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RESEARCH ARTICLE

The conjugacy class graph of some finite groups and its energy

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Abstract

The energy of a graph Γ, which is denoted by $\varepsilon(\Gamma)$, is defined to be the sum of the absolute values of the eigenvalues of its adjacency matrix. In this paper we present the concepts of conjugacy class graph of dihedral groups and introduce the general formula for the energy of the conjugacy class graph of dihedral groups. The energy of any dihedral group of order $2n$ in different cases, depends on the parity of n is proved in this paper. Also we introduce the general formula for the conjugacy class graph of generalized quaternion groups and quasidihedral groups.

Keywords: Energy of graph; Conjugacy class graph; adjacency matrix and eigenvalues.

INTRODUCTION

Let Γ, be a graph with set of vertex $V(\Gamma)=\{1,\ldots,n\}$ and set of edge $E(\Gamma)=\{e_1,\ldots,e_n\}$. The adjacency matrix of Γ, denoted by $A(\Gamma)$, is an $n\times n$ matrix defined as follows: the rows and the columns of $A(\Gamma)$ are indexed by $V(\Gamma)$. If $i\neq j$, then the (i,j)-entry of $A(\Gamma)$ is 0 and 1 for nonadjacent and adjacent vertices i and j respectively. The (i,i)-entry of $A(\Gamma)$ is 0 for $i=1,\ldots,n$ [1]. Let Γ be a simple graph, A be its adjacency matrix and $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of the graph Γ. By eigenvalues of the graph Γ we mean the eigenvalues of its adjacency matrix. The energy of Γ is defined as the sum of absolute values of its eigenvalues [2].

The energy of graph was first defined by Ivan Gutman in 1978. It is used in chemistry to approximate the total π-electron energy of molecules. The vertices of the graph represented the carbon atoms while the single edge between each pair of distinct vertices represented the hydrogen bonds between the carbon atoms [3]. There are many researches on the eigenvalues and energy of some kinds of graphs, for instance see [4-9]. Recently there are many researches in constructing a graph by a group, for instance one can refer to the work in [10, 11]. This paper consists of three parts. The first part is the introduction of the energy of the graph which is constructed by a group, followed by some fundamental concepts and definitions related to conjugacy classes and conjugacy class graph. The second part consists of some previous results which are used in this paper. Our main results are presented in the third part, in which we compute the eigenvalues and the energy of the conjugacy class graphs of dihedral groups of order $2n$, generalized quaternion groups of order $4n$ and quasidihedral groups, and lastly the general formulas for the energy of conjugacy class graphs of these groups are introduced.

Suppose G is a finite group. Two elements a and b of G are called conjugate if there exists an element $g\in G$ with $gag^{-1}=b$. The conjugacy class is an equivalence relation and therefore partition G into some equivalence classes. This means that every element of the group G belongs to precisely one conjugacy class. The equivalence class that contains the element $a\in G$ is $cl(a)=\{gag^{-1}:g\in G\}$ and is called the conjugacy class of a. The classes $\mathrm{cl}(a)$ and $cl(b)$ are equal if and only if a and b are conjugate. The class number of G is the number of distinct (non-equivalent) conjugacy classes and we denote it by $K(G)$.

PRELIMINARIES

In this section, some previous results and definitions on the conjugacy class graph and the energy of graph are presented which are used in this paper.

Definition 1 (Bapat, R. B. 2010.)

Suppose that G is a finite group and $Z(G)$ be the center of G. The vertices of conjugacy class graph of G are non-central conjugacy classes of G i.e. $|V(G)|=K(G)-|Z(G)|$, where $K(G)$ is the class number of G. Two vertices are adjacent if their cardinalities are not coprime (i.e. have common factor).

Definition 2 (Beineke, L. W., and Wilson, R. J. 2007)

Let Γ be a simple graph. Then Γ is called a complete graph with n vertices and is denoted by K_n if there is an edge between any two arbitrary vertices.

Proposition 1 (Balakrishnan, R. 2004.)

If the graph Γ consists of (disconnected) components Γ_1 and Γ_2 then the energy of Γ is $\varepsilon(\Gamma)=\varepsilon(\Gamma_1)+\varepsilon(\Gamma_2)$ and if one component of the graph Γ is Γ_1 and other components are isolated vertices, then $\varepsilon(\Gamma)=\varepsilon(\Gamma_1)$.

Proposition 2 (Gutman, I. and Zhou, B. 2006)

A totally disconnected graph has zero energy, while the complete graph K_n, with the maximum possible number of edges (among graph on *n* vertices) has an energy $2(n-1)$.

Proposition 3 (Bapat, 2010)

Let Γ be a complete graph namely K_n, then the eigenvalues of K_n, are $\lambda=n-1$ (with multiplicity 1) and $\lambda=-1$ (with multiplicity *n*-1).
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Theorem 1 (Samaila, D., Abba, B. I., and Pur, M. P. 2013)

The conjugacy classes in dihedral group D_{2n} are as follows, depending on the parity of *n*.

1. For odd *n*:

$$\{1\},\{a,a^{-1}\},...,\left\{a^{\frac{n-1}{2}},a^{-\left(\frac{n-1}{2}\right)}\right\},\{a^ib,0\le i\le n-1\}.$$

2. For even *n*:

$$\{1\},\left\{a^{\frac{n}{2}}\right\},\{a,a^{-1}\},\{a^2,a^{-2}\},...,\left\{a^{\frac{n-2}{2}},a^{-\left(\frac{n-2}{2}\right)}\right\},\left\{a^{2i}b,0\le i\le\frac{n-2}{2}\right\}$$

and $\left\{a^{2i+1}b,0\le i\le\dfrac{n-2}{2}\right\}$.

MAIN RESULTS

In this section we present our main results, namely the energy of the conjugacy class graph of all dihedral groups of order 2*n*, generalized quaternion groups and quasidihedral groups. First, we found the conjugacy class graph for all dihedral groups, D_{2n}.

Proposition 4

Suppose that D_{2n} be a dihedral group of order 2*n* where $n\ge 3, n\in\mathbb{N}$. Then the conjugacy class graphs of D_{2n}, are as follows:

Case 1: *n* odd

$\Gamma^{cl}_{D_{2n}}$ is a complete graph joined with isolated vertices, namely $\Gamma^{cl}_{D_{2n}}=K_{\frac{n-1}{2}}$.

Case 2: *n* and $\frac{n}{2}$ even

$\Gamma^{cl}_{D_{2n}}=K_{\frac{n+2}{2}}$.

Case 3: *n* even and $\frac{n}{2}$ odd

$\Gamma^{cl}_{D_{2n}}=K_{\frac{n-2}{2}}\cup K_2$.

Proof

Suppose D_{2n} is a dihedral group of order 2*n* and $\Gamma^{cl}_{D_{2n}}$ be its conjugacy class graph,

Case 1: n odd

By using Theorem 1, the set of vertex of the conjugacy class graph $\Gamma^{cl}_{D_{2n}}$ is $V(\Gamma^{cl}_{D_{2n}})=\left\{cl(a),cl(a^2),\ldots,cl\left(a^{\frac{n-1}{2}}\right),cl(b)\right\}$. For all vertices of the form $cl(a^k), k=1,2,\ldots,\frac{n-1}{2}$, their cardinalities are not coprime, hence they are adjacent and the greatest common diviser of $cl(a^k)$ and $cl(b)$, $\gcd(|cl(a^k)|,|cl(b)|)=1$, thus $\Gamma^{cl}_{D_{2n}}=K_{\frac{n-1}{2}}$, joined with isolated vertex $cl(b)$.

Case 2: *n* and $\frac{n}{2}$ even

By using Theorem 1, the set of vertex of the conjugacy class graph $\Gamma^{cl}_{D_{2n}}$ $V(\Gamma^{cl}_{D_{2n}})=\left\{cl(a),cl(a^2),\ldots,cl\left(a^{\frac{n-2}{2}}\right),cl(b),cl(ab)\right\}$. For all vertices of the form $cl(a^k)$, $k=1,2,\ldots,\frac{n-2}{2}$, their cardinalities are equal to 2 and are not coprime, hence they are adjacent and $|cl(b)|=|cl(ab)|$ = even integer thus all vertices are adjacent and $\Gamma^{cl}_{D_{2n}}$ is a complete graph. Now we have $\frac{n-2}{2}$ and two adjacent vertices, $\left(\frac{n-2}{2}+2=\frac{n+2}{2}\right)$, thus $\Gamma^{cl}_{D_{2n}}=K_{\frac{n+2}{2}}$.

Case 3: *n* even and $\frac{n}{2}$ odd

By using Theorem 2, the set of vertex of the conjugacy class graph $\Gamma^{cl}_{D_{2n}}$ is

$$V(\Gamma^{cl}_{D_{2n}})=\left\{cl(a),cl(a^2),\ldots,cl\left(a^{\frac{n-2}{2}}\right),cl(b),cl(ab)\right\}.$$

For all vertices of the form $cl(a^k)$, $k=1,2,\ldots,\frac{n-2}{2}$, their cardinalities are equal to 2 and are not coprime, hence they are adjacent and $|\mathrm{cl(b)}|=|\mathrm{cl(ab)}|$ = odd integer thus for all vertices of the form $|cl(b)|$ $cl(a^k), k=1,2,\ldots,\frac{n-2}{2}$, their cardinalities are coprime with and $|\mathrm{cl(ab)}|$ Thus $\Gamma^{cl}_{D_{2n}}=K_{\frac{n-2}{2}}\cup K_2$.

Next, the energy of the conjugacy class graphs of dihedral groups are introduce in the following three theorems according to the cases as in Proposition 4. ■

Theorem 2

Let *G* be a dihedral group of order 2*n*, where *n* is an odd, integer, $n\ge 3, n\in\mathbb{Z}^+$ and $\Gamma^{cl}_{D_{2n}}$ be its conjugacy class graph. Then the energy of the graph $\Gamma^{cl}_{D_{2n}}$ is $\varepsilon(\Gamma^{cl}_{D_{2n}})=n-3$.

Proof

Suppose G is a dihedral group of order $2n$ and $\Gamma_{D_{2n}}^{cl}$ be its conjugacy class graph, from Proposition 4 case 1, the conjugacy class graphs of the group G, are complete graphs joined with isolated vertices, namely $\Gamma_{D_{2n}}^{cl} = K_{\frac{n-1}{2}}$. The eigenvalues of the complete graph $K_{\frac{n-1}{2}}$ are $\lambda = \left(\frac{n-1}{2}\right)-1$ (with multiplicity 1) and $\lambda = -1$ (with multiplicity $\left(\frac{n-1}{2}\right)-1$) Hence by Proposition 2, $\varepsilon\left(\Gamma_{D_{2n}}^{cl}\right) = 2\left(\left(\frac{n-1}{2}\right)-1\right) = n-3.$

Suppose G is a dihedral group of order $2n$ and $\Gamma_{D_{2n}}^{cl}$ be its conjugacy class graph, from Proposition 4 case 1, the conjugacy class graphs of the group G, are complete graphs joined with isolated vertices, namely $\Gamma_{D_{2n}}^{cl} = K_{\frac{n-1}{2}}$. The eigenvalues of the complete graph $K_{\frac{n-1}{2}}$ are $\lambda = \left(\frac{n-1}{2}\right)-1$ (with multiplicity 1) and $\lambda = -1$ (with multiplicity) $\left(\frac{n-1}{2}\right)-1$) Hence by Proposition 2, $\varepsilon\left(\Gamma_{D_{2n}}^{cl}\right) = 2\left(\left(\frac{n-1}{2}\right)-1\right) = n-3.$ ■

The following example illustrate Theorem 2.

Example 1

If G is a dihedral group of order 10, i.e. $G = D_{10} \cong \left\langle a,b : a^5 = b^2 = 1, bab = a^{-1}\right\rangle$, and $\Gamma_{D_{10}}^{cl}$ be its conjugacy class graph. Then the conjugacy classes of D_{10} are, $cl(e) = \{e\}$ $cl(a) = \{a, a^4\}, cl(a^2) = \{a^2, a^3\}$ and $cl(b) = \{b, ab, a^2b, a^3b, a^4b\}$. Thus, $V\left(\Gamma_{D_{10}}^{cl}\right) = k(D_{10}) - |Z(D_{10})| = 3.$

The set of vertex of this graph is $V\left(\Gamma_{D_{10}}^{cl}\right) = \{cl(a), cl(a^2), cl(b)\}$ and the set of edge is $E\left(\Gamma_{D_{10}}^{cl}\right) = \{cl(a)cl(a^2)\}$. Thus the conjugacy class graph of the dihedral group D_{10} is $\Gamma_{D_{10}}^{cl} = K_2 \cup \{cl(b)\}$ and the adjacency matrix of the graph $\Gamma_{D_{10}}^{cl}$ is $A\left(\Gamma_{D_{10}}^{cl}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and the eigenvalues of $A\left(\Gamma_{D_{10}}^{cl}\right)$ are 0,1 and -1. By the definition of the energy of the graph, we have $\varepsilon\left(\Gamma_{D_{10}}^{cl}\right) = |0| + |1| + |-1| = 2$. Meanwhile by using Theorem 2, $\varepsilon\left(\Gamma_{D_{10}}^{cl}\right) = 5 - 3 = 2.$

Let G be a dihedral group of order 10, i.e. $G = D_{10} \cong \left\langle a,b : a^5 = b^2 = 1, bab = a^{-1}\right\rangle$, and $\Gamma_{D_{10}}^{cl}$ be its conjugacy class graph. The conjugacy classes of D_{10} are, $cl(e) = \{e\}$ $cl(a) = \{a, a^4\}, cl(a^2) = \{a^2, a^3\}$ and $cl(b) = \{b, ab, a^2b, a^3b, a^4b\}$. Thus, $V\left(\Gamma_{D_{10}}^{cl}\right) = k(D_{10}) - |Z(D_{10})| = 3.$

The set of vertex of this graph is $V\left(\Gamma_{D_{10}}^{cl}\right) = \{cl(a), cl(a^2), cl(b)\}$ and the set of edge is $E\left(\Gamma_{D_{10}}^{cl}\right) = \{cl(a)cl(a^2)\}$. Thus the conjugacy class graph of the dihedral group D_{10} is $\Gamma_{D_{10}}^{cl} = K_2 \cup \{cl(b)\}$ and the adjacency matrix of the graph $\Gamma_{D_{10}}^{cl}$ is $A\left(\Gamma_{D_{10}}^{cl}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and the eigenvalues of $A\left(\Gamma_{D_{10}}^{cl}\right)$ are 0,1 and -1. By the definition of the energy of the graph, we have $\varepsilon\left(\Gamma_{D_{10}}^{cl}\right) = |0| + |1| + |-1| = 2$. Meanwhile by using Theorem 2, $\varepsilon\left(\Gamma_{D_{10}}^{cl}\right) = 5 - 3 = 2.$

Theorem 3

Let G be a dihedral group of order $2n$, where n and $\frac{n}{2}$ are even integers and $n \geq 4$, and $\Gamma_{D_{2n}}^{cl}$ be its conjugacy class graph. Then the energy of the graph $\Gamma_{D_{2n}}^{cl}$, $\varepsilon\left(\Gamma_{D_{2n}}^{cl}\right) = n.$

Proof

Suppose G is a dihedral group of order $2n$ and $\Gamma_{D_{2n}}^{cl}$ be its conjugacy class graph, from Proposition 3.1 case 2, the conjugacy class graphs of G when n and $\frac{n}{2}$ are even integers, are complete graphs of the form $\Gamma_{D_{2n}}^{cl} = K_{\frac{n+2}{2}}$. By using Proposition 3, the eigenvalues of the complete graph $K_{\frac{n+2}{2}}$ are $\lambda = \left(\frac{n+2}{2}\right)-1$ (with multiplicity 1) and $\lambda = -1$ (with multiplicity $\left(\frac{n+2}{2}\right)-1$). Hence by using Proposition 2, $\varepsilon\left(\Gamma_{D_{2n}}^{cl}\right) = 2\left(\left(\frac{n+2}{2}\right)-1\right) = n.$ ■

The example below gives the illustration of the above theorem.

Example 2

Suppose $G = D_8 \cong \left\langle a,b : a^4 = b^2 = 1, bab = a^{-1}\right\rangle$ is a dihedral group of order 8, and $\Gamma_{D_8}^{cl}$ be its conjugacy class graph. Then the conjugacy classes of D_8 are, $cl(e) = \{e\}$, $cl(a) = \{a, a^3\}$, $cl(a^2) = \{a^2\}, cl(b) = \{b, a^2b\}, cl(ab) = \{ab, a^3b\}$, thus the set of vertex of this graph is $V\left(\Gamma_{D_8}^{cl}\right) = \{cl(a), cl(b), cl(ab)\}$ and the set of edge is $E\left(\Gamma_{D_8}^{cl}\right) = \{cl(a)cl(b), cl(a)cl(ab), cl(b)cl(ab)\}$. Thus the conjugacy class graph of dihedral group D_8 is $\Gamma_{D_8}^{cl} = K_3$. The adjacency matrix of the graph $\Gamma_{D_8}^{cl}$ is $A\left(\Gamma_{D_8}^{cl}\right) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and the eigenvalues of $A\left(\Gamma^{cl}_{D_8}\right)$ are $\lambda=2$ and $\lambda=-1$ (with multiplicity 2). By the definition of the energy of the graph, we have $\varepsilon\left(\Gamma^{cl}_{D_8}\right)=|2|+|-1|+|-1|=4.$ Meanwhile by using Theorem 3, $\varepsilon\left(\Gamma^{cl}_{D_8}\right)=4.$

Theorem 4

Let G be a dihedral group of order $2n$, where n is even integer and $\frac{n}{2}$ is odd integer, and $\Gamma^{cl}_{D_{2n}}$ be its conjugacy class graph, then the energy of the graph $\Gamma^{cl}_{D_{2n}}$ is $\varepsilon\left(\Gamma^{cl}_{D_{2n}}\right)=n-2.$

Proof

Suppose G is a dihedral group of order $2n$ and $\Gamma^{cl}_{D_{2n}}$ be its conjugacy class graph**,** from Proposition 4, case 3, the conjugacy class graphs of D_{2n}, when n is even integer and $\frac{n}{2}$ is odd integer, are the union of the complete graphs of the form $\Gamma^{cl}_{D_{2n}}=K_{\frac{n-2}{2}}\cup K_2,$ using Proposition 3, the eigenvalues of the complete graph $K_{\frac{n-2}{2}}$ are $\lambda=\left(\frac{n-2}{2}\right)-1$ (with multiplicity 1) and $\lambda=-1$ (with multiplicity $\left(\frac{n-2}{2}\right)-1$) and the eigenvalues of the graph K_2 are $\lambda=1$ and $\lambda=-1.$ Hence by Proposition 2 $\varepsilon\left(\Gamma^{cl}_{D_{2n}}\right)=2\left(\left(\frac{n-2}{2}\right)-1\right)+2(2-1)=n-2.$ ■

Example 3

If G is a dihedral group of order 28, $G=D_{28}\cong\left\langle a,b:a^{14}=b^2=1,bab=a^{-1}\right\rangle$ and $\Gamma^{cl}_{D_{28}}$ be its conjugacy class graph. Then the conjugacy classes of D_{28} are:

$cl(e)=\{e\},cl(a)=\{a,a^{13}\},cl(a^2)=\{a^2,a^{12}\},$
$cl(a^3)=\{a^3,a^{11}\},cl(a^4)=\{a^4,a^{10}\},$
$cl(a^5)=\{a^5,a^9\},cl(a^6)=\{a^6,a^8\},cl(a^7)=\{a^7\},$
$cl(b)=\{b,a^2b,a^4b,a^6b,a^8b,a^{10}b,a^{12}b\},$
$cl(ab)=\{ab,a^3b,a^5b,a^7b,a^9b,a^{11}b,a^{13}b\}.$

Thus, $k(D_{28})=10,$ and $Z(D_{28})=\{e,a^7\}$, thus $\left|V\left(\Gamma^{cl}_{D_{28}}\right)\right|=k(D_{28})-|Z(D_{28})|=8,$ the set of vertex of the graph $\Gamma^{cl}_{D_{28}}$ is $V\left(\Gamma^{cl}_{D_{2n}}\right)=\left\{\begin{matrix}cl(a),cl(a^2),cl(a^3),cl(a^4),cl(a^5),cl(a^6),\\ cl(b),cl(ab)\end{matrix}\right\}$ and the set of edge is

$$\begin{aligned}E\left(\Gamma^{cl}_{D_{28}}\right)=\{&cl(a)cl(a^2),cl(a)cl(a^3),cl(a)cl(a^4),cl(a)cl(a^5),\\ &cl(a)cl(a^6),cl(a^2)cl(a^3),\\ &cl(a^2)cl(a^4),cl(a^2)cl(a^5),cl(a^2)cl(a^6),cl(a^3)cl(a^4),\\ &cl(a^3)cl(a^5),cl(a^3)cl(a^6),\\ &cl(a^4)cl(a^5),cl(a^4)cl(a^6),cl(a^5)cl(a^6),cl(b)cl(ab)\}.\end{aligned}$$

Thus, $\Gamma^{cl}_{D_{28}}=K_6\cup K_2$ and the adjacency matrix of this graph is the following 8×8 matrix:

$$A\left(\Gamma^{cl}_{D_{28}}\right)=\begin{bmatrix}0&1&1&1&1&1&0&0\\1&0&1&1&1&1&0&0\\1&1&0&1&1&1&0&0\\1&1&1&0&1&1&0&0\\1&1&1&1&0&1&0&0\\1&1&1&1&1&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{bmatrix}.$$

Thus $\lambda=5,\lambda=-1$ (with multiplicity 5), $\lambda=1$ and $\lambda=-1.$ By the definition of the energy of the graph we have $\varepsilon\left(\Gamma^{cl}_{D_{28}}\right)=|5|+5|-1|+|1|+|-1|=12.$

Proposition 5

Let G be a generalized quaternion group of order $4n$ where $n\geq 2,n\in\mathbb{N}.$ Then the conjugacy class graphs of Q_{4n} are as follows:
Case 1: n odd
$\Gamma^{cl}_{Q_{4n}}=K_{n-1}\cup K_2.$
Case 2: n even
$\Gamma^{cl}_{Q_{4n}}=K_{n+1}.$

Proof

Suppose Q_{4n} be a generalized quaternion group of order $4n$ and $\Gamma^{cl}_{Q_{4n}}$ be its conjugacy class graph,
Case 1: n odd
The conjugacy classes in generalized quaternion group Q_{4n} when n is odd are $\{1\},\{a^n\},\{a,a^{-1}\},...,\{a^{n-1},a^{-(n-1)}\},\{a^{2i}b,0\leq i\leq n-1\}$ and $\{a^{2i+1}b,0\leq i\leq n-1\}.$ Then the set of vertex of the conjugacy class graph $\Gamma^{cl}_{Q_{4n}}$ is $V\left(\Gamma^{cl}_{Q_{4n}}\right)=\left\{cl(a),cl(a^2),\ldots,cl(a^{n-1}),cl(b),cl(ab)\right\}.$ For all vertices of the form $cl(a^k),k=1,2,\ldots,n-1,$ their cardinalities are not coprime, hence they are adjacent and $|cl(b)|=|cl(ab)|$ = odd integer. Thus for all vertices of the form $cl(a^k),k=1,2,\ldots,n-1,$ their cardinalities are coprime with $|cl(b)|$ and $|\mathrm{cl}(\mathrm{ab})|$ Thus $\Gamma^{cl}_{Q_{4n}}=K_{n-1}\cup K_2.$
Case 2: n even
The conjugacy classes in generalized quaternion group Q_{4n} when n is even are $\{1\},\{a^n\},\{a,a^{-1}\},\{a^2,a^{-2}\},...,\{a^{n-1},a^{-(n-1)}\},\{a^{2i}b,0\leq i\leq 2n-2\}$ and $\{a^{2i+1}b,0\leq i\leq 2n-1\}.$ Then the set of vertex of the conjugacy class graph $\Gamma^{cl}_{Q_{4n}}$ $V\left(\Gamma^{cl}_{Q_{4n}}\right)=\left\{cl(a),cl(a^2),\ldots,cl(a^{n-1}),cl(b),cl(ab)\right\}$. For all vertices of the form $cl(a^k)$, $k=1,2,\ldots,n-1,$ their cardinalities are equal to 2 and are not coprime, hence they are adjacent and $|cl(b)|=|cl(ab)|$ = even integer. Thus, all vertices are adjacent and $\Gamma^{cl}_{Q_{4n}}$ is a complete graph. Now we have $n-1$ and two adjacent vertices, $(n-1+2=n+1).$, thus $\Gamma^{cl}_{Q_{4n}}=K_{n+1}.$ ■

Theorem 5

Let G be a generalized quaternion group of order $4n$, where n is an odd, integer, $n \geq 3, n \in \mathbb{N}$ and $\Gamma^{cl}_{Q_{4n}}$ be its conjugacy class graph. Then the energy of the graph $\Gamma^{cl}_{Q_{4n}}$ is $\varepsilon\left(\Gamma^{cl}_{Q_{4n}}\right) = 2n - 2.$

Proof

Suppose G be a generalized quaternion group of order $4n$ and $\Gamma^{cl}_{Q_{4n}}$ be its conjugacy class graph, from Proposition 5, case 1, the conjugacy class graphs of Q_{4n}, when n is odd integer are the union of the complete graphs of the form $\Gamma^{cl}_{Q_{4n}} = K_{n-1} \cup K_2$. The eigenvalues of the complete graph K_{n-1} are $\lambda = (n-1)\text{-}1$ (with multiplicity 1) and $\lambda = \text{-}1$ (with multiplicity $(n-1)\text{-}1$) and the eigenvalues of the graph K_2 are $\lambda = 1$ and $\lambda = -1$. Hence, by Proposition 2, $\varepsilon\left(\Gamma^{cl}_{D_{2n}}\right) = 2\left((n-1)\text{-}1\right) + 2(2-1) = 2n-2.$ ■

The following example illustrate Theorem 5.

Example 4

Let G be a generalized quaternion group of order 12, i.e. $G = Q_{12} \cong \left\langle a,b : a^3 = b^2, a^6 = 1, b^{-1}ab = a^{-1} \right\rangle$, and $\Gamma^{\mathrm{cl}}_{\mathrm{D}_{10}}$ be its conjugacy class graph. The conjugacy classes of Q_{12} are, cl(e) = {e} $cl(a^3) = a^3$ $cl(a) = \{a, a^5\}$, $cl(a^2) = \{a^2, a^4\}$, $cl(b) = \{b, a^2b, a^4b,\}$, and $cl(ab) = \{ab, a^3b, a^5b,\}$.

The set of vertex of this graph is $V\left(\Gamma^{cl}_{Q_{12}}\right) = \left\{cl(a), cl(a^2),, cl(b), cl(ab)\right\}$ and the set edge is

$$E\left(\Gamma^{cl}_{Q_{12}}\right) = \left\{\begin{array}{l} cl(a)cl(a^2), cl(a)cl(a^3), cl(a)cl(b), \\ cl(a)cl(ab), cl(a^2)cl(a^3), cl(a^2)cl(b), \\ cl(a^2)cl(ab), cl(a^3)cl(b), cl(a^3)cl(ab), \\ cl(b), cl(ab) \end{array}\right\}.$$

Thus, the Q_{12} is conjugacy class graph of the generalized quaternion group $\Gamma^{cl}_{Q_{12}}$ is $\Gamma^{cl}_{Q_{12}} = K_2 \cup K_2$ and the adjacency matrix of the graph

$$A\left(\Gamma^{cl}_{Q_{12}}\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and the eigenvalues of $A\left(\Gamma^{cl}_{Q_{12}}\right)$ are $\lambda = 1$ with multiplicity 2and $\lambda = -1$ with multiplicity 2.

By the definition of the energy of the graph, we have $\varepsilon\left(\Gamma^{cl}_{Q_{12}}\right) = 2|1| + 2|-1| = 4.$. Meanwhile by using Theorem 3, $\varepsilon\left(\Gamma^{cl}_{Q_{12}}\right) = 2.3 - 2 = 4.$

Theorem 6

Let G be a generalized quaternion group of order $4n$, where n is even integers, $n \geq 2$, and $\Gamma^{cl}_{Q_{4n}}$ be its conjugacy class graph. Then the energy of the graph $\Gamma^{cl}_{Q_{4n}}$ $\varepsilon\left(\Gamma^{cl}_{Q_{4n}}\right) = 2n.$

Proof

Suppose G be a generalized quaternion group of order $4n$ and $\Gamma^{cl}_{Q_{4n}}$ be its conjugacy class graph, from Proposition 5, case 2, the conjugacy class graphs of G when n is even integer, is complete graphs of the form $\Gamma^{cl}_{Q_{4n}} = K_{n+1}$. By using Proposition 3, the eigenvalues of the complete graph K_{n+1} are $\lambda = (n+1)-1$ (with multiplicity 1) and $\lambda = -1$ (with multiplicity $(n+1)-1$). Hence by using Proposition 2, $\varepsilon\left(\Gamma^{cl}_{Q_{4n}}\right) = 2\left((n+1)-1\right) = 2n.$ ■

Example 5

Suppose G is a generalized quaternion group of order 16, i.e. $G = Q_{16} \cong \left\langle a,b : a^4 = b^2, a^8 = 1, b^{-1}ab = a^{-1} \right\rangle$, and $\Gamma^{\mathrm{cl}}_{\mathrm{D}_{10}}$ be its conjugacy class graph. The conjugacy classes of Q_{16} are, cl(e) = {e} $cl(a) = \{a, a^7\}$, $cl(a^2) = \{a^2, a^6\}$, $cl(a^3) = \{a^3, a^5\}$, $cl(b) = \{b, a^2b, a^4b, a^6b\}$, and $cl(ab) = \{ab, a^3b, a^5b, a^7b\}$.

The set of vertex of this graph is $V\left(\Gamma^{cl}_{Q_{16}}\right) = \left\{cl(a), cl(a^2), cl(a^3), cl(b), cl(ab)\right\}$ and the set of edge is

$$E\left(\Gamma^{cl}_{Q_{16}}\right) = \left\{\begin{array}{l} cl(a)cl(a^2), cl(a)cl(a^3), cl(a)cl(b), cl(a)cl(ab), \\ cl(a^2)cl(a^3), cl(a^2)cl(b), cl(a^2)cl(ab), cl(a^3)cl(b), \\ cl(a^3)cl(ab), cl(b), cl(ab) \end{array}\right\}.$$

Thus the conjugacy class graph of the generalized quaternion group Q_{16} is $\Gamma^{cl}_{Q_{16}} = K_5$ and the adjacency matrix of the graph $\Gamma^{cl}_{Q_{16}}$ is

$$A\left(\Gamma^{cl}_{Q_{16}}\right) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and the eigenvalues of $A\left(\Gamma^{cl}_{Q_{16}}\right)$ are $\lambda = 4$, and $\lambda = -1$ with multiplicity 4.

By the definition of the energy of the graph, we have $\varepsilon\left(\Gamma^{cl}_{Q_{16}}\right) = |4| + 4|-1| = 8$. Meanwhile by using theorem 3.2, $\varepsilon\left(\Gamma^{cl}_{Q_{16}}\right) = 2.4 = 8.$

Proposition 6

Let be a quasidihedral group of order 2^n where $n \geq 4, n \in \mathbb{Z}^+$. Then the conjugacy class graphs of QD_{2^n} is as follows: $\Gamma^{cl}_{QD_{2^n}} = K_{2^{n-2}+1}$

Proof

Suppose QD_{2^n} be a quasidihedral group of order 2^n and $\Gamma^{cl}_{QD_{2^n}}$ be its conjugacy class graph,
By using Proposition 4, the set of vertex of the conjugacy class graph $\Gamma^{cl}_{QD_{2^n}}$ is $V\left(\Gamma^{cl}_{Q_{4n}}\right)=\left\{cl(a),cl(a^2),\ldots,cl\left(a^{2^{n-2}-1}\right),cl(b),cl(ab)\right\}$. For all vertices of the form $cl(a^k), k=1,2,\ldots,2^{n-2}-1,$ their cardinalities are equal two and not coprime, hence they are adjacent and $|cl(b)|=|cl(ab)|$ = even integer. Thus, for all vertices of the form $cl(a^k)$, $k=1,2,\ldots,2^{n-2}-1,$ their cardinalities are not coprime with $|cl(b)|$ and $|\text{cl}(\text{ab})|$, thus all vertices are adjacent and $\Gamma^{cl}_{QD_{2^n}}$ is a complete graph. Now we have $2^{n-2}-1$ and two adjacent vertices, $\left(2^{n-2}-1+2=2^{n-2}+1\right)$, thus $\Gamma^{cl}_{QD_{2^n}}=K_{2^{n-2}+1}$. ■

Theorem 7

Let G be a quasidihedral group of order 2^n, where $n\geq 4$, and $\Gamma^{cl}_{QD_{2^n}}$ be its conjugacy class graph. Then the energy of the graph $\Gamma^{cl}_{QD_{2^n}}$ $\varepsilon\left(\Gamma^{cl}_{QD_{2^n}}\right)=2^{n-1}$.

Proof

Suppose G be a quasidihedral group of order 2^n and $\Gamma^{cl}_{QD_{2^n}}$ be its conjugacy class graph, from Proposition 6, the conjugacy class graphs of G, is complete graphs of the form $\Gamma^{cl}_{QD_{2^n}}=K_{2^{n-2}+1}$. By using Proposition 3, the eigenvalues of the complete graph $K_{2^{n-2}+1}$ are $\lambda=2^{n-2}+1-1=2^{n-2}$ (with multiplicity 1) and $\lambda=-1$ (with multiplicity $\left(2^{n-2}+1\right)-1=2^{n-2}$). Hence by using the definition of the energy of the graph, $\varepsilon\left(\Gamma^{cl}_{QD_{2^n}}\right)=\left|2^{n-2}\right|+2^{n-2}\left|-1\right|=2.2^{n-2}=2^{n-1}$. ■

Example 6

Let G be a quasidihedral group of order 32, i.e. $G=QD_{32}\cong\left\langle a,b:a^{16}=b^2=1,bab^{-1}=a^7\right\rangle$, and $\Gamma^{\text{cl}}_{\text{D}_{10}}$ be its conjugacy class graph. The conjugacy classes of QD_{32} are, cl(e) = {e} $cl(a^8)=a^8$

$cl(a)=\{a,a^7\}, cl(a^2)=\{a^2,a^{14}\}, cl(a^3)=\{a^3,a^5\},$

$cl(a^4)=\{a^4,a^{12}\}, cl(a^6)=\{a^6,a^{10}\}, cl(a^9)=\{a^9,a^{15}\},$

$cl(a^{11})=\{a^{11},a^{13}\},$

$cl(b)=\{b,a^2b,a^4b,a^6b,a^8b,a^{10}b,a^{12}b,a^{14}b\}$, and

$cl(ab)=\{ab,a^3b,a^5b,a^7b,a^9b,a^{11}b,a^{13}b,a^{15}b\}$.

The set of vertex of this graph is

$$V\left(\Gamma^{cl}_{Q_{16}}\right)=\begin{Bmatrix}cl(a),cl(a^2),cl(a^3),cl(a^4),cl(a^6),cl(a^9),cl(a^{11}),\\ cl(b),cl(ab)\end{Bmatrix}$$

and the set of edge is

$$\begin{aligned}E\left(\Gamma^{cl}_{QD_{32}}\right)=\{&cl(a)cl(a^2),cl(a)cl(a^3),cl(a)cl(a^4),cl(a)cl(a^6),\\ &cl(a)cl(a^9),cl(a)cl(a^{11}),cl(a)cl(b),cl(a)cl(ab),cl(a^2)cl(a^3),\\ &cl(a^2)cl(a^4),cl(a^2)cl(a^6),cl(a^2)cl(a^9),cl(a^2)cl(a^{11}),cl(a^2)cl(b),\\ &cl(a^2)cl(ab),cl(a^3)cl(a^4),cl(a^3)cl(a^6),cl(a^3)cl(a^9),\\ &cl(a^3)cl(a^{11}),cl(a^3)cl(b),cl(a^3)cl(ab),cl(a^4)cl(a^6),\\ &cl(a^4)cl(a^9),cl(a^4)cl(a^{11}),cl(a^4)cl(b),cl(a^4)cl(ab),cl(a^6)cl(a^9),\\ &cl(a^6)cl(a^{11}),cl(a^6)cl(b),cl(a^6)cl(ab),cl(a^9)cl(a^{11}),cl(a^9)cl(b),\\ &cl(a^9)cl(ab),cl(a^{11})cl(b),cl(a^{11})cl(ab),cl(b),cl(ab)\}\end{aligned}$$

Thus the conjugacy class graph of the quasidihedral group QD_{32} is $\Gamma^{cl}_{QD_{32}}=K_9$ and the adjacency matrix of the graph $\Gamma^{cl}_{QD_2}$ is

$$A\left(\Gamma^{cl}_{QD_{32}}\right)=\begin{bmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\1&1&0&1&1&1&1&1&1\\1&1&1&0&1&1&1&1&1\\1&1&1&1&0&1&1&1&1\\1&1&1&1&1&0&1&1&1\\1&1&1&1&1&1&0&1&1\\1&1&1&1&1&1&1&0&1\\1&1&1&1&1&1&1&1&0\end{bmatrix}$$

and the eigenvalues of $A\left(\Gamma^{cl}_{QD_{32}}\right)$ are $\lambda=8,$ and $\lambda=-1$ with multiplicity 8.
By the definition of the energy of the graph, we have $\varepsilon\left(\Gamma^{cl}_{Q_{16}}\right)=|8|+8|-1|=16$. Meanwhile by using theorem 3.2, $\varepsilon\left(\Gamma^{cl}_{QD_{32}}\right)=2^{5-1}=16.$

CONCLUSION

In this paper, the general formulas for the energy of conjugacy class graph of dihedral groups are introduced. For n an odd integer, $\varepsilon\left(\Gamma^{cl}_{D_{2n}}\right)=n$ - 3., while for n and $\dfrac{n}{2}$ even integers $\varepsilon\left(\Gamma^{cl}_{D_{2n}}\right)=n.$ and if n is even integer and $\dfrac{n}{2}$ is odd integer then, $\varepsilon\left(\Gamma^{cl}_{D_{2n}}\right)=n-2.$ Also we introduced the general formulas for the energy of conjugacy class graph of generalized quaternion groups and quasidihedral groups.

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