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RESEARCH ARTICLE

On the dominating number, independent number and the regularity of the relative co-prime graph of a group

Norarida Abd Rhani [a, b], Nor Muhainiah Mohd Ali [a,*], Nor Haniza Sarmin [a], Ahmad Erfanian [c]

[a] Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Malaysia
[b] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia
[c] Department of Pure Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

* Corresponding author: normuhainiah@utm.my

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Abstract

Let *H* be a subgroup of a finite group *G*. The co-prime graph of a group is defined as a graph whose vertices are elements of *G* and two distinct vertices are adjacent if and only if the greatest common divisor of order of *x* and *y* is equal to one. This concept has been extended to the relative co-prime graph of a group with respect to a subgroup *H*, which is defined as a graph whose vertices are elements of *G* and two distinct vertices *x* and *y* are joined by an edge if and only if their orders are co-prime and any of *x* or *y* is in *H*. Some properties of graph such as the dominating number, degree of a dominating set of order one and independent number are obtained. Lastly, the regularity of the relative co-prime graph of a group is found.

Keywords: Co-prime graph, relative co-prime graph, dominating number, independent number, regular graph

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INTRODUCTION

Let G be a finite group with identity element e and H be a subgroup of G. We consider the simple graphs which are undirected, with no loops or multiple edges. For any graph Γ, the sets of the vertices and the edges of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively.

The graph related to the prime elements of G has been discussed since 1981. Williams was the first person who introduced the prime graph of a group where the vertices are the primes dividing the order of G and two vertices p and q are joined by an edge if and only if G contains an element of order pq. In his paper, he proved that for any finite simple group, $t(G) \leq 6$, where t is the number of connected components in G. The significance of the prime graphs of finite groups can be found in Iiyori and Yamaki (1993) and Williams (1981).

Motivated by this research, Ma *et al.* (2014) introduced the co-prime graph of a group. The definition of the graph is stated as follows:

Definition 1 (Ma *et al.*, 2014) : The co-prime graph, denoted as Γ_G is a graph whose vertices are element of G and two distinct vertices x and y are adjacent if and only if $(|x|,|y|)=1$.

In this paper, they found some properties of the co-prime graph such as the diameter, planarity and clique number. Besides, some groups whose co-prime graphs are complete, planar, a star or regular are found. This graph was also studied by Dorbidi (2016) where some results were obtained. He found that for the co-prime graph, the clique number and chromatic number are equal. Also, a complete answer to the question which is asked in Ma *et al.* (2014) is provided which is "Is it possible to characterize all finite groups having the property that $Aut(\Gamma_G) \cong G$?"

In 2015, Rajkumar and Devi introduced the co-prime graph of subgroups of a group, which is defined as a graph whose vertex set is the set of all proper subgroups of G and two distinct vertices are adjacent if and only if the order of the corresponding subgroups are co-prime. They studied the relation between the properties of algebraic of a group and theoretic of a graph of its co-prime graph.

Recently in Abd Rhani *et al.* (2017), we introduced the relative co-prime graph of a group with respect to H, which is defined as a graph having the set of all elements of G as it vertices and two distinct vertices are adjacent if and only if their orders are co-prime and any of element is in H. Therefore, in this paper we determined some graph properties such as dominating number, degree of e and independent number by using those definition. Besides, we characterized a group and an order of subgroup whose the relative co-prime graph is regular.

PRELIMINARIES

In this section, we provide the definition of the relative co-prime graph of G and some basic properties in graph theory that are used throughout this study.

Definition 2 (Abd Rhani *et al.*, 2017) : The relative co-prime graph of a group G with respect to a subgroup H, denoted as $\Gamma_{copr}(H,G)$, is a graph whose vertices are elements of G and two distinct vertices x and y are adjacent if and only if $(|x|,|y|)=1$ and any of element x or y is in H.

Definition 3 (Bondy and Murty, 1982) : A non-empty set S of $V(\Gamma)$ is called an independent set of Γ if there is no adjacent between two elements of S in Γ. Thus the independent number is the number of vertices in maximum independent set and it is denoted by $\alpha(\Gamma)$.

In 2013, Tamizh Chelvan and Sattanathan found the lower bound of independent number of the power graph where the vertices are all elements of G and two distinct vertices x and y are adjacent if and only if either $x^i = y$ or $y^j = x$, where $2 \le i, j \le n.$ $2 \le i, j \le n$, denoted as $\Gamma_p(G)$.

The next definition is the dominating set and dominating number of a graph.

Definition 4 (Bondy and Murty, 1982) : The dominating set $X \subseteq V(\Gamma)$ is a set where for each v outside X, there exist $x \in X$ such that v is adjacent to x. The minimum size of X is called the dominating number and it is denoted by $\gamma(\Gamma)$.

Definition 5 (Godsil and Royle, 2001) : A graph is called a regular graph if all of its vertices have the same sizes.

In 2015, Doostabadi *et al.* proved that the reduced power graph of a group G is regular if and only if G is a cyclic p-group or $\exp(G) = p$ for some prime number p. The reduced power graph of a finite group G is obtained when we remove the identity element from the vertex set.

Definition 6 (Harary, 1965) : The degree of x, denoted by deg(x), is the number of edges incident with x.

Proposition 1 (Tamizh Chelvan and Sattanathan, 2013) : Let G be a finite group with n elements and $Z(G)$ be its center. If $\deg(x) = n-1$ in $\Gamma_p(G)$, then $x \in Z(G)$.

The main objectives of this paper is to find the dominating number, independent number and the regularity of the relative co-prime graph of a group. These works are discussed in the next section.

RESULTS AND DISCUSSION

The following theorem shows the dominating number of the relative co-prime graph of a group and the degree of a unique dominating set of order one.

Theorem 1:
Let H be a subgroup of a finite group G. Let $\{e\}$ be a unique dominating set of order 1 of $\Gamma_{copr}(H,G)$. Then $\gamma(\Gamma_{copr}(H,G)) = 1$ and $\deg(e) = |G| - 1$.

Proof. Let H be a subgroup of a group G. Assume that, $\{e\}$ is a unique dominating set of $\Gamma_{copr}(H,G)$. Now, we prove for uniqueness of $\{e\}$ Suppose $\{x\}$ is another dominating set of $\Gamma_{copr}(H,G)$ where $x \ne e$. Then, there is an edge between x and all elements in G. Let $|x| = n$. By Proposition 1, we have x is an element of center $Z(G)$. If not, then there exists an element $g \in G$ such that $g^{-1}xg$ and x are conjugate and $g^{-1}xg \ne x$. Thus, $|g^{-1}xg| = |x|$. Hence $g^{-1}xg$ and x are not adjacent which is a contradiction. Let y be other element in G such that $y \ne x$ and $y \ne e$. Take note $xy = yx$. Then $|xy| = |x||y|$. Thus, $(|xy|, |x|) = |x|$. Then, xy and x are not joined by an edge, which is a contradiction. Thus, $\{e\}$ is a dominating set of $\Gamma_{copr}(H,G)$ with $\gamma(\Gamma_{copr}(H,G)) = 1$

Since e is the only element of order 1 in $\Gamma_{copr}(H,G)$, then e is adjacent to all elements in G. So $\deg(e) = |G| - 1$. □

The next theorem shows an independent number of the relative co-prime graph of a group.

Theorem 2
Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ and H be a subgroup of order $p_1^{\beta_1} p_2^{\beta_2} \ldots p_k^{\beta_k}$, where α_i and β_i are positive integer, p_i's are prime number, $k \le r$ and $0 \le \beta_i \le \alpha_i$. Let A_i be an independent set. Then $\alpha(\Gamma_{copr}(H,G)) = \max\{|A_i|, |G \setminus H|\}$ where $1 \le i \le r$.

Proof. Suppose that $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ and $|H| = p_1^{\beta_1} p_2^{\beta_2} \ldots p_k^{\beta_k}$ where α_i and β_i are positive integer, p_i's are prime number, $k \le r$ and $0 \le \beta_i \le \alpha_i$. By Cauchy's Theorem, G contains an element of order p_i for $1 \le i \le r$.

Suppose that $p_i = Syl_{p_i}(G)$ and $A_i = p_i \cap H$. So, each A_i is the independent set. In G, we also have $G \setminus H$ as an independent set. Thus, $\alpha(\Gamma_{copr}(H,G)) = \max\{|A_i|, |G \setminus H|\}$. □

The next theorem give the characterization of a group and a subgroup whose the relative co-prime graph of a group is regular.

Theorem 3
Let H be a subgroup of a finite group G. Then $\Gamma_{copr}(H,G)$ is regular if and only if $G \cong \mathbb{Z}_2$ and $|H| = 1$.

Proof. By Theorem 1, we have $\deg(e) = |G| - 1$. Thus, for every element $s \in G$, we should have $\deg(s) = |G| - 1$. If $|G \setminus H| \ge 2$, then there exist elements $x_1, x_2 \in G \setminus H$. So, $\deg(x_1) < |G| - 2$ which is a contradiction. Hence $|G \setminus H| = 1$.

By Lagrange's Theorem, we have $|H| \mid |G|$. Hence, we have $|G| = k|H|$. So, $|G| - |H| = 1$ which implies that $k|H| - |H| = 1$. Then, $(k-1)|H| = 1$. So, we have $k = 2$ and $|H| = 1$. Hence, $|G| = 2$, $|H| = 1$. Therefore, $G \cong \mathbb{Z}_2$ and $|H| = 1$.

The converse is trivial. □

CONCLUSION

In this paper, some properties of the relative co-prime graph of a group such as dominating number, degree of a dominating set of order one and independent number are found. It is found that, the dominating number is equal to one and $\deg(e) = |G| - 1$. Furthermore, the independent number is equal to $\max\{|A_i|, |G \setminus H|\}$. Lastly, it is found that, the relative co-prime graph of a group is regular if and only if $G \cong \mathbb{Z}_2$ and $|H| = 1$.

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