l_3,\ g_7 = l_4,\ g_8 = l_5$ and $g_9 = l_6$. Based on Definition 2, there are five relations that need to be proven. For the first consistency check, $g_k(g_jg_i) = (g_kg_j)g_i$ for $k > j > i$, the following relations hold:

i) $c(ba) = (cb)a$,
ii) $l_1(cb) = (l_1c)b$,
iii) $l_1(ca) = (l_1c)a$,
iv) $l_1(ba) = (l_1b)a$,
v) $l_2(l_1c) = (l_2l_1)c$,
vi) $l_2(l_1b) = (l_2l_1)b$,
vii) $l_2(l_1a) = (l_2l_1)a$,
viii) $l_2(cb) = (l_2c)b$,
ix) $l_2(ca) = (l_2c)a$,
x) $l_2(ba) = (l_2b)a$,
xi) $l_3(l_2l_1) = (l_3l_2)l_1$,
xii) $l_3(l_2c) = (l_3l_2)c$,
xiii) $l_3(l_2b) = (l_3l_2)b$,
xiv) $l_3(l_2a) = (l_3l_2)a$,
xv) $l_3(l_1c) = (l_3l_1)c$,
xvi) $l_3(l_1b) = (l_3l_1)b$,
xvii) $l_3(l_1a) = (l_3l_1)a$,
xviii) $l_3(cb) = (l_3c)b$,
xix) $l_3(ca) = (l_3c)a$,
xx) $l_3(ba) = (l_3b)a$,
xxi) $l_4(l_3l_2) = (l_4l_3)l_2$,
xxii) $l_4(l_3l_1) = (l_4l_3)l_1$,
xxiii) $l_4(l_3c) = (l_4l_3)c$,
xxiv) $l_4(l_3b) = (l_4l_3)b$,
xxv) $l_4(l_3a) = (l_4l_3)a$,
xxvi) $l_4(l_2l_1) = (l_4l_2)l_1$,
xxvii) $l_4(l_2c) = (l_4l_2)c$,
xxviii) $l_4(l_2b) = (l_4l_2)b$,
xxix) $l_4(l_2a) = (l_4l_2)a$,
xxx) $l_4(l_1c) = (l_4l_1)c$,
xxxi) $l_4(l_1b) = (l_4l_1)b$,
xxxii) $l_4(l_1a) = (l_4l_1)a$,
xxxiii) $l_4(cb) = (l_4c)b$,
xxxiv) $l_4(ca) = (l_4c)a$,
xxxv) $l_4(ba) = (l_4b)a$,
xxxvi) $l_5(l_4l_3) = (l_5l_4)l_3$,
xxxvii) $l_5(l_4l_2) = (l_5l_4)l_2$,
xxxviii) $l_5(l_4l_1) = (l_5l_4)l_1$,
xxxix) $l_5(l_4c) = (l_5l_4)c$,
xl) $l_5(l_4b) = (l_5l_4)b$,
xli) $l_5(l_4a) = (l_5l_4)a$,
xlii) $l_5(l_3l_2) = (l_5l_3)l_2$,

xliii)	$l_5(l_3l_1) = (l_5l_3)l_1$,	lxiv)	$l_6(l_4l_3) = (l_6l_4)l_3$,
xliv)	$l_5(l_3c) = (l_5l_3)c$,	lxv)	$l_6(l_4l_2) = (l_6l_4)l_2$,
xlv)	$l_5(l_3b) = (l_5l_3)b$,	lxvi)	$l_6(l_4l_1) = (l_6l_4)l_1$,
xlvi)	$l_5(l_3a) = (l_5l_3)a$,	lxvii)	$l_6(l_4c) = (l_6l_4)c$,
xlvii)	$l_5(l_2l_1) = (l_5l_2)l_1$,	lxviii)	$l_6(l_4b) = (l_6l_4)b$,
xlviii)	$l_5(l_2c) = (l_5l_2)c$,	lxix)	$l_6(l_4a) = (l_6l_4)a$,
xlix)	$l_5(l_2b) = (l_5l_2)b$,	lxx)	$l_6(l_3l_2) = (l_6l_3)l_2$,
l)	$l_5(l_2a) = (l_5l_2)a$,	lxxi)	$l_6(l_3l_1) = (l_6l_3)l_1$,
li)	$l_5(l_1c) = (l_5l_1)c$,	lxxii)	$l_6(l_3c) = (l_6l_3)c$,
lii)	$l_5(l_1b) = (l_5l_1)b$,	lxxiii)	$l_6(l_3b) = (l_6l_3)b$,
liii)	$l_5(l_1a) = (l_5l_1)a$,	lxxiv)	$l_6(l_3a) = (l_6l_3)a$,
liv)	$l_5(cb) = (l_5c)b$,	lxxv)	$l_6(l_2l_1) = (l_6l_2)l_1$,
lv)	$l_5(ca) = (l_5c)a$,	lxxvi)	$l_6(l_2c) = (l_6l_2)c$,
lvi)	$l_5(ba) = (l_5b)a$,	lxxvii)	$l_6(l_2b) = (l_6l_2)b$,
lvii)	$l_6(l_5l_4) = (l_6l_5)l_4$,	lxxviii)	$l_6(l_2a) = (l_6l_2)a$,
lviii)	$l_6(l_5l_3) = (l_6l_5)l_3$,	lxxix)	$l_6(l_1c) = (l_6l_1)c$,
lix)	$l_6(l_5l_2) = (l_6l_5)l_2$,	lxxx)	$l_6(l_1b) = (l_6l_1)b$,
lx)	$l_6(l_5l_1) = (l_6l_5)l_1$,	lxxxi)	$l_6(l_1a) = (l_6l_1)a$,
lxi)	$l_6(l_5c) = (l_6l_5)c$,	lxxxii)	$l_6(cb) = (l_6c)b$,
lxii)	$l_6(l_5b) = (l_6l_5)b$,	lxxxiii)	$l_6(ca) = (l_6c)a$,
lxix)	$l_6(l_5a) = (l_6l_5)a$,	lxxxiv)	$l_6(ba) = (l_6b)a$.

By the polycyclic presentation of as given in (1),

For i),

$$
\begin{aligned}
c(ba) &= cabcl_1^{-1}l_2l_5^{-2}l_6^2 = acl_2l_4l_5^{-1}l_6bcl_1^{-1}l_2l_5^{-2}l_6^2\\
&= acl_2l_4l_5^{-1}l_6bcl_1^{-1}l_2l_5^{-2}l_6^2 = acl_2l_4bl_5^{-1}l_6cl_1^{-1}l_2l_5^{-2}l_6^2\\
&= acl_2bl_1^{-1}l_2l_3l_5^{-1}l_6cl_1^{-1}l_2l_5^{-2}l_6^2 = acbl_1^{-1}l_3^{-1}l_4^{-1}l_1^{-1}l_2l_3l_5^{-1}l_6cl_1^{-1}l_2l_5^{-2}l_6^2\\
&= abcl_1^{-2}l_2l_4^{-1}cl_5^{-1}l_6l_1^{-1}l_2l_5^{-2}l_6^2 = abcl_1^{-2}l_2cl_4l_5^{-1}l_6l_1^{-1}l_2l_5^{-2}l_6^2\\
&= abcl_1^{-2}cl_2^{-1}l_4l_5^{-3}l_6^3l_1^{-1}l_2 = abcl_1^{-1}cl_1l_2^{-1}l_4l_5^{-3}l_6^3l_1^{-1}l_2\\
&= abccl_1l_1l_2^{-1}l_4l_5^{-3}l_6^3l_1^{-1}l_2 = abccl_1l_1l_2^{-1}l_4l_5^{-3}l_6^3l_1^{-1}l_2\\
&= abc^2l_1l_4l_5^{-3}l_6^3 = abl_1l_4l_5^{-2}l_6^2,
\end{aligned}
$$

while,

$$
\begin{aligned}
(cb)a &= bca = bacl_2l_4l_5^{-1}l_6 = abcl_1^{-1}l_2l_5^{-2}l_6^2cl_2l_4l_5^{-1}l_6\\
&= abcl_1^{-1}l_2cl_5^{-2}l_6^2l_2l_4l_5^{-1}l_6 = abcl_1^{-1}cl_2^{-1}l_2l_4l_5^{-3}l_6^3\\
&= abccl_1l_4l_5^{-3}l_6^3 = abc^2l_1l_4l_5^{-3}l_6^3 = abl_5l_6^{-1}l_1l_4l_5^{-3}l_6^3\\
&= abl_1l_4l_5^{-2}l_6^2.
\end{aligned}
$$

Therefore, $c(ba) = (cb)a$.

For ii),

$$
\begin{aligned}
l_1(ca) &= l_1acl_2l_4l_5^{-1}l_6 = al_1^{-1}l_2l_4^{-1}cl_2l_4l_5^{-1}l_6 = al_1^{-1}l_2cl_4l_2l_4l_5^{-1}l_6\\
&= al_1^{-1}cl_2^{-1}l_4l_2l_4l_5^{-1}l_6 = acl_1l_4^2l_5^{-1}l_6.
\end{aligned}
$$

Whereas,

$$(l_1c)a = cl_1^{-1}a = cal_4l_2^{-1}l_1 = acl_2l_4l_5^{-1}l_6l_4l_2^{-1}l_1 = acl_1l_4^2l_5^{-1}l_6.$$

Hence, $l_1(ca) = (l_1c)a$.
The rest of the relations iii) until lxxxiv) can be shown in a similar way.

Next, the relations of $Q_2(6)$ are shown to satisfy the second consistency relation,

$$(g_j^{e_j})g_i = g_j^{e_j-1}(g_jg_i) \text{ for } j > i, j \in I,$$

The following relations hold:

i)	$b^2a = b(ba)$,	iii)	$c^2b = c(cb)$.
ii)	$c^2a = c(ca)$,		

For i),

$$
\begin{aligned}
b^2a &= cl_5l_6^{-1}a = cl_5al_5^{-1}\\
&= cal_6l_5^{-1} = acl_2l_4l_5^{-1}l_6l_6l_5^{-1}\\
&= acl_2l_4l_5^{-2}l_6^2.
\end{aligned}
$$

Besides,

$$
\begin{aligned}
b(ba) &= babcl_1^{-1}l_2l_5^{-2}l_6^2 = abcl_1^{-1}l_2l_5^{-2}l_6^2bcl_1^{-1}l_2l_5^{-2}l_6^2\\
&= abcl_1^{-1}l_2bcl_5^{-2}l_6^2l_1^{-1}l_2l_5^{-2}l_6^2 = abcl_1^{-1}bl_1^{-1}l_3^{-1}l_4^{-1}cl_5^{-2}l_6^2l_1^{-1}l_2l_5^{-2}l_6^2\\
&= abcbl_3l_1^{-1}l_3^{-1}l_4^{-1}cl_5^{-2}l_6^2l_1^{-1}l_2l_5^{-2}l_6^2 = ab^2cl_1^{-1}cl_4l_5^{-2}l_6^2l_1^{-1}l_2l_5^{-2}l_6^2\\
&= ab^2ccl_1l_4l_5^{-2}l_6^2l_1^{-1}l_2l_5^{-2}l_6^2 = acl_5l_6^{-1}l_5l_6^{-1}l_1l_4l_5^{-2}l_6^2l_1^{-1}l_2l_5^{-2}l_6^2\\
&= acl_2l_4l_5^{-2}l_6^2.
\end{aligned}
$$

Therefore, b^2a is shown to coincide with $b(ba)$ and the rest of the relations can be shown in a similar way.

Next, for the third consistency relations, $g_j(g_i^{e_i}) = (g_jg_i)g_i^{e_i-1}$ for $j > i, i \in I$, the following relations hold:

i)	$ba^2 = (ba)a$,	xii)	$l_3b^2=(l_3b)b$,
ii)	$ca^2 = (ca)a$,	xiii)	$l_4b^2=(l_4b)b$,
iii)	$l_1a^2=(l_1a)a$,	xiv)	$l_5b^2=(l_5b)b$,
iv)	$l_2a^2=(l_2a)a$,	xv)	$l_6b^2=(l_6b)b$,
v)	$l_3a^2=(l_3a)a$,	xvi)	$l_1c^2=(l_1c)c$,
vi)	$l_4a^2=(l_4a)a$,	xvii)	$l_2c^2=(l_2c)c$,
vii)	$l_5a^2=(l_5a)a$,	xviii)	$l_3c^2=(l_3c)c$,
viii)	$l_6a^2=(l_6a)a$,	xix)	$l_4c^2=(l_4c)c$,
ix)	$cb^2= (cb)b$,	xx)	$l_5c^2=(l_5c)c$,
x)	$l_1b^2= (l_1b)b$,	xxi)	$l_6c^2 = (l_6c)c$.
xi)	$l_2b^2= (l_2b)b$,		

For i), we want to prove that $ba^2=(ba)a$,

By (1),
$$ba^2 = bcl_2l_3l_4l_6.$$
Then, we have,

$$
\begin{aligned}
(ba)a &= abcl_1^{-1}l_2l_5^{-2}l_6^2a = abcl_1^{-1}l_2l_5^{-1}l_5^{-1}l_6l_6a = abcl_1^{-1}l_2l_5^{-1}l_5^{-1}l_6al_5\\
&= abcl_1^{-1}l_2l_5^{-1}l_5^{-1}al_5l_5 = abcl_1^{-1}l_2l_5^{-1}al_6^{-1}l_5^2 = abcl_1^{-1}l_2al_6^{-1}l_6^{-1}l_5^2\\
&= abcl_1^{-1}l_2al_5^2l_6^{-2} = abcl_1^{-1}al_1^{-1}l_2l_3l_5^2l_6^{-2} = abcal_1l_2^{-1}l_4l_1^{-1}l_2l_3l_5^2l_6^{-2}\\
&= abcal_3l_4l_5^2l_6^{-2} = abacl_2l_4l_5^{-1}l_6l_3l_4l_5^2l_6^{-2} = abacl_2l_3l_4^2l_5l_6^{-1}\\
&= aabcl_1^{-1}l_2l_5^{-2}l_6^2cl_2l_3l_4^2l_5l_6^{-1} = a^2bcl_1^{-1}l_2cl_5^{-2}l_6^2l_2l_3l_4^2l_5l_6^{-1}\\
&= a^2bcl_1^{-1}cl_2^{-1}l_2l_3l_4^2l_5^{-1}l_6 = a^2bccl_1l_3l_4^2l_5^{-1}l_6\\
&= a^2bl_5l_6^{-1}l_1l_3l_4^2l_5^{-1}l_6 = a^2bl_1l_3l_4^2 = cl_2l_3l_4l_6bl_1l_3l_4^2\\
&= cl_2l_3l_4bl_6l_1l_3l_4^2 = cl_2l_3bl_1^{-1}l_2l_3l_1l_3l_4^2l_6 = cl_2bl_1l_2l_3^2l_4^2l_6\\
&= cbl_1^{-1}l_3^{-1}l_4^{-1}l_1l_2l_3^2l_4^2l_6 = bcl_2l_3l_4l_6.
\end{aligned}
$$

Hence, it is proven that $ba^2=(ba)a$. For ii) until xxi), the relations are shown to be true.

Next, the relations of $Q_2(6)$ are shown to satisfy the forth consistency relation, $(g_i^{e_i})g_i = g_i(g_i^{e_i})$ for $i \in I$. The following relations hold:

i)	$a^2a = aa^2$,	iii)	$c^2c = cc^2$.
ii)	$b^2b = bb^2$,		

By relations (1),
For i),

$$
\begin{aligned}
a^2a &= cl_2l_3l_4l_6a = cl_2l_3l_4al_5 = cl_2al_2^{-1}l_3^{-1}l_4^{-1}l_1l_3l_4l_5 = cal_1^{-1}l_2l_3l_2^{-1}l_1l_5\\
&= cal_3l_5 = acl_2l_4l_5^{-1}l_6l_3l_5 = acl_2l_3l_4l_6.
\end{aligned}
$$

Next,
$$aa^2 = acl_2l_3l_4l_6.$$

The relations for ii) and iii) are shown to be true.

Lastly, the relations of $Q_2(6)$ are shown to satisfy the fifth consistency relation, $g_j = (g_j g_i^{-1})g_i$ for $j > i, i \notin I$. The following relations hold:

i)	$l_2=(l_2\, l_1^{-1})\, l_1$,	ix)	$l_6=(l_6\, l_2^{-1})\, l_2$,
ii)	$l_3=(l_3\, l_1^{-1})\, l_1$,	x)	$l_4=(l_4\, l_3^{-1})\, l_3$,
iii)	$l_4=(l_4\, l_1^{-1})\, l_1$,	xi)	$l_5=(l_5\, l_3^{-1})\, l_3$,
iv)	$l_5=(l_5\, l_1^{-1})\, l_1$,	xii)	$l_6=(l_6\, l_3^{-1})\, l_3$,
v)	$l_6=(l_6\, l_1^{-1})\, l_1$,	xiii)	$l_5=(l_5\, l_4^{-1})\, l_4$,
vi)	$l_3=(l_3\, l_2^{-1})\, l_2$,	xiv)	$l_6=(l_6\, l_4^{-1})\, l_4$,
vii)	$l_4=(l_4\, l_2^{-1})\, l_1$,	xv)	$l_6=(l_6\, l_5^{-1})\, l_5$.
viii)	$l_5=(l_5\, l_2^{-1})\, l_2$,		

All 15 relations above are true since l_1, l_2, l_3, l_4, l_5 and l_6 commute with each other. Since the presentation of $Q_2(6)$ satisfies the consistency relations given in Definition 2, then $Q_2(6)$ has a consistent polycyclic presentation. Using similar method, Theorem 2 and Theorem 3 can be proven.

Theorem 2
Let $Q_3(6)$ be the third crystallographic group of dimension six with quaternion point group of order eight and its polycyclic presentation is given as in the following:

$$\begin{aligned} Q_3(6) = \langle & a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \mid a^2 = cl_3^{-1}l_5, \\ & b^2 = cl_2^{-1}l_4^{-1}l_5^{-1}l_6^{-1},\ \ b^a = bcl_2l_4^2l_5^2l_6^2, \\ & c^2 = l_2^{-1}l_4^{-1}l_5^{-1}l_6^{-1},\ c^a = cl_3^{-1}l_4l_5l_6, \\ & c^b = c,\ l_1^a = l_3^{-1},\ l_1^b = l_4^{-1}, l_1^c = l_1^{-1},\ l_2^a = l_4,\ l_2^b = l_3^{-1}, \\ & l_2^c = l_2^{-1},\ l_3^a = l_1,\ l_3^b = l_2,\ l_3^c = l_3^{-1},\ l_4^a = l_2^{-1},\ l_4^b = l_1, \\ & l_4^c = l_4^{-1}, l_5^a = l_1l_4^{-1}l_6^{-1},\ l_5^b = l_2l_4l_5,\ l_5^c = l_1l_2l_3^{-1}l_4l_5, \\ & l_6^a = l_1^{-1}l_4^{-1}l_5^{-1},\ l_6^b = l_1^{-1}l_3l_6,\ l_6^c = l_1^{-1}l_2l_3l_4l_6, \\ & l_j^{l_i} = l_j,\ l_j^{l_i^{-1}} = l_j \text{ for } j > i,\ 1 \le i, j \le 6 \rangle \end{aligned} \tag{2}$$

Then, the polycyclic presentation is consistent.

Theorem 3
Let $Q_4(6)$ be the fourth crystallographic group of dimension six with quaternion point group of order eight and its polycyclic presentation is given as in the following:

$$\begin{aligned} Q_4(6) = \langle & a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \mid a^2 = cl_6,\ b^2 = c, \\ & b^a = bc^{-1}l_5^{-1},\ c^2 = l_5^{-1}l_6^{-1},\ c^a = c,\ c^b = c, \\ & l_1^a = l_3,\ l_1^b = l_4, l_1^c = l_1^{-1},\ l_2^a = l_4,\ l_2^b = l_3^{-1}, \\ & l_2^c = l_2^{-1},\ l_3^a = l_1^{-1},\ l_3^b = l_2,\ l_3^c = l_3^{-1},\ l_4^a = l_2^{-1}, \\ & l_4^b = l_1^{-1}, l_4^c = l_4^{-1},\ l_5^a = l_5,\ l_5^b = l_6,\ l_5^c = l_5,\ l_6^a = l_6, \\ & l_6^b = l_5,\ l_6^c = l_6,\ l_j^{l_i} = l_j,\ l_j^{l_i^{-1}} = l_j \text{ for } j > i,\ 1 \le i, j \le 6 \rangle \end{aligned} \tag{3}$$

Then, the polycyclic presentation is consistent.

CONCLUSION

In this research, the polycyclic presentations of the second, third and fourth crystallographic group of dimension six with quaternion point group of order eight are shown to be consistent. Therefore, the crystallographic groups with quaternion point group of order eight are polycyclic. These polycyclic presentations can be applied in finding the homological invariants of the groups.

ACKNOWLEDGEMENT

The authors would like to express their appreciation for the support of the sponsor; Ministry of Higher Education (MOHE) Malaysia for the financial funding for this research through Research University Grant (GUP), Vote No: 11J96 and Fundamental Research Grant Scheme (FRGS) Vote no: 4F898 from Research Management Centre (RMC) Universiti Teknologi Malaysia (UTM) Johor Bahru. The first author is also indebted to UTM for her Zamalah Scholarship. The third author is also grateful to UTM and MOHE for her postdoctoral scholarship in University of Leeds, United Kingdom.

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