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RESEARCH ARTICLE

The adjacency matrix of the conjugate graph of some metacyclic 2-groups

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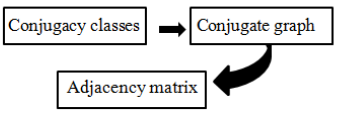
Graphical abstract



Abstract

Let G be a metacyclic 2-group and Γ_G^{conj} is the conjugate graph of G. The vertices of Γ_G^{conj} are non-central elements in which two vertices are adjacent if they are conjugate. The adjacency matrix of Γ_G^{conj} is a matrix $A = [a_{ij}]$ consisting of $0's$ and $1's$ in which the entry a_{ij} is 1 if there is an edge between the i^{th} and j^{th} vertices and 0 otherwise. In this paper, the adjacency matrix of a conjugate graph of some metacyclic 2-groups is presented.

Keywords: Adjacency matrix, conjugacy class, conjugate graph, metacyclic group.

INTRODUCTION

This section provides some basic concepts in group theory and graph theory that will be used throughout this paper.

Recently, groups are all-pervasive in mathematics. They are widely used in many branches of physical sciences. In one sense, this is to be expected because groups are quite often formed when an operation like multiplication or composition is applied to a set or system [1]. The following are some definitions on group theory.

Definition 1.1 [2] Cyclic Group
A group G is called cyclic if, for some $a \in G$, every element $x \in G$ is of the form a^n, where n is some integer. The element a is then called a generator of G.

Definition 1.2 [3] Metacyclic Group
A group is metacyclic if it has a cyclic normal subgroup H such that G/H is cyclic.

The following is the definition of the conjugate between two elements of a group G.

Definition 1.3 [4] Conjugate
Let a and b be two elements in finite group G, then a and b are called conjugate if there exist an element g in G such that $gag^{-1} = b$.

Definition 1.4 [5] Conjugacy Class
Let $x \in G$. The conjugacy class of x is the set $cl(x) = \{axa^{-1} | a \in G\}$.

Graph theory begins with very simple geometric ideas and has many powerful applications. Hence, graph theory has a very wide range applications in engineering, in computer sciences, in physical, social, and biological sciences, in linguistics and in numerous other areas [2]. The following are some basic concepts on graph theory.

Definition 1.3 [6] Graph
A graph Γ consists of a finite set V of objects called vertices, a finite set E of objects called edges, and a function f that assigns to each edge a subset $\{v_0, v_1\}$ where v_0 and v_1 are vertices. The vertices and edges are denoted as $V(\Gamma)$ and $E(\Gamma)$, respectively.

Definition 1.4 [6] Subgraph
A subgraph of graph Γ is a graph whose vertices and edges are subset of the vertices and edges of Γ.

In 2005, Beuerle [7] separated the classification into two parts, namely for the non-abelian metacyclic p-groups of class two and class three. Based on [7], the metacyclic p-groups of nilpotency class two are then partitioned into two families of non-isomorphic p-groups stated as follows :

1. $G \cong < a, b: a^{2^\alpha} = 1, b^2 = 1, [a, b] = a^{2^{\alpha-\lambda}} >$, where $\alpha, \beta, \lambda \in \mathbb{N}, \alpha \geq 2\lambda, \beta \geq \lambda \geq 1$.
2. $G \cong < a, b: a^4 = 1, b^2 = [a, b] = a^{-2} >$, a quaternion group of order 8, Q_8.

In this paper, we use group 1 in which $3 \leq \alpha \leq 4, \lambda = 1$ and group 2. Hence, the scope of this study are as follows :

1. $G_1 \cong < a, b: a^8 = 1, b^2 = 1, [a, b] = a^4 >$.

2. $G_2 \cong < a, b: a^{16} = 1, b^2 = 1, [a, b] = a^8 >$.
3. $G_3 \cong < a, b: a^4 = 1, b^2 = [a, b] = a^{-2} >$, a quaternion group of order 8, Q_8.

In this section, some works related to conjugate graph are stated.

In 2012, Erfanian and Toule [8] introduced a new graph called the conjugate graph denoted as Γ_G^{conj} in this paper. The vertices of this graph are non-central elements in which two vertices are adjacent if they are conjugate.

In 2016, the conjugacy classes of G_1, G_2 and G_3 have been determined by Bilhikmah [9]. The following are some theorems that have been proved by [9].

Theorem 2.1 [9] Let G_1 be a metacyclic 2-group of order 16, $G_1 \cong < a, b: a^8 = 1, b^2 = 1, [a, b] = a^4 >$. Then, the number of conjugacy classes, $K(G_1) = 10$.

The elements of group G_1 are stated as follows :

$G_1 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$.

The conjugacy classes of G_1 are found as follows :

$cl(e) = \{e\}, cl(a) = \{a, a^5\}, cl(a^2) = \{a^2\}$,
$cl(a^3) = \{a^3, a^7\}, cl(a^4) = \{a^4\}, cl(a^6) = \{a^6\}$,
$cl(b) = \{b, a^4b\}, cl(ab) = \{ab, a^5b\}, cl(a^2b) = \{a^2b, a^6b\}$,
$cl(a^3b) = \{a^3b, a^7b\}$.

From the lists of conjugacy classes above, we know that $|Z(G_1)| = 4$.

Theorem 2.2 [9] Let G_2 be a metacyclic 2-group of order at most 32, $G_2 \cong < a, b: a^{16} = 1, b^2 = 1, [a, b] = a^8 >$. Then, the number of conjugacy classes, $K(G_2) = 20$.

The elements of group G_2 are stated as follows :

$G_2 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, a^8b, a^9b, a^{10}b, a^{11}b, a^{12}b, a^{13}b, a^{14}b, a^{15}b\}$.

The conjugacy classes of G_2 are found as follows :

$cl(e) = \{e\}, cl(a) = \{a, a^9\}, cl(a^2) = \{a^2\}$,
$cl(a^3) = \{a^3, a^{11}\}, cl(a^4) = \{a^4\}, cl(a^5) = \{a^5, a^{13}\}$,
$cl(a^6) = \{a^6\}, cl(a^7) = \{a^7, a^{15}\}, cl(a^8) = \{a^8\}$,
$cl(a^{10}) = \{a^{10}\}, cl(a^{12}) = \{a^{12}\}, cl(a^{14}) = \{a^{14}\}$,
$cl(b) = \{b, a^8b\}, cl(ab) = \{ab, a^9b\}, cl(a^2b) = \{a^2b, a^{10}b\}$,
$cl(a^3b) = \{a^3b, a^{11}b\}, cl(a^4b) = \{a^4b, a^{12}b\}$,
$cl(a^5b) = \{a^5b, a^{13}b\}, cl(a^6b) = \{a^6b, a^{14}b\}$,
$cl(a^7b) = \{a^7b, a^{15}b\}$.

From the lists of conjugacy classes above, we know that $|Z(G_2)| = 8$.

Theorem 2.3 [9] Let G_3 be a quaternion group of order 8, $G_3 \cong < a, b: a^4 = 1, b^2 = [a, b] = a^{-2} >$. Then, the number of conjugacy classes of G_3, $K(G_3) = 5$.

The elements of group G_3 are stated as follows :

$G_3 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

The conjugacy classes of G_1 are found as follows :

$cl(e) = \{e\}, cl(a) = \{a, a^3\}, cl(a^2) = \{a^2\}$,
$cl(b) = \{b, a^2b\}, cl(ab) = \{ab, a^3b\}$.

From the lists of conjugacy classes above, we know that $|Z(G_3)| = 4$.

RESULTS AND DISCUSSION

In this section, the results on conjugacy classes of G_1, G_2 and G_3 are applied to conjugate graph.

Theorem 3.1 Let G_1 be a metacyclic 2-group. Then, the adjacency matrix of the conjugate graph is stated as follows :

$$A(\Gamma_{G_1}^{conj}) = \begin{bmatrix}
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0
\end{bmatrix}.$$

Proof : Based on Theorem 2.1, there are 16 elements in G_1. Since the number of vertices for $\Gamma_{G_1}^{conj}$ is $|G| - |Z(G)|$, thus the number of vertices of $\Gamma_{G_1}^{conj}$ is 12. There are six non-central conjugacy classes of order 2. Therefore, $\Gamma_{G_1}^{conj}$ consists of six components of K_2. The conjugate graph of G_1 is represented in Figure 3.1.

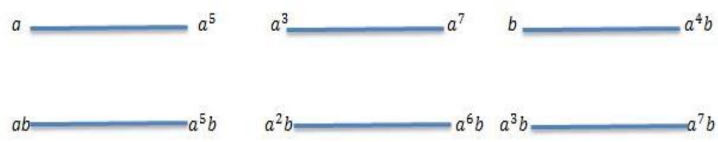


Figure 3.1 : The Conjugate Graph of G_1

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise.

Theorem 3.2 Let G_2 be a metacyclic 2-group. Then, the adjacency matrix of the conjugate graph is stated as follows :

$$A(\Gamma_{G_2}^{conj}) = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0
\end{bmatrix}.$$

Proof : Based on the Theorem 2.2, there are 32 elements in G_2 and 8 of them are the central elements. Thus, the number of vertices of the conjugate graph of G_2 is 24. The conjugate graph, $\Gamma_{G_2}^{conj}$ consists of 12 components of K_2. The conjugate graph of G_2 is represented in Figure 3.2.

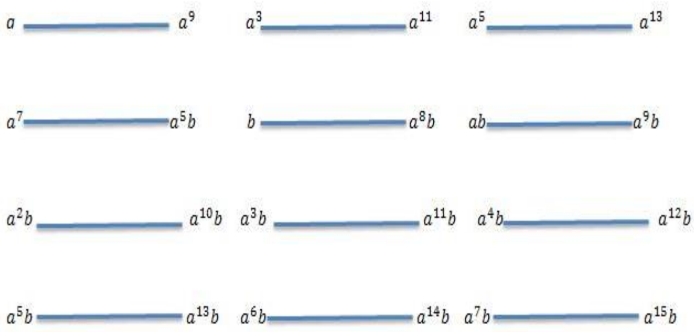

Figure 3.2 : The Conjugate Graph of G_2

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise.

Theorem 3.3 Let G_3 be a metacyclic 2-group. Then, the adjacency matrix of the conjugate graph is stated as follows :

$$A(\Gamma_{G_3}^{conj}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof : Based on Theorem 2.3, the quaternion group G_3 has 8 elements where two are the central elements. Thus the number of vertices of G_3 is 6 and since two vertices are adjacent if they are conjugate, then $\Gamma_{G_3}^{conj}$ consists of three components of K_2 as represented in Figure 3.3.

Figure 3.3 : The Conjugate Graph of G_3

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise.

CONCLUSION

In this paper, the conjugate graphs of groups G_1, G_2 and G_3 are determined based on the conjugacy classes of each elements of the groups. For group G_1, there are 12 non-central elements which make the vertices for the conjugate graph of G_1. Meanwhile, group G_2 consists of 32 elements where 8 are central. Hence, $\Gamma_{G_2}^{conj}$ has 24 vertices. For group G_3, it has 6 non-central elements. Therefore, $\Gamma_{G_3}^{conj}$ has 6 vertices. It is shown that the conjugate graphs of all three groups is a union of several components of K_2.

The adjacency matrix of each group has been stated in previous section where if the non-central elements are connected to each other in the conjugate graph, the element of matrix is 1 and 0 for otherwise.

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