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RESEARCH ARTICLE

Multiscale boundary element method for Poisson equation

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Abstract

This paper applied the multiscale boundary element method for the numerical solution of the Poisson equation. The multiscale technique coupling with boundary element method will be used to solve the problem of Poisson equation efficiently and faster. Numerical example is given to illustrate the efficiency of the propose method. The solution of proposed method will be compared with boundary element method and the former method show less iteration in computation.

***Keywords*:**Poisson equation, boundary element method, multiscale technique

INTRODUCTION

In the past, solving problems numerically often meant a great deal of programming and numerical problems. The solution of Poisson equation is a well-known problem in many fields of science and engineering. Several types of numerical methods exist, each with their advantages and disadvantages. Poisson equation is a second order partial differential equation of elliptic type with broad utility in mechanical engineering and theoretical physics. It is a generalization of Poisson equation, which is also frequently seen in physics. The equation is named after the French mathematician, geometer, and physicist Siméon Denis Poisson. This is often written as:

$$\nabla^2 u + f(x, y) = 0, \tag{1}$$

∇^2 is called Laplace operator, and u and $f(x, y)$ are real or complex-valued functions on a manifold. Usually, $f(x, y)$ is given and u is sought. There are many methods in solving the numerical computation of the Poisson equation such as the Finite Element Method (FEM) and the Boundary Element Method (BEM). The FEM does have similarities to the BEM in that it does use elements and nodes, but on the boundaries only. The FEM is a method of dividing a physical system to be analyzed into smaller pieces while the BEM is derived through the discretization of an integral equation. In the BEM, the discretization is done only at the boundary, and this will result in more efficient computation and easier to be used compared with the FEM (Liu, 2009).

This paper applied the multiscale boundary element method for the numerical solution of the Poisson equation. BEM has been widely used to solve the numerical problems, as it offers an excellent accuracy, efficient in modelling, an independent numerical method and easy mesh generation. This brings about the many advantages for the BEM. However, it suffers from well-known drawbacks with regard to the computational efficiency, since the conventional BEM leads to a linear system of equations with dense coefficient matrix (Liu, 2009). Moreover, it requires the knowledge of a suitable fundamental solution of differential equation. Problems with inhomogeneities or nonlinear differential equations are not accessible by pure BEM. To overcome this problem, we study on the application of multiscale boundary element method for the numerical solution of the Poisson equation with the help of Fortran. Solving the problem of Poisson equation using BEM is more slower since heavily use the numerical integration. Therefore we apply the Multiscale Boundary Element Method that will be able to solve the problems efficiently and fast.

MULTISCALE BOUNDARY ELEMENT METHOD

Multiscale technique

Multiscale modeling or multiscale mathematics is the field of solving problems which have important features at multiple scales of time and/or space. The past studies have demonstrated that all scale-born complexities can be effectively overcome, or drastically reduced by multiscale (multi-resolution, multilevel, multigrid, etc.) algorithms. Often, a combination of several multiscale approaches can benefit one particular problem in many different ways (Barth, 2001). An example of a combination of multiscale approachesis by Silvan-Cardenas and Wang, 2006 who has investigated using the multiscale Hermite transform as an approach to separate terrain elevations from feature heights.

In this paper, we attempt to make use of the conjugate gradient and interpolation as a multiscale technique coupling with BEM. For positive defined quadratic function of the form :

$$f(x) = \frac{1}{2} x^T Q x + b^T x + c, \tag{2}$$

where x is the unknown vector, b is the known right-hand-side vectorand c is a real number. $x, b \in \Re^n, Q = Q^T \rangle 0$, the gradient vectors $\{g^k\}$ are mutually orthogonal. That is,

$$\left(g^k\right)^T g^i = 0, \quad \text{for } i \neq k. \tag{3}$$

Moreover, the search direction vectors are mutually Q-conjugate (Joshi and Moudgalya, 2004). In other words,

$$\left(d^k\right)^T Q d^i = 0, \quad \text{for } i \neq k. \tag{4}$$

The basic conjugate gradient method which is designed using quadratic function is given as below,

Algorithm 1

Step 1: Set $k = 0$ select the initial point x^0.

Step 2: $g^0 = \nabla f(x^0)$. If $g^0 = 0$, stop; go to step 9: else, set $d^0 = -g^0$

Step 3: $\alpha^k = \dfrac{d^{k^T} g^k}{Q d^{k^T} d^k}$.

Step 4: $x^{k+1} = x^k + \alpha^k d^k$.

Step 5: $g^{k+1} = \nabla f(x^{k+1})$. If $g^{k+1} = 0$, stop; go to step 9.

Step 6: $\beta^k = \dfrac{g^{(k+1)^T} Q d^k}{d^{k^T} Q d^k}$.

Step 7: $d^{k+1} = -g^{k+1} + \beta^k d^k$.

Step 8: Set $k = k + 1$; go to step 3.

Step 9: End [7].

This method was first proposed for quadratic function and is developed into a method for the general functions (Chong and Zak, 2001). In this paper, we use piecewise Newton interpolation. This interpolation is for getting values at positions in between the data points. The points are simply joined by straight line segments. Each segment that bounded by two data points can be interpolated independently.

Boundary Element Method

Boundary Element Method (BEM) has emerged as a powerful alternative to finite elements particularly in cases where better accuracy is required due to problems such as stress concentration or where the domain extends to infinity. In the BEM is an important numerical technique, a method of great efficiency. BEM is a general numerical method for solving boundary-value or initial-value problems formulated by using of the Boundary Integral Equation. The BEM mesh much easier to generate for three dimensional problems or infinite domain problems using the dimension reduction in Boundary Integral Equation formulations.

Consider the following Poisson equation governing the potential field ϕ in domain V (either 2D or 3D, finite or infinite) and S is the boundary of the domain:

$$\nabla^2 \phi + f = 0, \quad \text{in } V \tag{5}$$

where f is a known function in domain V. Firstly must form an integral equation from the Poisson equation by using a weighted integral equation:

$$\int_V \left(\nabla^2 + f\right) w = 0. \tag{6}$$

The fundamental solution $G(x, y)$ of a particular equation is the weighting function that is used in the boundary element formulation of that equation. The fundamental solution for potential problems satisfies:

$$\nabla^2 G(x, y) + \delta(x, y) = 0, \ \forall x, y \in \Re^2 / \Re^3, \tag{7}$$

in which the derivatives are taken atpoint y, and $\Re^2$ and $\Re^3$ indicate the full 2D and 3D spaces, respectively. While the Dirac δ function $\delta(x, y)$ represents a unit source at the source point x, and $G(x, y)$ represents the response at the field point y that is due to that source.

The Dirac δ function $\delta(x, y)$ in 2D and 3D has following sifting properties:

$$\int_V f(y)\delta(x, y) dV(y) = \begin{cases} f(x), \text{ if } x \in V \\ 0, \text{ if } x \notin V \cup S' \end{cases} \tag{8}$$

$$\int_V f(y) \frac{\partial \delta(x, y)}{\partial x_i} dV(y) = \begin{cases} -\dfrac{\partial f(x)}{\partial x_i}, \text{ if } x \in V \\ 0, \text{ if } x \notin V \cup S. \end{cases} \tag{9}$$

The fundamental solution $G(x, y)$ is given by:

$$G(x, y) = \begin{cases} \dfrac{1}{2\pi} \log\left(\dfrac{1}{r}\right), \text{ for 2D}, \\ \dfrac{1}{4\pi r}, \text{ for 3D}, \end{cases} \tag{10}$$

where r is the distance between the source point x and field point y, and its normal derivative is:

$$F(x, y) \equiv \frac{\partial G(x, y)}{\partial n(y)} = \begin{cases} -\dfrac{1}{2\pi} \dfrac{\partial r}{\partial y_k} n_k(y), \text{ for 2D}, \\ -\dfrac{1}{4\pi r^2} \dfrac{\partial r}{\partial y_k} n_k(y), \text{ for 3D}, \end{cases} \tag{11}$$

Then use the Green-Gauss theorem. The multi-dimensional equivalent of integration by parts is the Green-Gauss theorem:

$$\int_V \left[u \nabla^2 v - v \nabla^2 u\right] dV = \int_S \left[u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right] dS, \tag{12}$$

for any two continuous functions u and v, where n is the component of the outward normal.

Let $v(y) = \phi(y)$, which satisfies Equation (5), and $u(y) = G(x, y)$, which satisfies Equation (7). From Equation (12), we have:

$$\int_V \left[G(x, y)\nabla^2 \phi(y) - \phi(y)\nabla^2 G(x, y)\right] dV(y)$$
$$= \int_S \left[G(x, y) \frac{\partial \phi(y)}{\partial n(y)} - \phi(y) \frac{\partial G(x, y)}{\partial n(y)}\right] dS(y). \tag{13}$$

Applying Equations (5), (7) and (8), we obtain:

$$\phi(x) = \int_S \left[G(x,y)q(y) - F(x,y)\phi(y)\right] dS(y)$$
$$+ \int_V G(x,y) f(y) dV(y), \ \forall x \in V, \tag{14}$$

where $q = \dfrac{\partial \phi}{\partial n}$.

Equation (14) is the representation integral of the solution ϕ inside the domain V for Equation (5). Once the boundary values of both ϕ and q are unknown on S, if needed, Equation (14) can be applied to calculate ϕ everywhere in V. This is the boundary integral equation generally used as a starting point for boundary elements.

NUMERICAL EXAMPLE

Numerical example in 2D are presented in this section to demonstrate the efficiency and fast of the multiscale boundary element method for the numerical computation of the Poisson equation compared with boundary element method. All computations were done using Fortran compiler.

Average error, $\overline{E}$ between mesh will be compared. The formula of average error:

$$\overline{E} = \frac{\int E dx}{\int 1 dx} \tag{15}$$

Consider the following the following Poisson equation governing the potential field ϕ in domain and S is the boundary of the domain:

$$\nabla^2 \phi + f = 0, \ \text{ in } V.$$

The boundary conditions to be considered are:

$$\phi = \overline{\phi}, \text{ on } S_\phi \quad \text{(Dirichlet boundary condition)}$$
$$q \equiv \frac{\partial \phi}{\partial n} = \overline{q}, \text{ on } S_q \quad \text{(Neumann boundary condition)}$$

in which the over bar indicates the prescribed value for the function. $S_\phi \cup S_q = S$ is the boundary of the domain and n is the outward normal of the boundary S. Figure 1 shows the boundary conditions and Figure 2 show the mesh of the graph that we will discretize the region in bigger mesh which produce small size by using multiscale technique.

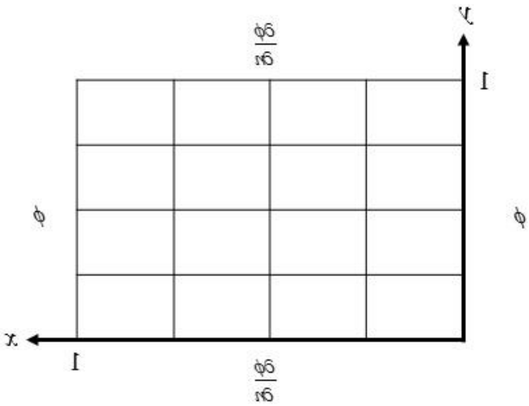

Fig. 1 Dirichlet and Neumann boundary conditions

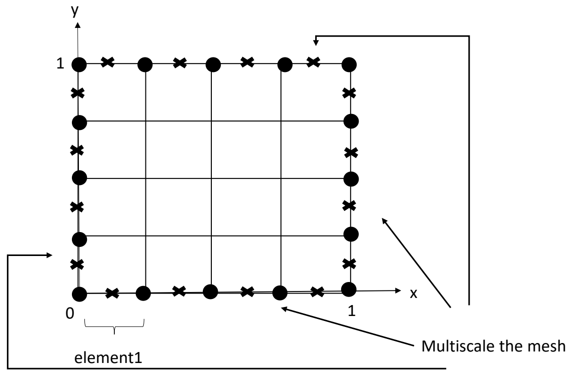

Fig. 2 Mesh of the problem.

Example:

$\nabla^2 \phi = 2 + 6y, \ \text{ in } V.$

Figure 3 shows the boundary conditions

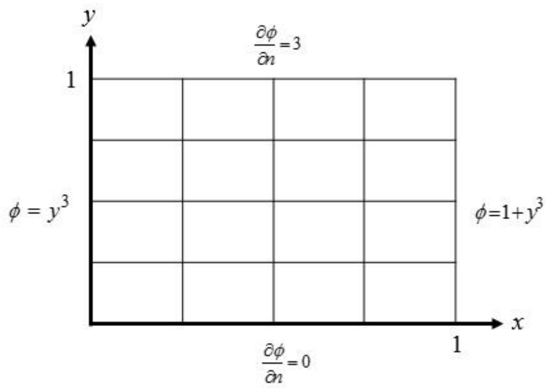

Fig. 3 Boundary conditions.

Results and discussion

Table 1 and 2 shows the iterationsbetween $32 \times 32, 64 \times 64$ and 128×128 sizes based on two methods and Table 3 shows the average error between mesh.

Table 1 Iteration count for Multiscale Boundary Element Method.

n size	No. of Iterations
32×32	115
64×64	224
128×128	414

Table 2 Iteration count for Boundary Element Method.

n size	No. of Iterations
32×32	139
64×64	250
128×128	424

Table 3 Average error between mesh.

n size	Average error, $\overline{E}$
4×4	0.03570
32×32	0.00370
64×64	0.00190
128×128	0.00097

The multiscale boundary element method for solving Poisson problem is presented in this paper. A multiscale technique approach, using a combination of the conjugate gradient and interpolation can significantly improve the conditioning of the boundary element method systems of equations and thus can facilitate faster convergence when the multiscale boundary element method is applied.

Based on Table 1 and Table 2, the number of iteration was compared. To compute the result 32×32 size by using the multiscale boundary element method, we need to compute 4×4 size first and then the result is used to compute the solution of 32×32 size. Therefore, the total number of iterations to compute 32×32 size solution is 115 iteration. However, the number of iteration needed to obtain the 32×32 size by boundary element method is 139 iteration. Moreover, the number of iterations for 64×64 and 128×128 sizes by using the multiscale boundary element method is less than boundary element method. Clearly, the multiscale boundary element method compute the solution faster than boundary element method.

Based on Table 3, the average error between mesh was compared. The average error state that the solution is close to the exact solution when the mesh larger. Then $32 \times 32, 64 \times 64, 128 \times 128$ elements is more efficient than initial 4×4 element.The numerical example are presented that clearly demonstrate the effiency of the developed the multiscale boundary element method for solving the Poisson Problem.

CONCLUSION

Based on numerical results, we conclude that the multiscale boundary element method is faster compared with boundary element method. This paper is expected to establish a numerical library for the solution of the numerical computation of Poisson equation.the proposed method can be used as a reference for the future studies in many fields of science and engineering. On the other hand, the numerical results obtained will serve as reference and can be used for validation purposes against other (future) experimental and numerical results. More research need to be done to improve the boundary element method. Wide spread applications of the boundary element method for solving large-scale engineering problems may not be far away.

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