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RESEARCH ARTICLE

A review on taxonomy of fuzzy graph

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Abstract

Fuzzy graph is a graph that incorporates fuzziness. Fuzzy graph originated from the concept of Fuzzy Set and Graph. In this paper, taxonomy of fuzzy graph is reviewed. Several results on fuzzy graph are presented together with some examples. The fuzzy graph introduced by Yeh and Bang is proven to be a special case of Rosenfeld's fuzzy graph. Two descriptions of isomorphic crisp graph are proven to be equivalent and every crisp isomorphism of two graphs is proven to be a fuzzy isomorphism. This paper provides an underlying knowledge on fuzzy graph which is an important foundation for further development and application of fuzzy graph.

***Keywords*:** Graph theory, fuzzy set theory, fuzzy graph, isomorphism

INTRODUCTION

Graph and Fuzzy Set are established theories that have been applied in numerous applications. Graph is one of many approaches to solve various problems involving relations and networks [1]. Leonhard Euler introduced graph in 1736 in order to solve the famous Konigsberg bridge problem [1]. The problem arose from citizens of Konigsberg city who questioned whether the seven bridges that connected various parts of the city can be traversed exactly once and return to the starting point. Euler then came up with an idea by representing the lands and bridges in the form of vertices and edges respectively.

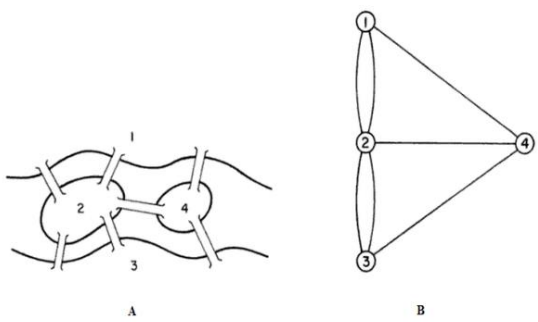

Fig. 1 Konigsberg bridge problem, A and representation in the form of vertices and edges, B [2].

Euler's idea of solving the Konigsberg bridge problem was in the form of vertices and edges which led to the development and advancement knowledge on graph [1]. The formal definition of a graph is as follows:

Definition 1. [1] A graph, $G=(V,E)$ contains a set of vertices, V and a set of edges, E such that for vertices $a,b\in V$ the edge is (a,b).

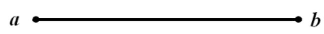

Fig. 2 A graph G with vertices, a,b and edge, (a,b).

A graph may include multiple edges and loops. A loop is an edge from a vertex to itself [3]. Graph in Fig. 2 is called a simple graph with no loops or multiple edges. A directed graph is a graph with directed edges. The directions of the edges are indicated by arrows.

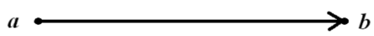

Fig. 3 A directed graph G with vertices, a,b and directed edge, (a,b).

Ever since the concept of graph was introduced, it is now becoming important and has been applied in various applications such as in network analysis, operations research and economics [4, 5].

However, in 1965, Zadeh [6] stated that some aspect in real world applications may involve with uncertainties and may results in inadequacy. Thus, Zadeh [6] introduced the concept of fuzzy set as a model with graded property in order to deal with the uncertainties. The formal definition of fuzzy set proposed by Zadeh is as follows:

Definition 2. A fuzzy set A is a set that consists of elements x_i characterized by a membership function $f_A(x_i)$ which associates the elements to a number of interval [0, 1] [6].

Definition 2 states that a fuzzy set A is a set that consists of elements x_i and each element has its own membership function $f_A(x_i)$ with value of [0, 1].

Additionally, a concept known as fuzzy relation was introduced as a natural extension of fuzzy set and is important and applicable to some applications involving relation [6]. A fuzzy relation is defined as a set of ordered pairs of elements in fuzzy sets [6].

FUZZY GRAPH

Graph is one of the mathematical concepts that usually occurs in networks and relational problems [4, 5].

Rosenfeld [7] introduced fuzzy graph in which the concept of fuzzy set is applied in graph. Rosenfeld [7] defined fuzzy graph as a graph that consists of vertices and edges with membership functions presented as follows:

Definition 3. Fuzzy graph $G=(\sigma,\mu)$ is a pair of functions $\sigma : S \to [0,1]$ and $\mu : S \times S \to [0,1]$, where for all $x, y \in S$ we have $\mu(x,y) \le \sigma(x) \wedge \sigma(y)$ [7].

Definition 3 states that a fuzzy graph consists of vertices and edges with membership value of [0, 1]. During the same year, Yeh and Bang [8] proposed a different version of fuzzy graph as follows:

Definition 4. A fuzzy graph $G=(V,R)$ where V is a set of vertices and R is a fuzzy relation on V in which the edges connecting the vertices in V have membership function $\mu_R : V \times V \to [0,1]$ [8].

Definition 4 states that only the edges are fuzzy while the vertices are crisp. Yeh and Bang [8] introduced the new version of fuzzy graph in order to be applied in clustering analysis, while Rosenfeld [7] defined it in a more general term.

Since the evolutionary work of fuzzy graph by Rosenfeld [7] and Yeh and Bang [8], Blue et al. [4, 5] studied the classification of fuzzy graphs. Blue et al. [4, 5] introduced five types of fuzziness possible in directed graphs. However, only possible situations are provided for each type with no clear examples are given in [4, 5]. Thus, the descriptions of the five types of fuzziness together with their examples are as follows:

i. Type I: Fuzzy set of crisp graphs

$G=\{(G_1,\mu(G_1)),(G_2,\mu(G_2)),\ldots,(G_n,\mu(G_n))\}$

A fuzzy set, G consist of crisp graphs G_i for $i=1,2,\ldots n$.

Every crisp graph in the set has a membership function belong to each graphs.

Example 1

Graphs G_1 and G_2 in Fig. 4 represent electrical plans of two different houses in a same block, but it is not known which electrical plan correspond to which house. The fuzziness of the two graphs are illustrated as $G=\{(G_1,0.5),(G_2,0.8)\}$.

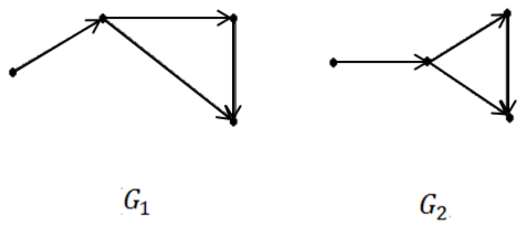

Fig. 4 Two crisp graphs G_1 and G_2.

ii. Type II: Crisp vertex and fuzzy edge

$G=\{V,E\}$ where

$V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{(e_1,\mu(e_1)),(e_2,\mu(e_2)),\ldots,(e_m,\mu(e_m))\}$

The graph G consists of crisp vertices and fuzzy edges.

Example 2

The vertices, V and edges, E in Fig. 5 represent cities and roads respectively. However some roads are closed due to construction such that:

$V=\{v_1,v_2,v_3,v_4\}$

$E=\{(e_1,0.1),(e_2,0.3),(e_3,0.2),(e_4,0.8),(e_5,0.9)\}$

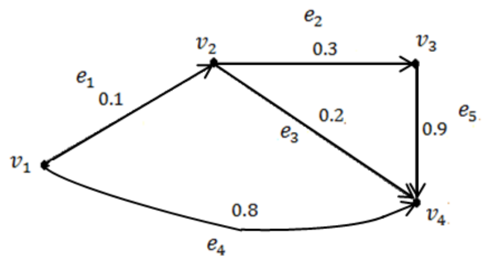

Fig. 5 A graph G with crisp vertices and fuzzy edges.

iii. Type III: Crisp vertices and edges with fuzzy connectivity.

$G=\{V,E\}$ where

$V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{e_1,e_2,\ldots,e_m\}$

$h_i=\{(h_{i,1},\sigma_{i,1}),(h_{i,2},\sigma_{i,2}),\ldots,(h_{i,n},\sigma_{i,n})\}$ for $i=1,2,3,\ldots m$

$t_i=\{(t_{i,1},\tau_{i,1}),(t_{i,2},\tau_{i,2}),\ldots,(t_{i,n},\tau_{i,n})\}$ for $i=1,2,3,\ldots m$

The edges connectivity are fuzzy such that the heads and tails of the edges have a membership value of [0, 1] while the vertices and edges are crisp.

Example 3

Graph in Fig. 6 represent the clinical incineration process as studied in [9]. The set of vertices, $V=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ represents the variables involved in the incineration process, while the edges represent the catalytic relationship between the variables. The thickness and color of the edges denote the connectivity strength between the vertices and different range of membership value for the fuzzy edge connectivity respectively.

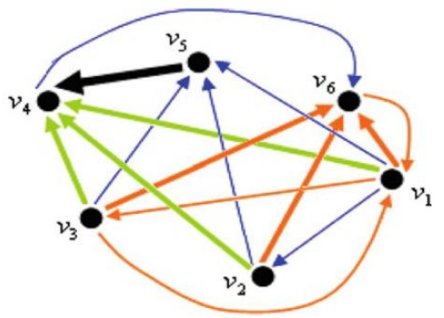

Fig. 6 Fuzzy graph Type III for the clinical incineration process [9].

iv. Type IV: Fuzzy vertices and crisp edges

$G = \{V, E\}$ where

$V = \{(v_1, \mu(v_1)), (v_2, \mu(v_2)), ..., (v_n, \mu(v_n))\}$ and $E = \{e_1, e_2, ..., e_m\}$

The graph G consists of fuzzy vertices and crisp edges.

Example 4

One has to find cost-effective travel plan to attend conferences, however the locations for the conferences are not yet revealed. The fuzzy vertices and crisp edges in Fig. 7 represent the locations and routes respectively.

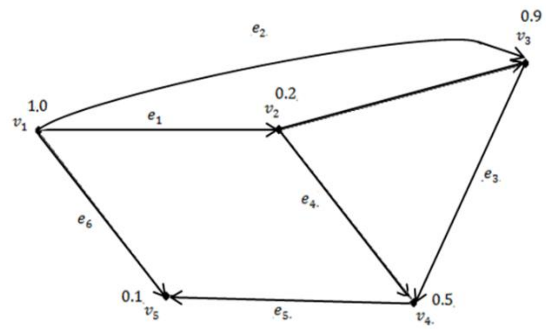

Fig. 7 A graph, G with fuzzy vertices and crisp edges.

v. Type V: Crisp Graph with Fuzzy Weight

$G = \{V, E\}$ where

$V = \{v_1, v_2, ..., v_n\}$ and $E = \{e_1, e_2, ..., e_m\}$

but the edges have fuzzy weights, defined by:

$w_i = \{(w_{i,1}, \mu(w_{i,1})), (w_{i,2}, \mu(w_{i,2})), ...\}$

Example 5

One has to find shortest time travel to a place, however the map only shows the distance of the paths but the time travel across the paths are unknown. The vertices and edges in Fig. 8 represent the place and paths respectively.

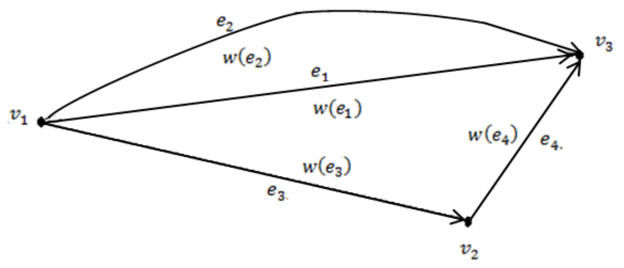

Fig. 8 A graph G with crisp vertices and edges, and weight on the edges.

The classification in [4, 5] was on only five types of fuzziness without the possibilities of their combinations. One type of fuzziness can be a special case of another type. For example, Type I can also be Type III fuzziness. Type I is a special case of Type III fuzziness as presented in example as follows:

Example 6

Let Type I fuzziness of graphs be

$G = \{(G_1, \mu(G_1)), (G_2, \mu(G_2)), ..., (G_n, \mu(G_n))\}$

There are few records of electrical plan of houses in one block, but one cannot distinguish which electrical plan belongs to which house.

The electrical plans are crisp graphs G_1, G_2 and G_3. Each crisp graph is assigned with a membership values defined by:

$\mu(G_1) = 0.2, \mu(G_2) = 0.4, \mu(G_3) = 0.3$

Thus, Type I graph is presented as follows:

$G = \{(G_1, 0.2), (G_2, 0.4), (G_3, 0.3)\}$

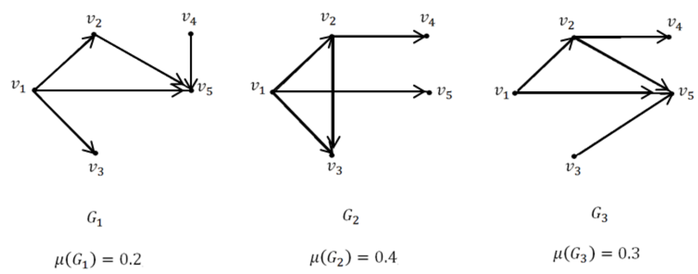

Fig. 9 Three electrical plans denoted by crisp graphs G_1, G_2 and G_3 with respective membership functions.

This type of fuzziness can also be a special case of Type III. Both vertex and edge sets are crisp but there may exist fuzziness on head and tail of the edges. The connectivity of the edges on the crisp graphs G_1, G_2 and G_3 above are crisp which have values of $\{0,1\} \subset [0,1]$ and thus satisfies fuzzy graph Type III. This implies that fuzzy graph Type I can be a special case of fuzzy graph Type III.

Next section discusses some properties of fuzzy graph such as degree of vertex, order and size of fuzzy graph, isomorphism of fuzzy graph and their relatedness.

SOME PROPERTIES AND ISOMORPHISMS OF FUZZY GRAPHS

The growing knowledge of fuzzy graph has led to the study of its properties and concept on isomorphism. Gani and Ahamed [10] introduced some properties of fuzzy graphs such as degree of vertex, order and size of fuzzy graph. Definitions on the properties of fuzzy graphs are presented as follows:

Throughout Definitions 5-7, let $G=(V,\mu,\rho)$ be a fuzzy graph where V is a vertex, μ is a mapping $\mu:V\to[0,1]$ and ρ is a mapping $\rho:V\times V\to[0,1]$.

Definition 5. Degree of vertex, v in G is defined as $d(v)=\sum\rho(v,u)$, where $u,v\in V$ and $u\neq v$ [10].
The minimum degree of G is $\delta(G)=\wedge\{d(v)/v\in V\}$
The maximum degree of G is $\Delta(G)=\vee\{d(v)/v\in V\}$

Definition 6. Order of G is defined as $O(G)=\sum\mu(v)$ where $v\in V$ [10].

Definition 7. Size of G is defined as $S(G)=\sum\rho(u,v)$ where $u,v\in V$ and $u\neq v$ [10].

Gani and Ahamed [10] proved the relationship between degree of vertex, order and size of fuzzy graphs in the following inequality: $\delta(G)\le\Delta(G)\le S(G)\le O(G)$.

However, Gani and Ahamed [10] did not illustrate any example in their paper. Here is an example of the relation.

Example 7
Let fuzzy graph $G=(\sigma,\mu)$, with set of vertices and membership function as follows:

$V=\{v_1,v_2,v_3,v_4,v_5\}$
$\sigma(v_1)=0.5,\sigma(v_2)=0.6,\sigma(v_3)=0.9,\sigma(v_4)=0.8,\sigma(v_5)=0.7$
$\mu(v_1,v_2)=0.5,\mu(v_1,v_3)=0.5,\mu(v_2,v_4)=0.2,\mu(v_3,v_4)=0.2,$
$\mu(v_2,v_5)=0.1,\mu(v_3,v_5)=0.3$

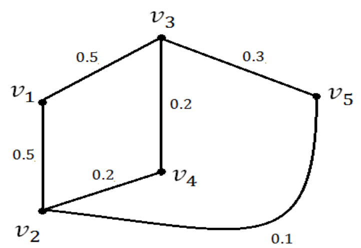

Fig. 10 Fuzzy Graph G

Degree of vertex, $d(v)$ are:
$d(v_1)=\mu(v_1,v_2)+\mu(v_1,v_3)=0.5+0.5=1.0$
$d(v_2)=\mu(v_1,v_2)+\mu(v_2,v_4)+\mu(v_2,v_5)=0.5+0.2+0.1=0.8$
$d(v_3)=\mu(v_1,v_3)+\mu(v_3,v_4)+\mu(v_3,v_5)=0.5+0.2+0.3=1.0$
$d(v_4)=\mu(v_2,v_4)+\mu(v_3,v_4)=0.2+0.2=0.4$
$d(v_5)=\mu(v_2,v_5)+\mu(v_3,v_5)=0.1+0.3=0.4$

Thus, minimum degree of G is: $\delta(G)=0.4$.
The maximum degree of G is: $\Delta(G)=1.0$.

Order, $O(G)$ and size, $S(G)$ of fuzzy graph are:
$O(G)=\sigma(v_1)+\sigma(v_2)+\sigma(v_3)+\sigma(v_4)+\sigma(v_5)$
$=0.5+0.6+0.9+0.8+0.7$
$=3.5$

$S(G)=\mu(v_1,v_2)+\mu(v_1,v_3)+\mu(v_2,v_4)+\mu(v_3,v_4)+\mu(v_2,v_5)+\mu(v_3,v_5)$
$=0.5+0.5+0.2+0.2+0.1+0.3$
$=1.8$

Thus the inequality:
$\delta(G)\le\Delta(G)\le S(G)\le O(G)=0.4\le1.0\le1.8\le3.5$

Additionally, two different graphs may have the same properties or equivalent. According to Chartrand [11], two equal graphs can be referred as isomorphic graphs. Isomorphism is a concept in which two graphs are equivalent [11]. The idea of isomorphism of crisp and fuzzy graphs is slightly different as fuzzy graph involves membership function. Bondy and Murty [12] described isomorphism of crisp graphs as a pair of bijection mappings between two graphs as follows:

Definition 8. Let two graphs $G=(V(G),E(G))$ and $H=(V(H),E(H))$, where $V(G)$ and $V(H)$ are set of vertices of G and H, while $E(G)$ and $E(H)$ are set of edges of G and H. The two graphs are said to be isomorphic denoted by $G\cong H$ if there exist bijections $\theta:V(G)\to V(H)$ and $\phi:E(G)\to E(H)$ such that vertices u_1 and u_2 are incident with edge u_1u_2 in G if and only if the mapped vertices $\theta(u_1)$ and $\theta(u_2)$ are incident with mapped edge $\phi(u_1u_2)$ in H [12].

Furthermore, Wilson [3] described isomorphism as follows:

Definition 9. Two graphs G and H are isomorphic if there is one to one correspondence between vertices of G and H such that the number of edges joining the vertices in G is equal to the number of edges joining the corresponding vertices in H [3].

Moreover, Chartrand [11] stated that if two graphs are isomorphic, then the number of vertices and edges of both graphs are equal.

Isomorphism of fuzzy graphs is slightly different with crisp graph as it involves membership function. Bhutani [13] defined isomorphism for fuzzy graphs, $G=(\sigma,\mu)$ and $G'=(\sigma',\mu')$ with their respective vertices set S and S' as follows:

Definition 10. Homomorpism of fuzzy graphs is a map $h:G\to G'$ and also $h:S\to S'$ such that;
$\sigma(x)\le\sigma'(h(x))$ for all $x\in S$
$\mu(x,y)\le\mu'(h(x),h(y))$ for all $x,y\in S$ [13]

Definition 11. Isomorphism of fuzzy graphs, $h:G\to G'$ is a bijective map $h:S\to S'$ such that;
$\sigma(x)=\sigma'(h(x))$ for all $x\in S$
$\mu(x,y)=\mu'(h(x),h(y))$ for all $x,y\in S$ [13]

Definition 12. Weak isomorphism of fuzzy graphs, $h: G \to G'$ is a homomorphism such that the map $h: S \to S'$ is bijective and satisfies; $\sigma(x) = \sigma'(h(x))$ for all $x \in S$ [13].

Definition 13. Co-weak isomorphism of fuzzy graphs, $h: G \to G'$ is a homomorphism such that the map $h: S \to S'$ is bijective and satisfies; $\mu(x, y) = \mu'(h(x), h(y))$ for all $x, y \in S$ [13].

Both isomorphisms in Definition 12 and 13 involve bijective mapping, but weak isomorphism only preserves the membership function of the vertices and not necessarily the membership function of the edges. The co-weak isomorphism preserves the membership function of the edges and not necessarily the membership function of the vertices.

Gani and Malarvizhi [14] discussed the relation of the order, size and degree of the vertices with isomorphism of fuzzy graph. Additionally, Chartrand [11] stated that the order and size of two isomorphic crisp graphs are the same. According to Gani and Malarvizhi [14], the case for fuzzy graphs is analogous to crisp graphs. Gani and Malarvizhi [14] described the relation of the order, size and degree of the vertices with isomorphic fuzzy graphs in the following theorems:

Theorem 1. If any two fuzzy graphs are isomorphic, then their order and size are same [14].

Theorem 2. If two fuzzy graphs are isomorphic, then the degree of their vertices is preserved [14].

Furthermore, Gani and Malarvizhi [14] stated that for weak isomorphic fuzzy graphs, their orders are the same while for co-weak isomorphic fuzzy graphs, their sizes are the same. But conversely, if the fuzzy graphs have the same orders or sizes, they are not necessarily be weak or co - weak isomorphic fuzzy graphs [14].

RESULTS AND DISCUSSION

Several trivial findings on fuzzy graph are presented in the form of theorems and lemma, and proven together with some examples are provided.

From the definition of fuzzy graph, Yeh and Bang [8] defined fuzzy graph in such a way that it can be applied in clustering analysis while Rosenfeld [7] defined fuzzy graph in a more general term.

Theorem 3. A fuzzy graph which consists of a crisp set of vertices and a set of fuzzy edges is a special case of the general fuzzy graph with fuzzy set of vertices and edges.

Proof:

From Definition 4 by Yeh and Bang [8], let $G = (V, R) = (\sigma, \mu_R)$ be a fuzzy graph with V as a set of vertices with membership function $\sigma : V \to \{0,1\} \subset [0,1]$ and R is a fuzzy relation on V with the edges connecting the vertices in V with membership function $\mu_R : V \times V \to [0,1] \subseteq [0,1]$.

This fuzzy graph consists of vertices with crisp value $\{0,1\} \subset [0,1]$ and fuzzy edges with value $[0,1] \subseteq [0,1]$. Thus, this fuzzy graph G also fulfils the Definition 3 of Rosenfeld [7] which states that a fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : S \to [0,1]$ and $\mu : S \times S \to [0,1]$, where for all $x, y \in S$ we have $\mu(x, y) \le \sigma(x) \wedge \sigma(y)$.

Since the crisp vertices V has the membership function, σ with value $\{0,1\} \subset [0,1]$ and the fuzzy edges has the membership function, μ_R with value $[0,1] \subseteq [0,1]$, thus the fuzzy graph which consists of a crisp set of vertices and fuzzy set of edges is a special case of the general fuzzy graph with fuzzy set of vertices and edges. ■

Example 8

Consider fuzzy graph $G = (V, R) = (\sigma, \mu_R)$.

Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ with membership function, σ with crisp value as follows:

$$\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = \sigma(v_4) = \sigma(v_5) = 1$$

R is a fuzzy relation on V which means each paired elements in V or edges in G is characterized by a membership function defined by μ_R as follows:

$$\mu_R(v_1, v_2) = 0.2, \mu_R(v_1, v_3) = 0.2, \mu_R(v_2, v_4) = 0.4,$$
$$\mu_R(v_3, v_4) = 0.3, \mu_R(v_4, v_5) = 0.8$$

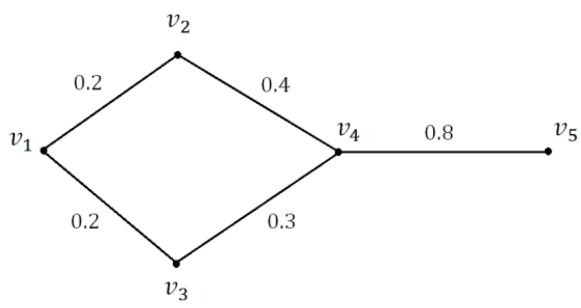

Fig. 11 Fuzzy Graph $G = (V, R)$

This fuzzy graph G satisfies both Definition 3 and 4. The vertices v_1, v_2, v_3, v_4, v_5 have membership value of 1 which satisfies Definition 4, and since $1 \in [0,1]$, it also satisfies Definition 3. The edges in the fuzzy graph are characterized by membership value of $[0,1]$, satisfies both definitions. Thus, $G = (V, R)$ is considered as fuzzy graph by Yeh and Bang [8] and a special case of fuzzy graph by Rosenfeld [7].

The concept of isomorphism of fuzzy graph was introduced by Bhutani [13] which analogous to the concept of isomorphic crisp graph but differed by their membership functions. Based on the descriptions of both isomorphism of crisp and fuzzy graphs, the following lemma and theorem are deduced:

Lemma 1. The following statements on isomorphism of crisp graphs are equivalent:

Suppose $G = (V(G), E(G))$ and $H = (V(H), E(H))$:

(i) There exist bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that vertices u_1 and u_2 are incident with edge u_1u_2 in G if and only if the mapped vertices $\theta(u_1)$ and $\theta(u_2)$ are incident with mapped edge $\phi(u_1u_2)$ in H [12].

(ii) There is one to one correspondence between the vertices of G and H such that the number of edges joining the vertices in G is equal to the number of edges joining the corresponding vertices in H [3].

Proof:

(i) ⇒ (ii)

Suppose $G=(V(G),E(G))$ and $H=(V(H),E(H))$, let:

$$V(G)=\{u_1,u_2,...,u_{n-1},u_n\}$$
$$V(H)=\{v_1,v_2,...,v_{n-1},v_n\}$$
$$E(G)=\{u_1u_2,...,u_{n-1}u_n\}$$
$$E(H)=\{v_1v_2,...,v_{n-1}v_n\}$$

There exist bijections $\theta:V(G)\to V(H)$ and $\phi:E(G)\to E(H)$:

$$V(H)=\{\theta(u_1),\theta(u_2),...,\theta(u_{n-1}),\theta(u_n)\}=\{v_1,v_2,...,v_{n-1},v_n\}$$
$$E(H)=\{\phi(u_1u_2),...,\phi(u_{n-1}u_n)\}=\{v_1v_2,...,v_{n-1}v_n\}$$

Which implies that there is one to one correspondence between the vertices and edges of both graphs respectively such that vertices u_1 and u_2 are incident with edge u_1u_2 in G if and only if the mapped vertices $\theta(u_1)$ and $\theta(u_2)$ are incident with mapped edge $\phi(u_1u_2)$ in H thus implies that u_1 and u_2 are adjacent in G if and only if v_1 and v_2 are adjacent in H. In other words, u_1u_2 is the edge in G if and only if v_1v_2 is the edge in H. This implies that the edges joining the vertices in G are corresponding to the edges joining the corresponding vertices in H, and since bijection involves one to one and onto relations, each edge in G is mapped with exactly one edge in H and there are no unmapped edges. Thus the number of edges in G is equal to the number of edges in H. Hence, statement (i) implies statement (ii) where there is one to one correspondence between the vertices of G and H such that the number of edges joining the vertices in G is equal to the number of edges joining the corresponding vertices in H.

(ii) ⇒ (i):

Suppose $G=(V(G),E(G))$ and $H=(V(H),E(H))$, let:

$$V(G)=\{u_1,u_2,...,u_{n-1},u_n\}$$
$$V(H)=\{v_1,v_2,...,v_{n-1},v_n\}$$
$$E(G)=\{u_1u_2,...,u_{n-1}u_n\}$$
$$E(H)=\{v_1v_2,...,v_{n-1}v_n\}$$

There is one to one correspondence between $V(G)$ and $V(H)$ which implies that there exist bijection between $V(G)$ and $V(H)$, such that the number of edges joining the vertices in G is equal to the number of edges joining the corresponding vertices in H.
Since each vertex in G is corresponding to exactly one vertex in H without any exclusion, the edges joining the vertices in G are also corresponding to the edges joining the corresponding vertices in H without any exclusion as the number of edges are the same. Thus, consider maps:
$\theta:V(G)\to V(H)$ and $\phi:E(G)\to E(H)$,
where each element in G is mapped to exactly one element in H and there are no unmapped elements. Thus these mappings are considered bijective.
Moreover, since the connected vertices in G are corresponding to connected vertices in H, thus if u_1u_2 is an edge in G where vertices u_1 and u_2 are incident with it, then $\phi(u_1u_2)=v_1v_2$ is an edge in H where vertices $\theta(u_1)=v_1$ and $\theta(u_2)=v_2$ are incident with it. Hence, statement (ii) implies statement (i) where there exist bijections $\theta:V(G)\to V(H)$ and $\phi:E(G)\to E(H)$ such that vertices u_1 and u_2 are incident with edge u_1u_2 in G if and only if the mapped vertices $\theta(u_1)$ and $\theta(u_2)$ are incident with mapped edge $\phi(u_1u_2)$ in H. Therefore, since each statement can implies the other statement in sequence, thus all of the descriptions of isomorphism of crisp graph in Lemma 1 are equivalent. Thus each description can be used to define isomorphism of crisp graph. ■

Isomorphism of crisp graph and fuzzy graph are differed by the membership function. Isomorphism of fuzzy graph involved bijective mapping of the vertices and edges between any two fuzzy graphs while preserving the membership function of both vertices and edges. Hence, by Lemma 1 and isomorphism of fuzzy graphs, the following theorem is deduced:

Theorem 4. Every crisp isomorphism of two graphs is a fuzzy isomorphism.

Proof:
By Lemma 1, let the isomorphism of crisp graph is defined by statement (ii), where there is one to one correspondence between the vertices of G and H such that the number of edges joining the vertices in G is equal to the number of edges joining the corresponding vertices in H. Let;

$$V(G)=\{v_1,v_2,...,v_{n-1},v_n\}$$
$$V(H)=\{v_1^{'},v_2^{'},...,v_{n-1}^{'},v_n^{'}\}$$
$$E(G)=V(G)\times V(G)=\{(v_1,v_2),...,(v_{n-1},v_n)\}$$
$$E(H)=V(H)\times V(H)=\{(v_1^{'},v_2^{'}),...,(v_{n-1}^{'},v_n^{'})\}$$

By Lemma 1 (i.e (ii) ⇒ (i)), there exist bijective relation between the two isomorphic crisp graphs. Let the isomorphism of the crisp graphs as a mapping $h:G\to H$ where there exist bijective mapping on their vertex sets $h:V(G)\to V(H)$ defined by:

$$h(v_1)=v_1^{'},\ h(v_2)=v_2^{'},\ \ldots,\ h(v_{n-1})=v_{n-1}^{'},\ h(v_n)=v_n^{'} \tag{1.1}$$

Such that the edges joining the vertices in G correspond to the edges joining the corresponding vertices in H, thus,

$$(h(v_1),h(v_2))=(v_1^{'},v_2^{'}),\ldots,(h(v_{n-1}),h(v_n))=(v_{n-1}^{'},v_n^{'}) \tag{1.2}$$

Since every crisp graph is a fuzzy graph [9], crisp graphs $G=(V(G),E(G))$ and $H=(V(H),E(H))$ can be considered as fuzzy graphs $G=(\sigma,\mu)$ and $H=(\sigma^{'},\mu^{'})$ which consist of vertices and edges with membership function of crisp value $\{0,1\}\subset[0,1]$ denoted by:

$$\sigma:V(G)\to\{0,1\}$$
$$\sigma^{'}:V(H)\to\{0,1\}$$
$$\mu:V(G)\times V(G)\to\{0,1\}$$
$$\mu^{'}:V(H)\times V(H)\to\{0,1\}$$

Such that,

$$\sigma(v_1),\ \sigma(v_2),\ \ldots,\ \sigma(v_{n-1}),\ \sigma(v_n)\in\{0,1\}$$
$$\sigma^{'}(v_1^{'}),\ \sigma^{'}(v_2^{'}),\ \ldots,\ \sigma^{'}(v_{n-1}^{'}),\ \sigma^{'}(v_n^{'})\in\{0,1\}$$
$$\mu(v_1,v_2),\ \ldots,\ \mu(v_{n-1},v_n)\in\{0,1\}$$
$$\mu^{'}(v_1^{'},v_2^{'}),\ldots,\ \mu^{'}(v_{n-1}^{'},v_n^{'})\in\{0,1\}$$

for all $v_1,v_2,...,v_{n-1},v_n\in V(G)$ and $v_1^{'},v_2^{'},...,v_{n-1}^{'},v_n^{'}\in V(H)$

Hence, by (1.1) and (1.2), and since the membership functions have the same crisp values of $\{0,1\}$ we can determine that:

$$\sigma(v_1)=\sigma^{'}(v_1^{'})=\sigma^{'}(h(v_1))\in\{0,1\}$$
$$\sigma(v_2)=\sigma^{'}(v_2^{'})=\sigma^{'}(h(v_2))\in\{0,1\}$$
$$\vdots$$
$$\sigma(v_{n-1})=\sigma^{'}(v_{n-1}^{'})=\sigma^{'}(h(v_{n-1}))\in\{0,1\}$$
$$\sigma(v_n)=\sigma^{'}(v_n^{'})=\sigma^{'}(h(v_n))\in\{0,1\}$$

and

$$\mu(v_1,v_2)=\mu^{'}(v_1^{'},v_2^{'})=\mu^{'}(h(v_1),h(v_2))\in\{0,1\}$$
$$\vdots$$
$$\mu(v_{n-1},v_n)=\mu^{'}(v_{n-1}^{'},v_n^{'})=\mu^{'}(h(v_{n-1}),h(v_n))\in\{0,1\}$$

for all $v_1,v_2,...,v_{n-1},v_n\in V(G)$ and $v_1^{'},v_2^{'},...,v_{n-1}^{'},v_n^{'}\in V(H)$

Therefore, isomorphism of crisp graphs G and H also satisfies the definition of isomorphism of fuzzy graph in [13] whereby isomorphism $h:G\to H$ is a bijective map $h:V(G)\to V(H)$ such that:

$$\sigma(v_1)=\sigma^{'}(h(v_1)) \text{ for all}$$
$$\mu(v_1,v_2)=\mu^{'}(h(v_1),h(v_2)) \text{ for all } v_1,v_2,...,v_{n-1},v_n\in V(G)$$
$$v_1^{'},v_2^{'},...,v_{n-1}^{'},v_n^{'}\in V(H)$$

Since the membership functions of crisp graphs always equal to crisp value of $\{0,1\}\subset[0,1]$ and the isomorphism of crisp graphs preserves the bijective relation between both graphs, therefore every crisp isomorphism of two graphs is a fuzzy isomorphism. ∎

Example 9

Consider crisp graphs $G=(V(G),E(G))$ and $H=(V(H),E(H))$ with respective vertex set and edge set as follows:

$V(G)=\{x,y,z\}$ and $E(G)=\{(x,y),(y,z),(x,z)\}$

$V(H)=\{a,b,c\}$ and $E(H)=\{(a,b),(b,c),(a,c)\}$

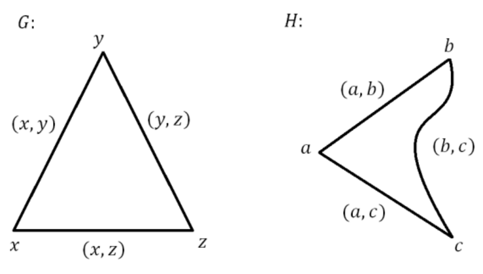

Fig. 12 Crisp graphs G and H

There is one to one correspondence between the vertices of G and H, $x\leftrightarrow a, y\leftrightarrow b, z\leftrightarrow c$, such that the number of edges joining the vertices in G is equal to the number of edges joining the corresponding vertices in H, $|E(G)|=|E(H)|=3$. Thus crisp graphs G and H are isomorphic.

Moreover, since every crisp graph is a fuzzy graph with crisp value of $\{0,1\}$, thus $G=(V(G),E(G))$ and $H=(V(H),E(H))$ can be considered as fuzzy graphs $G=(\sigma,\mu)$ and $H=(\sigma^{'},\mu^{'})$ which consist of vertices and edges with membership function of crisp value $\{0,1\}\subset[0,1]$. Let the membership function defined by

$$\sigma(x)=\sigma(y)=\sigma(z)=1$$
$$\mu(x,y)=\mu(y,z)=\mu(x,z)=1$$
$$\sigma^{'}(a)=\sigma^{'}(b)=\sigma^{'}(c)=1$$
$$\mu^{'}(a,b)=\mu^{'}(b,c)=\mu^{'}(a,c)=1$$

Thus by definition of isomorphism of fuzzy graph, let the bijective mapping of the vertex set $h:V(G)\to V(H)$ and by (1.1) and (1.2),

$$h(x)=a,\quad h(y)=b,\quad h(z)=c$$
$$(h(x),h(y))=(a,b),\ (h(y),h(z))=(b,c),\quad (h(x),h(z))=(a,c).$$

Furthermore it also satisfies

$$\sigma(x)=\sigma^{'}(h(x))=\sigma^{'}(a)=1$$
$$\sigma(y)=\sigma^{'}(h(y))=\sigma^{'}(b)=1$$
$$\sigma(z)=\sigma^{'}(h(z))=\sigma^{'}(c)=1$$
$$\mu(x,y)=\mu^{'}(h(x),h(y))=\mu^{'}(a,b)=1$$
$$\mu(y,z)=\mu^{'}(h(y),h(z))=\mu^{'}(b,c)=1$$
$$\mu(x,z)=\mu^{'}(h(x),h(z))=\mu^{'}(a,c)=1$$

Thus isomorphism of crisp graphs G and H is also isomorphism of fuzzy graphs since it also satisfies Definition 11.

Since G and H are isomorphic fuzzy graphs, hence by Theorem 1 and 2,

the orders of isomorphic fuzzy graphs G and H are the same:

$$\sigma(x)+\sigma(y)+\sigma(z)=3$$
$$\sigma^{'}(a)+\sigma^{'}(b)+\sigma^{'}(c)=3.$$

The sizes of isomorphic fuzzy graphs G and H are the same,

$$\mu(x,y)+\mu(y,z)+\mu(x,z)=3$$
$$\mu^{'}(a,b)+\mu^{'}(b,c)+\mu^{'}(a,c)=3.$$

The degrees of their vertices are preserved,

$$d(x)=\mu(x,y)+\mu(x,z)=1+1=2$$
$$d(a)=\mu^{'}(a,b)+\mu^{'}(a,c)=1+1=2$$
$$\therefore d(x)=d(a)=2$$
$$d(y)=\mu(x,y)+\mu(y,z)=1+1=2$$
$$d(b)=\mu^{'}(b,c)+\mu^{'}(a,b)=1+1=2$$
$$\therefore d(y)=d(b)=2$$
$$d(z)=\mu(x,z)+\mu(y,z)=1+1=2$$
$$d(c)=\mu^{'}(a,c)+\mu^{'}(b,c)=1+1=2$$
$$\therefore d(z)=d(c)=2$$

CONCLUSION

The taxonomy of fuzzy graph is reviewed in this paper. Some findings on fuzzy graph are presented whereby fuzzy graph introduced by Yeh and Bang [8] is proven to be a special case of fuzzy graph by Rosenfeld [7]. In addition, two descriptions of isomorphic crisp graph are proven to be equivalent and every crisp isomorphism is proven to be a fuzzy isomorphism. These new findings and taxonomy of fuzzy graphs are presented to provide underlying knowledge of fuzzy graph concept for further development purposes.

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