RESEARCH ARTICLE

Heat transfer in free convection flow of micropolar fluids over an oscillating vertical plate

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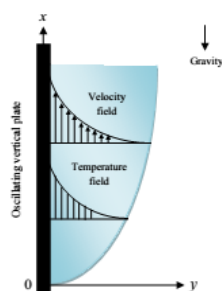[c] *Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Mayalsia Johor, Malaysia*

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Graphical abstract

Abstract

An analytical investigation is carried out to study the unsteady free convection stream of micropolar fluids over an oscillating vertical plate. Wall couple stress is promised at the bounding plate with isothermal temperature. Problem is displayed in terms of combined partial differential equations together with certain physical conditions and then written in non-dimensional form. Exact solutions are achieved using the Laplace transform technique. Analytical results of velocity, microrotation and temperature are plotted in graphs and debated for different inserted parameters. Excellent validation of present results is acquired with existing results in literature. It is observed that, the velocity is smaller for micropolar fluids than for Newtonian fluids.

Keywords: Micropolar fluids, microrotaions, free convection, exact solutions

INTRODUCTION

Rheological characteristics of non-Newtonian fluids are termed by their constitutive equations. Because of complex nature of non-Newtonian fluids, several models or constitutive equations have been suggested based on their empirical observations. Amongst them, one model was proposed by (Eringen, 1966) using micromorphic fluid theory, known as micropolar fluids. Micropolar fluids pose microrotation and microinertia effect and constitute a significant branch of fluids. In literature micropolar fluids are used to describe flow characteristics of colloidal suspensions, geomor-phological sediments, liquid crystals, polymeric additives, haematological suspensions, lubricants and many other biological fluids (Ariman *et al.*, 1973). Eringen and Lukaszewicz discussed fascinating characteristics of theory and applications of micropolor fluids (Erigen, 2001; Lukaszewicz 1999).

(Nazar *et al.*, 2002) examined boundary layer flow of micropolar fluid past an isothermal sphere by taking the free convection. (Cheng, 2008) discussed natural convection for micropolar fluids in combined transfer of heat and mass over a sphere accepting constant wall temperature together with constant wall concentration. (Sherief *et al.*, 2011) used Laplace transform method and studied unsteady flow of a micropolar fluid from with rapidly stimulated plate. Analysis of heat transfer from moving surfaces with internal heat generation in a micropolar fluid is carried out by (El-Hakiem, 2014). (Hassanien *et al.*, 1997) investigated the natural convection boundary layer flow of a micropolar fluid. (Ishak *et al.*, 2008) discussed the micropolar fluids with heat transfer over a stretching surface with variable heat flux. (Lok *et al.*, 2005) reported the steady mixed convection flow near

the stagnation point on a vertical surface of a micropolar fluid. Boundary layer stagnation point flow past a moving wall of a micropolar fluid has been studied by (Gorla, 1983). (Damesh *et al.*, 2007) illustrated the micropolar fluid with unsteady natural convection heat transfer over a vertical surface with constant heat flux using numerical technique, whereas conjugate transfer of heat and mass in unsteady flow of a micropolar fluids with wall couple stress studied by (Khalid *et al.*, 2015).

(Abo-Dahab *et al.*, 2013) investigated the unsteady flow of rotating and chemically reacting slip-flow regime with heat generation MHD micropolar fluid. The aim of the present work is to provide exact solutions for the unsteady free convection flow of an incompressible micropolar fluid over an infinite vertical plate oscillating in its own plan. More exactly, heat transfer when the bounding plate takes wall couple stress with isothermal temperature is studied.

PROBLEM FORMULATION

Assume, unsteady boundary layer stream of an incompressible micropolar fluid in the region $y > 0$ determine by a plane surface positioned at $y = 0$ with a fixed end at $x = 0$. It is expected that at the initial moment $t = 0,$ both the plate and the fluid are at rest at the constant temperature T_∞. At time $t = 0^+$ the plate begins to oscillate in its plane ($y = 0$) according to

$$V = UH(t)\cos(\omega t)\,\mathbf{i};\ t > 0, \tag{1}$$

where $H(t)$ is the unit step function, U is the amplitude of the motion, $\mathbf{i}$ is the unit vector in the vertical flow direction and ω is the frequency of oscillation of the plate. At the same time, the plate temperature level are raised to T_w which are thereafter maintained constants. Assume that, the velocity, microrotation and temperature are functions of y and t only. Taking usual Boussinesq's approximation, the unsteady flow is governed by the following set of partial differential equations.

$$\rho\frac{\partial u}{\partial t} = (\mu+\alpha)\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial N}{\partial y} + \rho g\beta_f(T-T_\infty), \tag{2}$$

$$\rho j\frac{\partial N}{\partial t} = \gamma_0\frac{\partial^2 N}{\partial y^2}, \tag{3}$$

$$\rho c_p\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial y^2}. \tag{4}$$

The corresponding initial and boundary conditions are

$$u(y,0)=0,\ N(y,0)=0,\ T(y,0)=T_\infty \text{ for all } y\geq 0, \tag{5}$$

$$u(0,t)=H(t)U\cos(\omega t),\ N(0,t)=-n\frac{\partial u}{\partial y}(0,t), \tag{6}$$

$$T_\infty(0,t)=T_w,\ t>0,$$
$$u(\infty,t)\to 0,\ N(\infty,t)\to 0,\ T(\infty,t)\to T_\infty \text{ as } y\to\infty. \tag{7}$$

Here u is velocity, μ is dynamic viscosity, ρ is density, g is gravitational acceleration, α is vortex viscosity, t is time, T is temperature, β_T is volumetric coefficient of thermal expansion, N is the microrotation whose direction of rotation is in the xy-plane, j is microinertia per unit mass, γ_0 is spin gradient viscosity, c_p is heat capacity at constant pressure, k is thermal conductivity and ωt is phase angle. The spin gradient viscosity γ_0, measures the relationship between the coefficients of viscosity and micro-inertia, is defined as

$$\gamma_0 = \left(\mu+\frac{\alpha}{2}\right)j.$$

To moderate the above equations (2-8) into their non-dimensional forms, we establish the succeeding non-dimensional quantities,

$$y^* = \frac{U}{v}y,\ t^* = \frac{U^2}{v}t,\ u^* = \frac{u}{U},\ N^* = \frac{v}{U^2}N,$$
$$\theta = \frac{T-T_\infty}{T_w-T_\infty},\ \omega^* = \frac{v}{U^2}\omega,\ j^* = \frac{U^2}{v^2}j. \tag{8}$$

Employing equation (8) into equations (2-4), we achieve the following non-dimensional partial differential equations (* symbol is omitted for simplicity)

$$\frac{\partial u}{\partial t} = (1+\beta)\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial N}{\partial y} + Gr\theta, \tag{9}$$

$$\frac{\partial N}{\partial t} = \frac{1}{\eta}\frac{\partial^2 N}{\partial y^2}, \tag{10}$$

$$\Pr\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial y^2}. \tag{11}$$

The consistent initial and boundary conditions take the following non-dimensional forms:

$$u(y,0)=0,\ N(y,0)=0,\ \theta(y,0)=0 \text{ for all } y\geq 0, \tag{12}$$

$$u(0,t)=H(t)\cos(\omega t),\ N(0,t)=-n\frac{\partial u}{\partial y}(0,t), \tag{13}$$

$$\theta(0,t)=1,\ t>0,$$
$$u(\infty,t)\to 0,\ N(\infty,t)\to 0,\ \theta(\infty,t)\to 0 \text{ as } y\to\infty, \tag{14}$$

where

$$\beta=\frac{\alpha}{\mu},\ \Pr=\frac{\mu c_p}{k},\ \eta=\frac{\mu j}{\gamma_0},\ Gr=\frac{g\beta_T(T_w-T_\infty)v}{U^3},$$

are the microrotation parameter, Prandtl number, dimensionless spin gradient and Grashof number, respectively.

SOLUTION OF THE PROBLEM

Applying the Laplace transforms to equations (9-11), and using conditions (12-14), the following solutions in the transformed plane are obtained

$$\bar{u}(y,q) = a_4\frac{q}{(q^2+\omega^2)}e^{-y\sqrt{\beta_0 q}} + a_5\frac{q}{(q^2+\omega^2)}e^{-y\sqrt{\eta q}}$$
$$+a_6\frac{1}{q^2}e^{-y\sqrt{\beta_0 q}} + a_7\frac{1}{q^2}e^{-y\sqrt{\eta q}}$$
$$-a_8\frac{1}{q^2}e^{-y\sqrt{\Pr q}}, \tag{15}$$

$$\overline{N}(y,q) = \frac{a_1}{\sqrt{q}}e^{-y\sqrt{\eta q}} - \frac{a_1\omega}{2i}\frac{1}{\sqrt{q}(q+i\omega)}e^{-y\sqrt{\eta q}}$$
$$+\frac{a_1\omega}{2i}\frac{1}{\sqrt{q}(q-i\omega)}e^{-y\sqrt{\eta q}} + \frac{a_2}{q\sqrt{q}}e^{-y\sqrt{\eta q}}, \tag{16}$$

$$\bar{\theta}(y,q) = \frac{1}{q}e^{-y\sqrt{\Pr q}}. \tag{17}$$

By taking the inverse Laplace transforms of above equations, we obtain

$$u(y,t) = u_1(y,t) + u_2(y,t) - u_3(y,t) - u_4(y,t)$$
$$+u_5(y,t) + u_6(y,t) - u_7(y,t), \tag{18}$$

$$N(y,t) = N_1(y,t) + N_2(y,t), \tag{19}$$

$$\theta(y,t) = \theta_1(y,t), \tag{20}$$

with

$$u_1(y,t) = \left(\frac{a_4}{4}\right)H(t)e^{-i\omega t}\left[\begin{array}{l} e^{-y\sqrt{-i\omega\beta_0}}\,\mathrm{erfc}\left(\dfrac{y}{2}\sqrt{\dfrac{\beta_0}{t}}-\sqrt{-i\omega t}\right) \\ +e^{y\sqrt{-i\omega\beta_0}}\,\mathrm{erfc}\left(\dfrac{y}{2}\sqrt{\dfrac{\beta_0}{t}}+\sqrt{-i\omega t}\right) \end{array}\right],$$

$$u_2(y,t) = \left(\frac{a_4}{4}\right)H(t)e^{i\omega t}\left[\begin{array}{l} e^{-y\sqrt{i\omega\beta_0}}\,\mathrm{erfc}\left(\dfrac{y}{2}\sqrt{\dfrac{\beta_0}{t}}-\sqrt{i\omega t}\right) \\ +e^{y\sqrt{i\omega\beta_0}}\,\mathrm{erfc}\left(\dfrac{y}{2}\sqrt{\dfrac{\beta_0}{t}}+\sqrt{i\omega t}\right) \end{array}\right],$$

$$u_3(y,t) = \left(\frac{a_5}{4}\right) H(t) e^{-i\omega t} \begin{bmatrix} e^{-y\sqrt{-i\omega\eta}} \text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} - \sqrt{-i\omega t}\right) \\ + e^{y\sqrt{-i\omega\eta}} \text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} + \sqrt{-i\omega t}\right) \end{bmatrix},$$

$$u_4(y,t) = \left(\frac{a_5}{4}\right) H(t) e^{i\omega t} \begin{bmatrix} e^{-y\sqrt{i\omega\eta}} \text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} - \sqrt{i\omega t}\right) \\ + e^{y\sqrt{i\omega\eta}} \text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} + \sqrt{i\omega t}\right) \end{bmatrix},$$

$$u_5(y,t) = a_6 \left[\left(t + \frac{y^2\beta_0}{2}\right)\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\beta_0}{t}}\right) - y\sqrt{\beta_0}\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\beta_0}{4t}}\right],$$

$$u_6(y,t) = a_7 \left[\left(t + \frac{y^2\eta}{2}\right)\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right) - y\sqrt{\eta}\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\eta}{4t}}\right],$$

$$u_7(y,t) = a_8 \left[\left(t + \frac{y^2 \text{Pr}}{2}\right)\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\text{Pr}}{t}}\right) - y\sqrt{\text{Pr}}\sqrt{\frac{t}{\pi}}e^{-\frac{y^2 \text{Pr}}{4t}}\right],$$

$$N_1(y,t) = a_1\left[\frac{1}{\sqrt{\pi t}}e^{-\frac{y^2\eta}{4t}}\right] - a_3 \begin{bmatrix} \frac{1}{2\sqrt{i\omega}}e^{i\omega t} \\ \begin{bmatrix} e^{-y\sqrt{i\omega\eta}}\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} - \sqrt{i\omega t}\right) \\ -e^{y\sqrt{i\omega\eta}}\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} + \sqrt{i\omega t}\right) \end{bmatrix} \end{bmatrix},$$

$$N_2(y,t) = a_3 \begin{bmatrix} \frac{1}{2\sqrt{-i\omega}}e^{-i\omega t} \\ \begin{bmatrix} e^{-y\sqrt{-i\omega\eta}}\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} - \sqrt{-i\omega t}\right) \\ -e^{y\sqrt{-i\omega\eta}}\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}} + \sqrt{-i\omega t}\right) \end{bmatrix} \end{bmatrix} + a_2\left[2\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\eta}{4t}} - y\sqrt{\eta}\,\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\eta}{t}}\right)\right],$$

$$\theta_1(y,t) = \text{erfc}\left(\frac{y}{2}\sqrt{\frac{\text{Pr}}{t}}\right),$$

where

$$a_1 = \frac{\left(n\sqrt{\beta_0}\right)(\eta - \beta_0)}{\left(\eta - \beta_0 + n\beta\beta_0\sqrt{\eta\beta_0} - n\beta_0\beta\eta\right)},$$

$$a_2 = A_1 n Gr\sqrt{\beta_0}\frac{(\eta - \beta_0)}{\left(\eta - \beta_0 + n\beta\beta_0\sqrt{\eta\beta_0} - n\beta_0\beta\eta\right)}$$
$$- A_1 n Gr\sqrt{\text{Pr}}\frac{(\eta - \beta_0)}{\left(\eta - \beta_0 + n\beta\beta_0\sqrt{\eta\beta_0} - n\beta_0\beta\eta\right)},$$

$$a_3 = \frac{a_1\omega}{2i}, \qquad a_4 = 1 - A_0 a_1,$$
$$a_5 = A_0 a_1, \qquad a_6 = Gr A_1 - a_2 A_0,$$
$$a_7 = a_2 A_0, \qquad a_8 = Gr A_1,$$
$$A_0 = \frac{\beta\beta_0\sqrt{\eta}}{\eta - \beta_0}, \quad A_1 = \frac{\beta_0}{\text{Pr} - \beta_0},$$
$$\beta_0 = \frac{1}{1 + \beta}.$$

Limiting case

In this case, the stream in the fluid is convinced due to spontaneous motion of the plate. Hence taking $\omega t = \beta = \eta = 0$ into equation (18), which correspond the spontaneous motion of the plate as

$$u(y,t) = \frac{1}{4}\left[\text{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right]$$
$$+ a_6\left[\left(t + \frac{y^2}{2}\right)\text{erfc}\left(\frac{y}{2\sqrt{t}}\right) - y\sqrt{\frac{t}{\pi}}e^{-\frac{y^2}{4t}}\right]$$
$$- a_8\left[\left(t + \frac{y^2\text{Pr}}{2}\right)\text{erfc}\left(\frac{y}{2}\sqrt{\frac{\text{Pr}}{t}}\right) - y\sqrt{\text{Pr}}\sqrt{\frac{t}{\pi}}e^{-\frac{y^2\text{Pr}}{4t}}\right].$$

(21)

It is important to note that equation (21) is found identical to those obtained by (Chudhary *et al.*, 2007) equation (19), in the absence of MHD and porosity effects. Hence, this verifies the correctness of the present work. This fact is also shown in Figure 7.

RESULTS AND DISCUSSION

In this section, final results are computed for different physical parameters which are presented by mean of graphs. Parameters of physical interest are microrotation parameter β, Prandtl number Pr and Grashof number *Gr*. Influence of microrotation parameter β on velocity and the microrotation is depicted in Fig.s 1 and 2. These graphs show that velocity decreases whereas microrotation increases with increasing β. Obviously, velocity satisfies the imposed boundary conditions in equations (13) and (14). On the other hand, the microrotation takes the negative values of the gradient of velocity at the plate surface and approaching to zero as one move away from the plate surface as shown in Fig. 2. This fact totally aggresses with imposed conditions on microrotation (see equations (13) and (14)). Fig.s 3 and 4 present plots for velocity and microrotation for different values of Prandtl number Pr. These graphs show that the influence of increasing values of Pr result in decreasing of the velocity, magnitude of the microrotation and temperature as well.

Fig.s 5 and 6 show deviations in velocity and microrotation outlines for different values of Grashof number *Gr*. It is observed that an increase in *Gr* leads to an increase in velocity due to enhancement in the buoyancy force. Besides that magnitude of microrotation decreases for large values of *Gr*. For positive values of *Gr* correspond to freezing of the surface by natural convection however *Gr* = 0 shows the lack of heat transfer due to free convection.
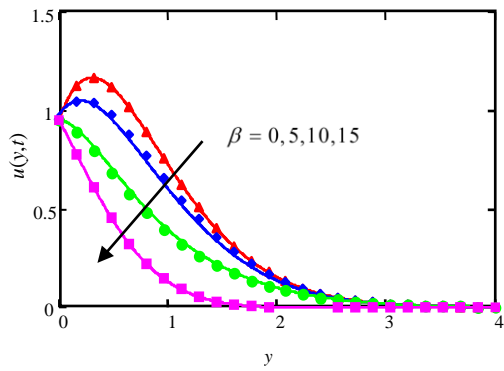
Fig. 1 Velocity profiles for different values of β when $\mathrm{Pr} = 10, \eta = 1.5, n = 0.6, Gr = 5, \omega = \pi / 4 \ \& \ t = 0.6.$
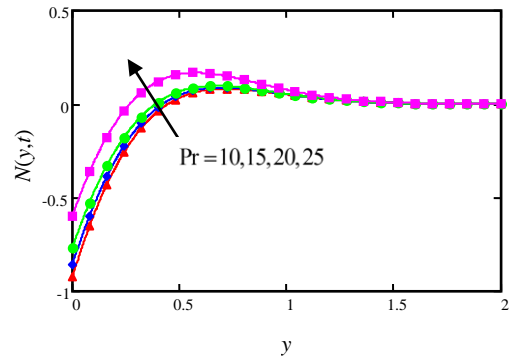

Fig. 2 Microrotations for different β when $\mathrm{Pr} = 10, \eta = 1.5, n = 0.6, Gr = 5, \omega = \pi / 4 \ \& \ t = 0.6.$

Fig. 3 Velocity profiles for different values of Pr when $\beta = 0.5, \eta = 1.5, n = 0.6, Gr = 5, \omega = \pi / 4 \ \& \ t = 0.6.$

Fig. 4 Microrotations for different Pr when $\beta = 0.5, \eta = 1.5, n = 0.6, Gr = 5, \omega = \pi / 4 \ \& \ t = 0.6.$
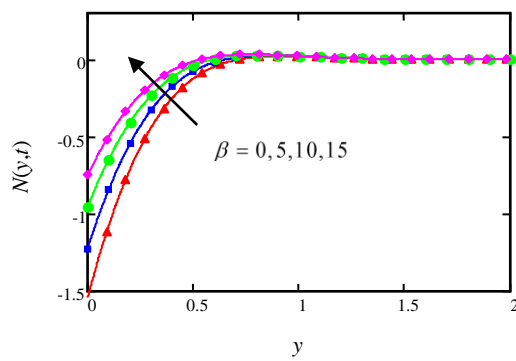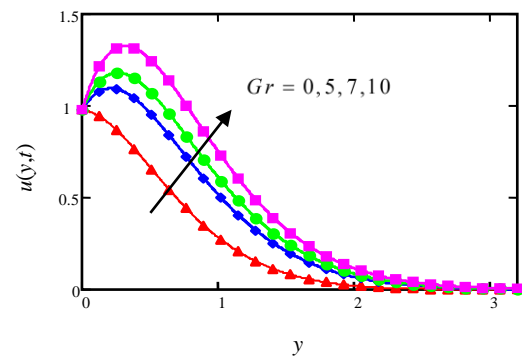
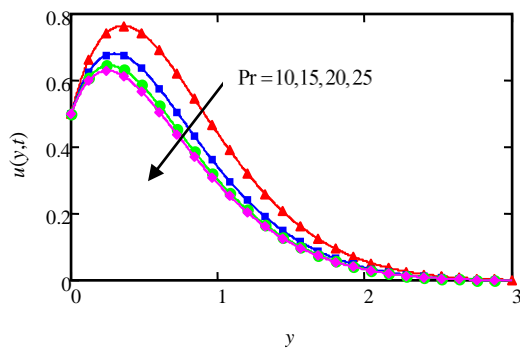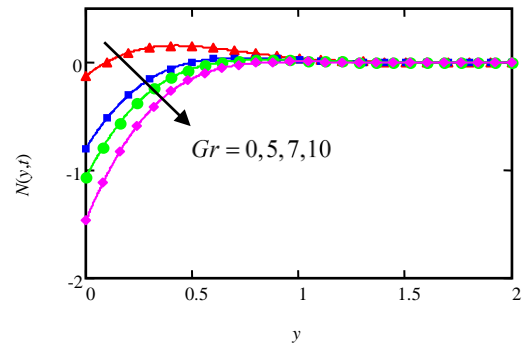**Fig. 5** Velocity profiles for different *Gr* when $\mathrm{Pr} = 10, \beta = 0.5, \eta = 1.5, n = 0.6, \omega = \pi / 4 \ \& \ t = 0.6.$

Fig. 6 Microrotaions for different *Gr* when $\mathrm{Pr} = 10, \beta = 0.5, \eta = 1.5, n = 0.6, \omega = \pi / 4 \ \& \ t = 0.6.$

In order to check the correctness of present results, the velocity profiles of present result, equation (21) is compared with existing results in literature (Chaudhary *et al.*, 2007). This comparison is shown in Fig. 7.
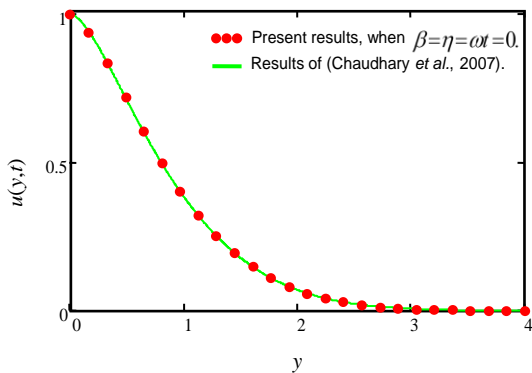

Fig. 7 Comparison of micropolar fluid velocity (when $\beta = \eta = 0$), with results obtained by (Chaudhary *et al.*, 2007) see Equation (19), when $t = 0.2, \omega t = 0$ and $M' = 0$.

CONCLUSION

An analytical study is performed to study explore the unsteady free convection stream of micropolar fluids over an oscillating vertical plate with wall couple stress. Problem is demonstrated in terms of coupled partial differential equations together with some physical conditions and then written in non-dimensional form. Laplace transform technique is used to find the exact solutions. It is concluded that velocity across the boundary layer increases with increasing Pr whereas decreases with increasing values of β and Pr. Magnitude of microrotation on the plate is decreases with increasing *Gr* while rises with increasing β and Pr. Solution (21) is found in excellent agreement with those obtain by (Chaudhary *et al.*, 2007).

ACKNOWLEDGEMENT

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