MJFAS MALAYSIAN JOURNAL OF FUNDAMENTAL AND APPLIED SCIENCES
PRINT ISSN: 2289-5981 | ONLINE ISSN: 2289-599X

RESEARCH ARTICLE

Operator inequalities involving Khatri-Rao sums and Moore-Penrose inverses

Arnon Ploymukda, Pattrawut Chansangiam*

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

* *Corresponding author: pattrawut.ch@kmitl.ac.th*

Article history
Received 14 June 2017
Accepted 2 October 2017

Abstract

We establish relations between the Khatri-Rao sum of Hilbert space operators and ordinary products, powers, ordinary inverses, and Moore-Penrose inverses in terms of inequalities, including arithmetic-geometric mean inequality and Kantorovich type inequalities. In particular, such relations hold for the tensor sum of operators and the Khatri-Rao sum of complex matrices.

Keywords: Tensor product, Khatri-Rao sum (product), Tracy-Singh sum (product), Moore-Penrose inverse, operator inequality

INTRODUCTION

In mathematics, there are many kinds of matrix products/sums which have rich theory and numerous applications. Such matrix products include the Kronecker (tensor) product, the Tracy-Singh product, and the Khatri-Rao product. Recall that the Kronecker product of two complex matrices A and B is defined by

$$A \hat{\otimes} B = \left[a_{ij} B\right]_{ij}, \tag{1}$$

that is, the (i,j)th block of $A \hat{\otimes} B$ is given by $a_{ij}B$. The Kronecker sum of an n x n matrix A and an m x m matrix B is defined as

$$A \hat{\oplus} B = A \otimes I_n + I_m \otimes B. \tag{2}$$

Here, I_k denotes the identity matrix of size k x k for any natural number k. The Tracy-Singh product, introduced in [1], is a generalization of the Kronecker product. Indeed, partition $A = [A_{ij}]$ and $B = [B_{kl}]$, where the submatrices A_{ij} and B_{kl} can be of arbitrary sizes. Then the Tracy-Singh product of A and B is defined as

$$A \hat{\boxtimes} B = \left[\left[A_{ij} \hat{\otimes} B_{kl}\right]_{kl}\right]_{ij}.$$

If A and B have the same form of partitioning, we can define their Khatri-Rao product by [2]

$$A \hat{\boxdot} B = \left[A_{ij} \hat{\otimes} B_{ij}\right]_{ij}.$$

See more information about theory of matrix products in [3-6].

The notions of Tracy-Singh sum and Khatri-Rao sum for matrices are respectively defined by (see [7])

$$A \hat{\boxplus} B = A \hat{\boxtimes} I_n + I_m \hat{\boxtimes} B, \tag{3}$$

$$A \hat{\circledast} B = A \hat{\boxdot} I_n + I_m \hat{\boxdot} B. \tag{4}$$

Here, we partition I_m and I_n so that their diagonal blocks are identity matrices. If A and B are of only one block, then the Tracy-Singh sum (3) and the Khatri-Rao sum (4) are reduced to the Kronecker sum (2).

A significant development in operator theory is to introduce the tensor product of Hilbert space operators, generalizing the Kronecker product of matrices. From now on, let $\mathcal{H}$ and $\mathcal{K}$ be complex separable Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, denote by $\mathbb{B}(\mathcal{X},\mathcal{Y})$ the algebra of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and abbreviate $\mathbb{B}(\mathcal{X},\mathcal{X})$ to $\mathbb{B}(\mathcal{X})$. The identity operator on the space $\mathcal{X}$ is denoted by I_X or I if the underlying space is clear from the context. Using the universal mapping property, the tensor product of $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ is the unique bounded linear operator from the tensor space $\mathcal{H} \otimes \mathcal{K}$ into itself satisfying

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. The tensor sum of $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ is defined to be (see [8])

$$A \oplus B = A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B. \tag{5}$$

Recently, the Tracy-Singh product, the Khatri-Rao product, and the Khatri-Rao sum for Hilbert space operators were investigated by the authors [9-13]. The notion of Khatri-Rao sum of operators includes the tensor sum of operators and the Khatri-Rao sum of complex matrices as special cases.

In this paper, we develop further theory of operator products/sums by establishing certain inequalities for Khatri-Rao sums of operators. These inequalities involve ordinary products and powers, ordinary and

Moore-Penrose inverses. We also deduce Kantorovich type inequalities concerning Khatri-Rao sums. Our results generalize some matrix inequalities in [7]. In operator case, we require some mild assumptions such as the closeness of their ranges. Moreover, new operator inequalities are established by means of block partitioning technique.

The paper is organized as follows. The next section supplies preliminaries on Tracy-Singh products/sums and Khatri-Rao products/sums for Hilbert space operators. The third section deals with certain operator inequalities concerning Khatri-Rao sums and Moore-Penrose inverses. In the final section, we establish Kantorovich type operator inequalities involving Khatri-Rao sums.

PRELIMINARIES

For Hermitian operators $A, B \in \mathbb{B}(\mathcal{H})$, the partial order $A \geqslant B$ means that $A-B$ is a positive operator.

Tracy-Singh products/sums for operators

To define the Tracy-Sing product of operators, we first fix the decompositions of Hilbert spaces (this can be done by using the projection theorem):

$$\mathcal{H} = \bigoplus_{i=1}^{m} \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{k=1}^{n} \mathcal{K}_k$$

where $\mathcal{H}_i$ and $\mathcal{K}_k$ are Hilbert spaces for all *i*, *k*. It follows that any operator $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ can be uniquely represented as operator matrices

$$A = \left[A_{ij} \right]_{i,j=1}^{m,m} \text{ and } B = \left[B_{kl} \right]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$ and $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$ for each i, j, k, l. Basic algebraic operations for operator matrices, such as, the addition, the scalar multiplication, the usual multiplication, and the adjoints can be performed in the same way as those of block matrices.

Definition 1. According to the previous setup, the Tracy-Singh product of A and B is defined to be the bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$ into itself, represented by the operator matrix

$$A \boxtimes B = \left[\left[A_{ij} \otimes B_{kl} \right]_{kl} \right]_{ij}. \tag{6}$$

Recall that a Moore-Penrose inverse of $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$ is an operator $A^{\dagger} \in \mathbb{B}(\mathcal{K}, \mathcal{H})$ satisfying the following conditions (see [14]):

$$AA^{\dagger}A = A, \; A^{\dagger}AA^{\dagger} = A^{\dagger}, \; (AA^{\dagger})^* = AA^{\dagger}, \; (A^{\dagger}A)^* = A^{\dagger}A.$$

The existence of $A^{\dagger}$ is equivalent to the closeness of the range of *A*, and in this case the operator $A^{\dagger}$ is unique (see, e.g., [15]).

Lemma 2 ([9]). The Tracy-Singh product fulfills the following properties, assuming that all operators are compatible:

$$(A \boxtimes B)^* = A^* \boxtimes B^*, \tag{7}$$

$$(\alpha A) \boxtimes B = \alpha (A \boxtimes B) = A \boxtimes (\alpha B), \tag{8}$$

$$A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C, \tag{9}$$

$$(B + C) \boxtimes A = B \boxtimes A + C \boxtimes A, \tag{10}$$

$$(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD, \tag{11}$$

$$(A \boxtimes B)^{\dagger} = A^{\dagger} \boxtimes B^{\dagger}. \tag{12}$$

Lemma 3 ([9]). Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices. If $A \geqslant 0$ and $B \geqslant 0$, then $A \boxtimes B \geqslant 0$.

Definition 4. Let $A = \left[A_{ij} \right]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{kl} \right]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$. We define the Tracy-Singh sum of A and B as follows:

$$A \boxplus B = A \boxtimes I_{\mathcal{K}} + I_{\mathcal{H}} \boxtimes B \tag{13}$$

which belongs to $\mathbb{B}\left(\bigoplus_{i,j=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_j \right)$.

Khatri-Rao Products/Sums for Operators

From now on, fix the following Hilbert space direct sums:

$$\mathcal{H} = \bigoplus_{i=1}^{n} \mathcal{H}_i, \quad \mathcal{K} = \bigoplus_{i=1}^{n} \mathcal{K}_i.$$

Definition 5. Let $A = \left[A_{ij} \right]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{ij} \right]_{i,j=1}^{n,n} \in \mathbb{B}(\mathcal{K})$ be operator matrices. We define the Khatri-Rao product of A and B to be the bounded linear operator from $\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_i$ into itself, represented as follows:

$$A \boxdot B = \left[A_{ij} \otimes B_{ij} \right]_{i,j=1}^{n,n}. \tag{14}$$

Lemma 6 ([11]). There is an isometry Z such that $ZZ^* \leqslant I$ and

$$A \boxdot B = Z^* \left(A \boxtimes B \right) Z \tag{15}$$

for any $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. We call Z the selection operator associated with the ordered tuple $\left(\mathcal{H}, \mathcal{K} \right)$.

Definition 7. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. We define the Khatri-Rao sum of A and B as follows:

$$A \circledast B = A \boxdot I_{\mathcal{K}} + I_{\mathcal{H}} \boxdot B \tag{16}$$

which belongs to $\mathbb{B}\left(\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_i \right)$.

Note that if both A and B are 1×1 block operator matrices, their Khatri-Rao sum $A \circledast B$ becomes the tensor sum. If $\mathcal{H}_i = \mathcal{K}_i = \mathbb{C}$, the Khatri-Rao sum $A \circledast B$ reduces to the Hadamard sum of complex matrices (see e.g. [7]).

Lemma 8 ([13]). There is an isometry Z such that $ZZ^* \leqslant I$ and

$$A \circledast B = Z^* \left(A \boxplus B \right) Z \tag{17}$$

for any $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$.

OPERATOR INEQUALITIES INVOLVING KHATRI-RAO SUMS AND MOORE-PENROSE INVERSES

In this section, we derive certain operator inequalities involving Khatri-Rao sums and Moore-Penrose inverses. Roughly speaking, we may consider the Khatri-Rao sum and the Khatri-Rao product as the "sum" and the "product", respectively. The Moore-Penrose inverse plays a role like the "inverse" for operators. To ensure the existence of the Moore-Penrose inverse of an operator, we must impose the closeness of its range. The results in this section include those for the tensor sum of operators and the Khatri-Rao sum of complex matrices as special cases.

To derive operator inequalities in this section, we apply a block-partitioning technique, which is explained in the next lemma.

Lemma 9 (see e.g. [16]) Let $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{bmatrix}$ be an operator in $\mathbb{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that T_{11} has a closed range. Then $T \geqslant 0$ if and only if

(i) $T_{11} \geqslant 0$,

(ii) $T_{12} = T_{11} T_{11}^{\dagger} T_{12}$,

(iii) $T_{22}^* = T_{22} \geqslant T_{12}^* T_{11}^{\dagger} T_{12}$.

Recall that for any positive real numbers *a* and *b*, we have

$$\frac{(a+b)^2}{ab} \le \frac{a}{b} + \frac{b}{a} + 2$$

(indeed, both sides are equal). Let us generalize this fact to operators in which we consider the Khatri-Rao sum as the "sum" and the Moore-Penrose inverse as the "inverse".

Theorem 10. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive invertible operators such that $A \boxdot B$ has a closed range. Then

$$\left(A \circledast B\right)\left(A \boxdot B\right)^{\dagger}\left(A \circledast B\right) \leqslant A \boxdot B^{-1} + A^{-1} \boxdot B + 2I. \tag{18}$$

Proof. Denote

$$S = \begin{bmatrix} A^{1/2} \boxtimes B^{1/2} & A^{1/2} \boxtimes B^{-1/2} + A^{-1/2} \boxtimes B^{1/2} \end{bmatrix}$$

and $X = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$ where Z is the selection operator associated with $(\mathcal{H}, \mathcal{K})$. Using Lemma 2, we get

$$\begin{aligned} 0 &\leqslant S^* S \\ &= \begin{bmatrix} A \boxtimes B & A \boxtimes I + I \boxtimes B \\ A \boxtimes I + I \boxtimes B & A \boxtimes B^{-1} + A^{-1} \boxtimes B + 2I \boxtimes I \end{bmatrix} \\ &= \begin{bmatrix} A \boxtimes B & A \boxplus B \\ A \boxplus B & A \boxtimes B^{-1} + A^{-1} \boxtimes B + 2I \boxtimes I \end{bmatrix}. \end{aligned}$$

Pre- and post-multiplying $S^* S$ by X^* and X, respectively, we obtain

$$\begin{aligned} 0 &\leqslant X^* S^* S X \\ &= \begin{bmatrix} Z^*\left(A \boxtimes B\right)Z & Z^*\left(A \boxplus B\right)Z \\ Z^*\left(A \boxplus B\right)Z & Z^*\left(A \boxtimes B^{-1} + A^{-1} \boxtimes B + 2I \boxtimes I\right)Z \end{bmatrix} \\ &= \begin{bmatrix} A \boxdot B & A \circledast B \\ A \circledast B & A \circledast B^{-1} + A^{-1} \boxdot B + 2I \boxdot I \end{bmatrix}. \end{aligned}$$

Applying Lemma 9, we get the inequality (18).

Recall that for any positive real numbers *a* and *b*, we have

$$\frac{(a+1)^2}{a} \le a + 2 + \frac{1}{a}$$

(indeed, both sides are equal). The next theorem generalize this fact to operators; it is also an extension of [7, Corollary 3.8].

Theorem 11. Let $A \in \mathbb{B}(\mathcal{H})$ be a positive operator such that A and $A \boxdot I$ have closed ranges, and $A \boxtimes I = I \boxtimes A$. Then

$$\left(A \circledast AA^{\dagger}\right)\left(A \boxdot I\right)^{\dagger}\left(A \circledast AA^{\dagger}\right) \leqslant A \circledast A^{\dagger} + 2AA^{\dagger} \boxdot I. \tag{19}$$

Proof. Note first that $A^{\dagger}$ and $\left(A \boxdot I\right)^{\dagger}$ exist due to the assumption that A and $A \boxdot I$ have closed ranges. Now, denote

$$S = \begin{bmatrix} A^{1/2} \boxtimes I & A^{1/2} \boxtimes I + (A^{\dagger})^{1/2} \boxtimes I \end{bmatrix} \text{ and } X = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$$

where Z is the selection operator. Since $A \boxtimes I = I \boxtimes A$, we have, by Lemma 2, $A^{\dagger} \boxtimes I = I \boxtimes A^{\dagger}$ and $AA^{\dagger} \boxtimes I = I \boxtimes AA^{\dagger}$. Then

$$\begin{aligned} 0 &\leqslant S^* S \\ &= \begin{bmatrix} A \boxtimes I & A \boxtimes I + AA^{\dagger} \boxtimes I \\ A \boxtimes I + AA^{\dagger} \boxtimes I & A \boxtimes I + A^{\dagger} \boxtimes I + 2AA^{\dagger} \boxtimes I \end{bmatrix} \\ &= \begin{bmatrix} A \boxtimes I & A \boxtimes I + I \boxtimes AA^{\dagger} \\ A \boxtimes I + I \boxtimes AA^{\dagger} & A \boxtimes I + I \boxtimes A^{\dagger} + 2AA^{\dagger} \boxtimes I \end{bmatrix} \\ &= \begin{bmatrix} A \boxtimes I & A \boxplus AA^{\dagger} \\ A \boxplus AA^{\dagger} & A \boxplus A^{\dagger} + 2AA^{\dagger} \boxtimes I \end{bmatrix}. \end{aligned}$$

Pre- and post-multiplying by X^* and X, respectively, we obtain

$$\begin{aligned} 0 &\leqslant X^* S^* S X \\ &= \begin{bmatrix} Z^*\left(A \boxtimes I\right)Z & Z^*\left(A \boxplus AA^{\dagger}\right)Z \\ Z^*\left(A \boxplus AA^{\dagger}\right)Z & Z^*\left(A \boxplus A^{\dagger} + 2AA^{\dagger} \boxtimes I\right)Z \end{bmatrix} \\ &= \begin{bmatrix} A \boxdot I & A \circledast AA^{\dagger} \\ A \circledast AA^{\dagger} & A \circledast A^{\dagger} + 2AA^{\dagger} \boxdot I \end{bmatrix}. \end{aligned}$$

The proof is complete by using Lemma 9.

An equivalent form the arithmetic-geometric mean inequality is that for any real number $a > 0$ we have

$$a + \frac{1}{a} \geq 2.$$

The next theorem is a generalization of this fact; it is also an operator extension of [7, Corollary 3.6].

Theorem 12. Let $A \in \mathbb{B}(\mathcal{H})$ be a positive operator such that A has a closed range and $A^{\dagger} \boxdot I = I \boxdot A^{\dagger}$. Then

$$A \circledast A^{\dagger} \geqslant 2AA^{\dagger} \boxdot I.$$

Proof. The Moore-Penrose inverse of A exists since its range is closed. Now, denote $S = A \boxtimes I \geqslant 0$. The spectral theorem implies that $S + S^{\dagger} \geqslant 2SS^{\dagger}$. We have by using Lemma 2 that

$$\begin{aligned} A \boxplus A^{\dagger} &= A \boxtimes I + A^{\dagger} \boxtimes I \\ &= A \boxtimes I + (A \boxtimes I)^{\dagger} \\ &\geqslant 2(A \boxtimes I)(A \boxtimes I)^{\dagger} \\ &= 2AA^{\dagger} \boxtimes I. \end{aligned}$$

We get the desire result by pre- and post-multiplying by Z^* and Z, respectively.

KANTOROVICH TYPE INEQUALITIES INVOLVING KHATRI-RAO SUMS

In this section, Kantorovich type inequalities involving Khatri-Rao sums are established. We begin with an auxilliary lemma.

Lemma 13. Let S be a positive invertible operator in $\mathbb{B}(\mathcal{K})$ with $mI \leqslant S \leqslant MI$ where m, M are positive constants. The following inequalities hold:

$$S \leqslant (m+M)I - mMS^{-1}, \tag{20}$$

$$S^2 \leqslant (m+M)S - mMI. \tag{21}$$

Proof. Since $mI \leqslant S \leqslant MI$, we have $(MI-S)(mI-S) \leqslant 0$ and hence

$$(MI-S)(mI-S)\,S^{-1} \;=\; S^{-1/2}(MI-S)(mI-S)S^{-1/2} \;\leqslant\; 0$$

Now, the desired inequalities follow easily.

The next lemma provides certain operator inequalities, generalizing matrix results in [17].

Lemma 14. Let S be a positive invertible operator in $\mathbb{B}(\mathcal{K})$ with $mI \leqslant S \leqslant MI$ where m, M are positive constants. For any $X \in \mathbb{B}(\mathcal{H},\mathcal{K})$ such that $X^*X = I$, we have

$$\left(X^*SX\right)^2 \leqslant X^*S^2X \leqslant \frac{(M+m)^2}{4Mm}\left(X^*SX\right)^2, \tag{22}$$

$$\left(X^*SX\right)^{-1} \leqslant X^*S^{-1}X \leqslant \frac{(M+m)^2}{4Mm}\left(X^*SX\right)^{-1}. \tag{23}$$

Proof. Since $X^*X = I$, we have $X^\dagger = X^*$. Since $XX^\dagger$ is Hermitian and idempotent, it is a projection and thus $XX^\dagger \leqslant I$. Then

$$\left(X^*SX\right)^2 \;=\; (SX)^* XX^*(SX) \;\leqslant\; (SX)^* I (SX) \;=\; X^*S^2X.$$

It follows from (21) in Lemma 13 that

$$\begin{aligned} X^*S^2X &\leqslant (m+M)X^*SX - mMXX^* \\ &= \frac{(m+M)^2}{4mM}\left(X^*SX\right)^2 - \left(\frac{m+M}{2\sqrt{mM}}X^*SX - \sqrt{mM}\,I\right)^2 \\ &\leqslant \frac{(m+M)^2}{4mM}\left(X^*SX\right)^2. \end{aligned}$$

Since $S^{1/2}X\left(X^*SX\right)^{-1}X^*S$ is Hermitian and idempotent, it is a projection and thus $S^{1/2}X\left(X^*SX\right)^{-1}X^*S \leqslant I$. It follows from (20) that

$$\begin{aligned} X^*S^{-1}X &\leqslant \frac{(m+M)}{mM}XX^* - mMXSX^* \\ &= \frac{(m+M)^2}{4mM}\left(X^*SX\right)^{-1} \\ &\quad - \left[\frac{m+M}{2\sqrt{mM}}\left(X^*SX\right)^{-1/2} - \frac{1}{\sqrt{mM}}\left(X^*SX\right)^{1/2}\right]^2 \\ &\leqslant \frac{(m+M)^2}{4mM}\left(X^*SX\right)^{-1}. \end{aligned}$$

The next theorem establishs Kantorovich type inequalities concerning ordinary powers and inverses of operators.

Theorem 15. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive invertible operators and $mI \leqslant A \boxtimes I \oplus I \boxtimes B \leqslant MI$ where m, M are positive constants. Then

$$\left(A \circledast B\right)^2 \leqslant A^2 \circledast B^2 \leqslant \frac{(M+m)^2}{Mm}\left(A \circledast B\right)^2. \tag{24}$$

If
$A \circledast B$ is invertible, then

$$\left(A \circledast B\right)^{-1} \leqslant A^{-1} \circledast B^{-1} \leqslant \frac{(M+m)^2}{Mm}\left(A \circledast B\right)^{-1}. \tag{25}$$

Proof. Denote

$$S = \begin{bmatrix} A \boxtimes I & 0 \\ 0 & I \boxtimes B \end{bmatrix} \text{ and } X = \frac{1}{\sqrt{2}}\begin{bmatrix} Z \\ Z \end{bmatrix}$$

where Z is the selection operator associated with $(\mathcal{H},\mathcal{K})$. Then $X^*X = I$. By Lemma 3, $A \boxtimes I \geqslant 0$ and $I \boxtimes B \geqslant 0$. We have $S \geqslant 0$ by Lemma 14. Since A and B are invertible, we conclude that S is invertible. Using Lemma 8, we have

$$\begin{aligned} X^*SX &= \frac{1}{2}\begin{bmatrix} Z^* & Z^* \end{bmatrix}\begin{bmatrix} A \boxtimes I & 0 \\ 0 & I \boxtimes B \end{bmatrix}\begin{bmatrix} Z \\ Z \end{bmatrix} \\ &= \frac{1}{2}\left[Z^*\left(A \boxtimes I + I \boxtimes B\right)Z\right] \\ &= \frac{1}{2}\left[Z^*\left(A \boxplus B\right)Z\right] \\ &= \frac{1}{2}\left(A \circledast B\right), \\ X^*S^{-1}X &= \frac{1}{2}\begin{bmatrix} Z^* & Z^* \end{bmatrix}\begin{bmatrix} A^{-1} \boxtimes I & 0 \\ 0 & I \boxtimes B^{-1} \end{bmatrix}\begin{bmatrix} Z \\ Z \end{bmatrix} \\ &= \frac{1}{2}\left[Z^*\left(A^{-1} \boxtimes I + I \boxtimes B^{-1}\right)Z\right] \\ &= \frac{1}{2}\left[Z^*\left(A^{-1} \boxplus B^{-1}\right)Z\right] \\ &= \frac{1}{2}\left(A^{-1} \circledast B^{-1}\right), \\ X^*S^2X &= \frac{1}{2}\begin{bmatrix} Z^* & Z^* \end{bmatrix}\begin{bmatrix} A^2 \boxtimes I & 0 \\ 0 & I \boxtimes B^2 \end{bmatrix}\begin{bmatrix} Z \\ Z \end{bmatrix} \\ &= \frac{1}{2}\left[Z^*\left(A^2 \boxtimes I + I \boxtimes B^2\right)Z\right] \\ &= \frac{1}{2}\left[Z^*\left(A^2 \boxplus B^2\right)Z\right] \\ &= \frac{1}{2}\left(A^2 \circledast B^2\right). \end{aligned}$$

Substitution in (22) and (23) of Lemma 14 leads to the results.

Lemma 16. Let S be a positive invertible operator in $\mathbb{B}(\mathcal{K})$ with $mI \leqslant S \leqslant MI$ where m,M are positive constants. For any $X \in \mathbb{B}(\mathcal{H},\mathcal{K})$ such that $X^*X = I$, we have

$$X^*SX - (X^*S^{-1}X)^{-1} \leqslant (\sqrt{M}-\sqrt{m})^2 I \tag{26}$$

$$X^*S^2X - (X^*SX)^2 \leqslant \frac{1}{4}(M-m)^2 I \tag{27}$$

$$(X^*S^2X)^{1/2} - X^*SX \leqslant \frac{(M-m)^2}{4(M+m)} I. \tag{28}$$

Proof. Using (20), we obtain

$$\begin{aligned} X^*SX &- (X^*S^{-1}X)^{-1} \\ &\leqslant (m+M)I - mMX^*S^{-1}X - \left(X^*S^{-1}X\right)^{-1} \\ &= \left(\sqrt{M}-\sqrt{m}\right)^2 I - \left[\sqrt{mM}\left(X^*S^{-1}X\right)^{1/2} - \left(X^*S^{-1}X\right)^{-1/2}\right]^2 \\ &\leqslant \left(\sqrt{M}-\sqrt{m}\right)^2 I. \end{aligned}$$

It follows from (21) that

$$\begin{aligned} X^*S^2X &- (X^*SX)^2 \\ &\leqslant (m+M)X^*SX - mMI - \left(X^*SX\right)^2 \\ &= \frac{1}{4}(M-m)^2 I - \left[X^*SX - \frac{1}{2}(m+M)I\right]^2 \\ &\leqslant \frac{1}{4}(M-m)^2 I. \end{aligned}$$

On the other hand, we have from (21) that

$$\begin{aligned} \left(X^*S^2X\right)^{1/2} &- X^*SX \\ &\leqslant \left(X^*S^2X\right)^{1/2} - \frac{1}{m+M}S^2 - \frac{mM}{m+M}I \\ &= \frac{(M-m)^2}{4(m+M)}I - \left[\frac{1}{\sqrt{mM}}\left(X^*S^2X\right)^{1/2} - \frac{\sqrt{m+M}}{2}I\right]^2 \\ &\leqslant \frac{(M-m)^2}{4(m+M)}I. \end{aligned}$$

Our last theorem provides another forms of Kantorovich inequality concerning ordinary powers and inverses.

Theorem 17. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be positive invertible operators and $mI \leqslant (A \boxtimes I) \oplus (I \boxtimes B) \leqslant MI$ where m,M are positive constants. Then

$$2\left(A^2 \circledast B^2\right) - \left(A \circledast B\right)^2 \leqslant (M-m)^2 I, \tag{29}$$

$$\left(A^2 \circledast B^2\right)^{1/2} - \frac{1}{\sqrt{2}}\left(A \circledast B\right) \leqslant \frac{(M-m)^2}{2\sqrt{2}} I. \tag{30}$$

In addition, if $A^{-1} \circledast B^{-1}$ is invertible, then

$$\frac{1}{2}\left(A \circledast B\right) - 2\left(A^{-1} \circledast B^{-1}\right)^{-1} \leqslant \left(\sqrt{M}-\sqrt{m}\right)^2 I. \tag{31}$$

Proof. The proof can be omiited since it is similar to that of Theorem 15. Instead of Lemma 14, we apply Lemma 16.

CONCLUSION

We provide relations between the Khatri-Rao sum of operators and various kinds of operator operations, namely, ordinary products and powers, ordinary and Moore-Penrose inverses. These relations appear in terms of inequalities, including arithmetic-geometric mean inequality and Kantorovich type inequalities. The results involving Moore-Penrose inverses are valid under the assumption of closedness of certain operators. Our results show that the Khatri-Rao sum and the Khatri-Rao product can be regarded as the "sum" and the "product" of operators, respectively.

ACKNOWLEDGEMENT

The second author would like to thank King Mongkut's Institute of Technology Ladkrabang Research Fund for financial supports.

REFERENCES

[1] Tracy, D. S. and Singh, R. P. 1972. A new matrix product and its applications in partitioned matrix differentiation. *Statistica Neerlandica*. 26(4), 143-157.

[2] Khatri, C. G. and Rao, C. R. 1968. Solutions to some functional equations and their applications to characterization of probability distributions. *Sankhya*. 26(4), 167-180.

[3] Al Zhour, Z. A. and Kilicman, A. 2006. Extension and generalization inequalities involving the Khatri-Rao product of several positive matrices, *Journal of Inequalities and Applications*, Article ID 80878.

[4] Liu, S. and Trenkler, G. 2008. Hadamard, Khatri-Rao, Kronecker and other matrix products, *International Journal of Information and Systems Sciences*, 4(1), 160-177.

[5] Shuangzhe, L. 1999. Matrix results on the Khatri-Rao and Tracy-Singh products, *Linear Algebra and its Appications.* 289(1-3), 267-277.

[6] Van Loan, C. F. 2000. The ubiquitous Kronecker product, *Journal of Computational and Applied Mathematics*. 123(1-2), 85-100.

[7] Al Zhour, Z. A. and Kilicman, A. 2006. Matrix equalities and inequalities involving Khatri-Rao and Tracy-Singh sums, *Journal of Inequalities in Pure and Applied Mathematics.* 7(1), 11-17.

[8] Kubrusly, C. S. and Levan, N. 2011. Preservation of tensor sum and tensor product, *Acta Mathematica Universitatis Comenianae*. 80(1), 133-142 .

[9] Ploymukda, A. Chansangiam, P. and Lewkeeratiyutkul, W. 2017. Algebraic and order properties of Tracy-Singh products for operator matrices, *Journal of Computational Analysis and Applications*, 24(4), 656-664.

[10] Ploymukda, A. Chansangiam, P. and Lewkeeratiyutkul, W. 2017. Analytic properties of Tracy-Singh products for operator matrices, *Journal of Computational Analysis and Applications*, 24(4), 665-674

[11] Ploymukda, A. and Chansangiam, P. 2016. Khatri-Rao products for operator matrices acting on the direct sum of Hilbert spaces, *Journal of Mathematics*, Article ID 8301709.

[12] Ploymukda, A. and Chansangiam, P. 2017. Several inequalities for Khatri-Rao products of Hilbert space operators, *Communications in Mathematics and Applications.* 8(1), 45-60.

[13] Ploymukda, A. and Chansangiam, P. 2018. Khatri-Rao sums for Hilbert space operators, *Songklanakarin Journal of Science and Technology*. 40(5), (in press).

[14] Penrose, R. 1955. A generalized inverse for matrices, *Mathematical Proceedings of the Cambridge Philosophical Society*. 51(3), 406-413.

[15] Caradus, S. R. 1978. *Generalized inverses and operator theory*, Queen's Papers in Pure and Applied Mathematics no. 50, Queen's University, Kingston, Ontario.

[16] Xu, Q. and Sheng, L. 2008. Positive semi-definite matrices of adjointable operators on Hilbert-modules, *Linear Algebra and its Applications.* 428(4), 992-1000.

[17] Liu, S. and Neudecker, H. 1999. A survey of Cauchy-Schwarz and Kantorovich-type matrix inequalities, *Statistical Papers.* 40(1), 55–73.