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RESEARCH ARTICLE

Derivation of stochastic Taylor methods for stochastic differential equations

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Abstract

This paper demonstrates a derivation of stochastic Taylor methods for stochastic differential equations (SDEs). The stochastic Taylor series is extended and truncated at certain terms to achieve the order of convergence of stochatsic Taylor methods for SDEs. The systematic derivation of the expansion of stochastic Taylor series formula is presented. Numerical methods of Euler, Milstein scheme and stochastic Taylor methods of order 2.0 are proposed.

Keywords: *Stochastic Taylor method, stochastic differential equations, stochastic Taylor expansion*

INTRODUCTION

Most of the biological and physical phenomena happened can be modeled mathematically by using stochastic differential equations (SDEs). Previously, ordinary differential equations (ODEs) often used to described those systems. Due to the fact that most of the physical phenomenon are influenced by the environmental noise and disturbances, therefore SDEs are used to represent the systems. SDEs were perturbed randomly by the unpredictable movement of white noise, which then contribute to the difficulty in finding the analytical solutions of SDEs. This leads to the development of numerical method in order to approximate the solution for SDEs. Ito (1951) was the first whom introduces SDEs and became a catalyst for the development in the SDEs field (Kloeden & Platen, 1992). Recently, numerical methods for solving SDEs undergo an intensive research. There are many researchers whom discussed the topic on the numerical computations of SDEs such as Oksendal (1985), Kloeden & Platen (1992), Milstein (1995), Burrage & Burrage (1999), Carletti (2006) and Norhayati (2010).

The earliest numerical method for SDEs named as an Euler-Maruyama method had been introduced by Maruyama in 1950 as stated by Carletti (2006). This simplest stochastic numerical approximation has a strong order of convergence 0.5 for multiplicative noise as well as 1.0 for additive noise (Burrage, 1999). This low order of convergence will result in inaccurate numerical computations. Consequently, the more efficient numerical method needed and one of the best approach is to use the truncation of stochastic Taylor series expansion (Burrage, 1999). Next contribution was made by Milstein (1974) when he proposed a so-called Milstein scheme from the truncation of stochastic Taylor series. Milstein scheme is then proved to have the strong order of convergence 1.0 (Milstein, 1995).

In this paper, we present a derivation of stochastic Taylor expansion up to 2.0 order of convergence. The numerical example will be carried out and the result will be discussed.

DERIVATION OF STOCHASTIC TAYLOR EXPANSIONS FOR SDEs

In this section, we present a systematic derivation of stochastic Taylor expansion for SDEs. The derivation of strong Taylor expansion which approximates up to 2.0 order of convergence were set up.

Stochastic Taylor expansion for autonomous SDEs

Let consider SDEs

$$dx(t) = f(x(t))dt + g(x(t))dW(t) \tag{1}$$

where $t \in [t_0, T]$. Equation (1) can be expressed in the integral form

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(x(t))dt + \int_{t_n}^{t_{n+1}} g(x(t))dW(t) \tag{2}$$

For simplicity the following notation is introduced

$$f = f(x(t_n))$$

$$g = g(x(t_n))$$

$$f' = \frac{\partial f}{\partial x_{t_n}}(x(t_n))$$

$$g' = \frac{\partial g}{\partial x_{t_n}}(x(t_n))$$

$$f'' = \frac{\partial^2 f}{\partial x^2_{t_n}}(x(t_n))$$

$$g'' = \frac{\partial^2 g}{\partial x^2_{t_n}}(x(t_n))$$

$$g''' = \frac{\partial^3 f}{\partial x^3_{t_n}}(x(t_n))$$

The derivation of stochastic Taylor expansion for SDE is done by replacing the integrals in (2) with their corresponding Taylor expansions about x_{t_n}, where $x_{t_n} = x(t_n)$. The methods considered here are based on Rao (1974). By applying Taylor expansion for drift function f and diffusion function g we therefore obtain

$$\begin{aligned} f(x(t)) = f + (x(t)-x(t_n)) f' + 1/2(x(t) \\ -x(t_n))^2 f'' + O_f(|x(t)-x(t_n)|^3) \end{aligned} \tag{3}$$

$$\begin{aligned} g(x(t)) = g + (x(t)-x(t_n))g' + 1/2(x(t)-x(t_n))^2 g'' \\ +1/6(x(t)-x(t_n))^3 g''' + O_g(|x(t)-x(t_n)|^4) \end{aligned} \tag{4}$$

where $O_f(|x(t)-x(t_n)|^3)$ and $O_g(|x(t)-x(t_n)|^4)$ representing higher order term for drift and diffusion functions respectively. Substituting (3) and (4) into (2) hence

$$\begin{aligned} x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} \{ & f + (x(t)-x(t_n)) f' \\ & +1/2(x(t)-x(t_n))^2 f'' \\ & +O_f(|x(t)-x(t_n)|^3)\} dt \\ & +\int_{t_n}^{t_{n+1}} \{ g + (x(t)-x(t_n)) g' \\ & +1/2(x(t)-x(t_n))^2 g'' \\ & +1/6(x(t)-x(t_n))^3 g''' \\ & +O_g(|x(t)-x(t_n)|^4)\} dW(t) \end{aligned} \tag{5}$$

Generally, equation (5) can be written as

$$\begin{aligned} x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} \sum_{j=0}^{M_1} \left\{ \frac{1}{j!} \left[(x(t)-x(t_n)) \frac{\partial}{\partial z_0} \right]^j \times f(z_0) \right\} dt \\ + \int_{t_n}^{t_{n+1}} \sum_{j=0}^{M_2} \left(\frac{g^{(j)}}{j!} (x(t)-x(t_n))^j \right) dW(t) \end{aligned} \tag{6}$$

We then expand and rearrange (5) in order to get higher order numerical schemes to the solution of SDEs. We then obtain

$$\begin{aligned} x(t_{n+1}) = x(t_n) + f\int_{t_n}^{t_{n+1}} dt + g\int_{t_n}^{t_{n+1}} dW(t) \\ + f'\int_{t_n}^{t_{n+1}} (x(t)-x(t_n)) dt \\ + g'\int_{t_n}^{t_{n+1}} (x(t)-x(t_n)) dW(t) \\ +1/2 f''\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))^2 dt \\ +1/2 g''\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))^2 dW(t) \\ +1/6 g'''\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))^3 dW(t) \\ +\int_{t_n}^{t_{n+1}} O_f(|x(t)-x(t_n)|^3) dt \\ +\int_{t_n}^{t_{n+1}} O_g(|x(t)-x(t_n)|^4) dW(t) \end{aligned} \tag{7}$$

From equation (7) we identify the multiple integrals together with their elementary functions as follow

(a) $f\int_{t_n}^{t_{n+1}} dt = f(t_{n+1}-t_n) = f\cdot\Delta$

(b) $g\int_{t_n}^{t_{n+1}} dW(t) = g(W(t_{n+1})-W(t_n)) = g\cdot\Delta W(t)$

(c) $f'\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))dt$

(d) $g'\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))dW(t)$

(e) $1/2 f''\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))^2 dt$

(f) $1/2 g''\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))^2 dW(t)$

(g) $1/6 g'''\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))^3 dW(t)$

To solve (c), $x(t)-x(t_n)$ is expanded in the form of Taylor series which lead to the following representation

$$\begin{aligned} x(t)-x(t_n) = f\int_{t_n}^{t} dt + g\int_{t_n}^{t} dW(t) \\ + f'\int_{t_n}^{t} (x(t)-x(t_n))dt \\ + g'\int_{t_n}^{t} (x(t)-x(t_n))dW(t) \\ +1/2 f''\int_{t_n}^{t} (x(t)-x(t_n))^2 dt \\ +1/2 g''\int_{t_n}^{t} (x(t)-x(t_n))^2 dW(t) \\ +1/6 f'''\int_{t_n}^{t} (x(t)-x(t_n))^3 dt \\ + higher\ order\ term \end{aligned} \tag{8}$$

Then we have

$$\begin{aligned} f'\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))dt = f' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} ds dt \\ + f' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s) dt \\ + f' f'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (x(s)-x(s_n)) ds dt \\ + f' g'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (x(s)-x(s_n)) dW(s) dt \\ + higher\ order\ term \end{aligned} \tag{9}$$

The term $x(s)-x(s_n)$ in (9) is written as a lower order Taylor method as

$$\begin{aligned} x(s)-x(s_n) &= f(x(s_n))(s-s_n) \\ &\quad + g(x(s_n))(W(s)-W(s_n)) \\ &= f(s-s_n) + g(s-s_n) \end{aligned} \tag{10}$$

Substituting (10) into (9), the following is obtained

$$\begin{aligned} & f'\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))dt \\ &= f' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} ds dt \\ &+ f' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s) dt \\ &+ f' g' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (W(s)-W(s_n)) dW(s) dt \\ &+ higher\ order\ term \end{aligned} \tag{11}$$

With the same technique as in (c), the term (d) can be expanded as

$$\begin{aligned}
& g'\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))\,dW(t) \\
&= g'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} ds\,dW(t) \\
&+g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)\,dW(t) \\
&+g'f'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(x(s)-x(s_n))ds\,dW(t) \\
&+g'g'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(x(s)-x(s_n))dW(s)\,dW(t) \\
&+1/2g'g''\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(x(s)-x(s_n))^2 dW(s)\,dW(t) \\
&+higher\ order\ term
\end{aligned} \tag{12}$$

Substituting lower order form of Taylor series to replace $x(s)-x(s_n)$, equation (12) then can be written as

$$\begin{aligned}
& g'\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))\,dW(t) \\
&= g'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} ds\,dW(t) \\
&+g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)\,dW(t) \\
&+g'f'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))ds\,dW(t) \\
&+g'g'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(s-s_n)dW(s)\,dW(t) \\
&+g'g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))dW(s)\,dW(t) \\
&+g'g'g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{t}^{u}(W(u)-W(u_n))dW(u)dW(s)\,dW(t) \\
&+1/2g'g''gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))^2 dW(s)\,dW(t)
\end{aligned} \tag{13}$$

In order to solve (e), we employed the same technique as previously mention. We obtain

$$\begin{aligned}
& 1/2f''\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^2\,dt \\
&=1/2f''\int_{t_n}^{t_{n+1}}(f(t-t_n)+g(W(t)-W(t_n))^2\,dt \\
&=1/2f''ff\int_{t_n}^{t_{n+1}}(t-t_n)^2\,dt \\
&+1/2f''fg\int_{t_n}^{t_{n+1}}(t-t_n)(W(t)-W(t_n))\,dt \\
&+1/2f''gg\int_{t_n}^{t_{n+1}}(W(t)-W(t_n))^2 dt
\end{aligned} \tag{14}$$

By solving (f), we have

$$\begin{aligned}
& 1/2g''\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^2\,dW(t) \\
&=1/2g''f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(x(t)-x(t_n))\,dt\,dW(t) \\
&+1/2g''g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(x(t)-x(t_n))\,dW(t)\,dW(t) \\
&+1/2g''f'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(x(t)-x(t_n))^2\,dt\,dW(t) \\
&+1/2g''g'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(x(t)-x(t_n))^2\,dW(t)\,dW(t)
\end{aligned} \tag{15}$$

By applying lower form of Taylor expansion, equation (15) can be expanded as

$$\begin{aligned}
& 1/2g''\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^2\,dW(t) \\
&=1/2g''fg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))\,dt\,dW(t) \\
&+1/2g''gf\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(t-t_n)\,dW(t)\,dW(t) \\
&+1/2g''gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))\,dW(t)\,dW(t) \\
&+1/2g''g'gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))^2\,dW(t)\,dW(t)
\end{aligned} \tag{16}$$

By expanding the Taylor expansion to solve (g) and then substituting the lower order form of stochastic taylor expansion, we then have

$$\begin{aligned}
& 1/6g'''\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^3\,dW(t) \\
&=1/6g'''ggg\int_{t_n}^{t_{n+1}}(W(t)-W(t_n))^3\,dW(t)
\end{aligned} \tag{17}$$

Adding together (a)-(g), the stochastic Taylor expansion for SDE is

$$\begin{aligned}
x(t_{n+1}) &= x(t_n)+f\int_{t_n}^{t_{n+1}}dt+g\int_{t_n}^{t_{n+1}}dW(t)+f'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dsdt \\
&+f'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dW(s)dt \\
&+f'g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))dW(s)dt \\
&+g'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}ds\,dW(t) \\
&+g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dW(s)\,dW(t) \\
&+g'f'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))ds\,dW(t) \\
&+g'g'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(s-s_n)dW(s)\,dW(t) \\
&+g'g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))dW(s)\,dW(t) \\
&+g'g'g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{t}^{u}(W(u)-W(u_n))dW(u)dW(s)\,dW(t) \\
&+1/2g'g''gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))^2 dW(s)\,dW(t) \\
&+1/2f''gg\int_{t_n}^{t_{n+1}}(W(t)-W(t_n))^2 dt \\
&+1/2g''fg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))\,dt\,dW(t) \\
&+1/2g''gf\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(t-t_n)\,dW(t)\,dW(t) \\
&+1/2g''gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))\,dW(t)\,dW(t) \\
&+1/2g''g'gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))^2\,dW(t)\,dW(t) \\
&+1/6g'''ggg\int_{t_n}^{t_{n+1}}(W(t)-W(t_n))^3\,dW(t) \\
&+\int_{t_n}^{t_{n+1}}O_f(|x(t)-x(t_n)|^3)dt \\
&+\int_{t_n}^{t_{n+1}}O_g(|x(t)-x(t_n)|^4)dW(t)
\end{aligned} \tag{18}$$

Strong Taylor methods for SDEs

Stochastic Taylor expansions are the key to the development of numerical methods for SDEs. Stochastic Taylor expansion can be truncating up to certain order of convergence so that the numerical schemes of SDEs can be developed. The numerical scheme of Euler-Maruyama method with 0.5 order of convergence is obtained by truncating the stochastic Taylor expansion (18) at third terms. Hence we have

$$x(t_n+1)=x(t_n)+f\int_{t_n}^{t_{n+1}}dt+g\int_{t_n}^{t_{n+1}}dW(t)+R1 \qquad (19)$$

where $\int_{t_n}^{t}dt=\Delta$ and $\int_{t_n}^{t}dW(t)=\Delta W(t)$.

Then, Euler-Maruyama scheme is given as

$$x(t_{n+1})=x(t_n)+f\cdot\Delta+g\cdot(\Delta W(t))+R1 \qquad (20)$$

where $R1$ is remainder term. Expanding the Taylor expansion and truncating the equations (18) at fifth term, we then obtain a Milstein scheme which can be presented as

$$\begin{aligned}x(t_n+1)&=x(t_n)+f\int_{t_n}^{t_{n+1}}dt+g\int_{t_n}^{t_{n+1}}dW(t)\\&+g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dW(s)dW(t)+R2\end{aligned} \qquad (21)$$

In addition, the scheme of strong Taylor method with the order of convergence 2.0 can as well be presented as

$$\begin{aligned}x(t_{n+1})&=x(t_n)+f\int_{t_n}^{t_{n+1}}dt+g\int_{t_n}^{t_{n+1}}dW(t)+f'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dsdt\\
&+f'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dW(s)dt\\
&+f'g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))dW(s)dt\\
&+g'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}ds\,dW(t)\\
&+g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dW(s)\,dW(t)\\
&+g'f'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))ds\,dW(t)\\
&+g'g'f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(s-s_n)dW(s)\,dW(t)\\
&+g'g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))dW(s)\,dW(t)\\
&+g'g'g'g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{t}^{u}(W(u)-W(u_n))dW(u)dW(s)\,dW(t)\\
&+1/2g'g''gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))^2dW(s)\,dW(t)\\
&+1/2f''gg\int_{t_n}^{t_{n+1}}(W(t)-W(t_n))^2dt\\
&+1/2g''fg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))\,dt\,dW(t)\\
&+1/2g''gf\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(t-t_n)\,dW(t)\,dW(t)\\
&+1/2g''gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))\,dW(t)\,dW(t)\\
&+1/2g''g'gg\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(t)-W(t_n))^2\,dW(t)\,dW(t)\\
&+1/6g'''ggg\int_{t_n}^{t_{n+1}}(W(t)-W(t_n))^3\,dW(t)\\
&+R3\end{aligned} \qquad (22)$$

where $R3$ is the remainder term. Numerical scheme above improved the rate of convergence with order of convergence 2.0. The numerical approximations to the multiple stratonovich integrals have been introduced as in Table 1 below in a way to simplify the above schemes.

Table 1 Numerical approximations to the statonovich integrals.

Stratonovich Integrals	Numerical Approximations
$\int_{t_n}^{t_{n+1}}dt$	Δ
$\int_{t_n}^{t_{n+1}}dW(t)$	ΔW
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dsdt$	$\frac{\Delta^2}{2}$
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dW(s)dt$	ΔZ
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}ds\,dW(t)$	$(\Delta W)\Delta-\Delta Z$
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dW(s)\,dW(t)$	$\frac{1}{2}(\Delta W)^2$
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))dW(s)\,dW(t)$	$\frac{1}{6}(\Delta W)^3$
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))ds\,dW(t)$	$\frac{1}{6}\Delta((\Delta W)^2-\Delta)$
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(s-s_n)dW(s)\,dW(t)$	$\frac{1}{6}\Delta(\Delta W)^2+\frac{1}{12}\Delta^2$
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(s_n))dW(s)dt$	$\frac{1}{6}\Delta(\Delta W)^2+\frac{1}{12}\Delta^2$
$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\int_{t}^{u}(W(u)-W(u_n))$ $dW(u)dW(s)dW(t)$	$\frac{1}{24}(\Delta W)^4$

where ΔW and ΔZ are random variables which are normally distributed with $\Delta W\sim N(0,\Delta)$, $\Delta Z\sim N(0,\frac{1}{3}\Delta^3)$ and $E(\Delta W\Delta Z)=\frac{1}{2}\Delta^2$. Therefore, by applying the numerical approximation to the stratonovich integrals, Milstein scheme in (21) can be presented as

$$\begin{aligned}x(t_{n+1})&=x(t_n)+f\cdot\Delta+g\cdot(\Delta W(t))\\&+1/2g'g(\Delta W(t))^2+R2\end{aligned} \qquad (23)$$

Both Euler-Maruyama scheme and Milstein scheme have order of convergence 1.0. Then we have strong Taylor method with order of convergence 2.0 as follow:

$$
\begin{aligned}
x(t_{n+1}) = x(t_n) &+ f\cdot\Delta(t) \\
&+ g\cdot(\Delta W(t)) + f' f\cdot(\Delta^2/2) \\
&+ f' g\cdot(\Delta Z(t)) \\
&+ f' g' g\cdot((\Delta(t)(\Delta W(t))^2/6)+((\Delta(t))^2/12)) \\
&+ g' f\cdot((\Delta W(t))\Delta(t)-\Delta Z(t)) \\
&+ g' g\cdot((\Delta W(t))^2/2) \\
&+ g' f' g\cdot(((\Delta W(t))^2\Delta(t)-(\Delta(t))^2)/6) \\
&+ g' g' f\cdot((\Delta(t)(\Delta W(t))^2/6)+((\Delta(t))^2/12)) \\
&+ g' g' g\cdot((\Delta W(t))^3/6) \\
&+ g' g' g' g\cdot((\Delta W(t))^4/24) \\
&+ 1/2g' g'' gg\cdot((\Delta W(t))^4/24) \\
&+ 1/2 f'' gg\cdot((\Delta(t)(\Delta W(t))^2/6)+((\Delta(t))^2/12)) \\
&+ 1/2g'' fg\cdot(((\Delta W(t))^2\Delta(t)-(\Delta(t))^2)/6) \\
&+ 1/2g'' gf\cdot((\Delta(t)(\Delta W(t))^2/6)+((\Delta(t))^2/12)) \\
&+ 1/2g'' gg\cdot((\Delta W(t))^3/6) \\
&+ 1/2g'' g' gg\cdot((\Delta W(t))^4/24) \\
&+ 1/6g''' ggg\cdot((\Delta W(t))^4/24) + R3
\end{aligned} \tag{24}
$$

RESULTS AND DISCUSSION

The following linear SDE taken from Küchler(2000) is used to compare the performance of 2.0 strong Taylor method, Euler-Maruyama and Milstein schemes. Let us consider

$$dX(t) = aX(t)dt + bX(t)dW(t), \qquad t\in[0,T] \tag{25}$$

The exact solution of (25) is

$$X(t) = \Phi_{t,t_0}(X_0) \tag{26}$$

where $\Phi_{t,t_0} = \exp((a-b/2)(t-t_0)+b(W(t)-W(t_0)))$

In this numerical example, we have set the coefficient as

$$a=-2.0,\ \ b=0.5,\ \ T=2.0,\ \ X(0)=1.0 \quad \text{and} \quad \Delta=0.01$$

We simulate 200 sample paths of strong solution SDE via Euler-Maruyama, Milstein and stochastic Taylor method with order 2.0. The mean-square error between numerical solution and exact solution has been calculated. The perfomance of Euler-Maruyama, Milstein and Taylor method with order 2.0 are presented in Figure 1, Figure 2 and Figure 3 respectively.

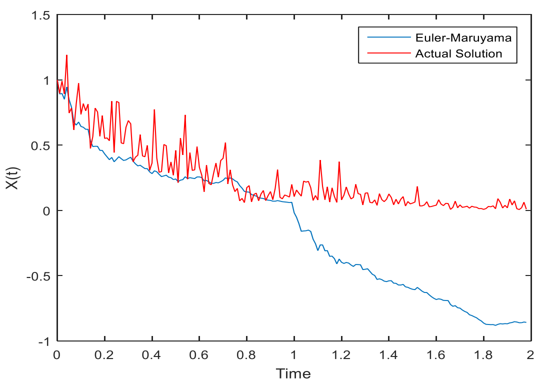

Figure 1 Strong approximations of SDEs via Euler-Maruyama.

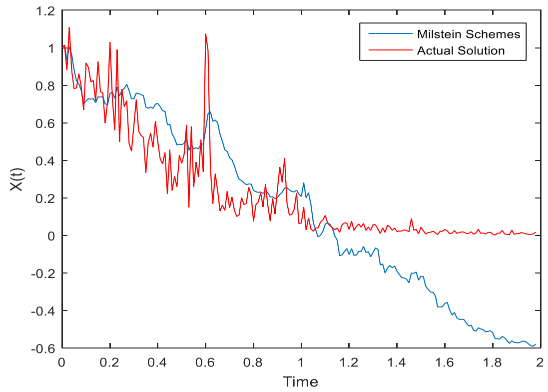

Figure 2 Strong approximation of SDEs via Milstein scheme.

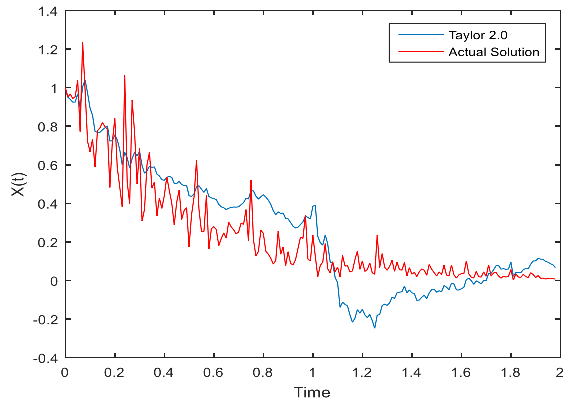

Figure 3 Strong approximation of SDEs via Taylor method order 2.0.

Based on Figure 1, Figure 2 and Figure 3, it shows that Figure 3 gives a better performance than the result presented in Figure 1 and Figure 2. Table 2 shows mean-square error between numerical solution and the exact solution for SDE. Stochastic Taylor method with order of convergence 2.0 gives a better solution compare to Euler-Maruyama and Milstein methods.

Table 2 Mean-Square Error of Numerical and Exact Solution.

Numerical Scheme	Euler-Maruyama	Milstein	2.0 Stochastic Taylor Method
MSE	0.247702	0.084398	0.024104

CONCLUSION

In this paper, the derivation of higher order numerical schemes to the solution of SDEs truncated from the stochastic Taylor expansions have been presented. It shows that the lower order numerical schemes show low accuracy to solve the system of SDEs.

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