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Estimation of 2- and 3-parameter Burr Type XII distributions using EM algorithm

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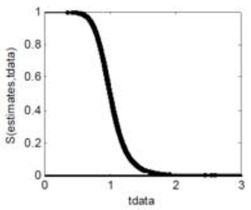
GRAPHICAL ABSTRACT



ABSTRACT

The Burr Type XII distribution is one of the systems of continuous distributions and is widely known because the distribution includes the characteristics of various well known distributions such as Weibull and gamma distributions. Maximum likelihood estimation (MLE) has been a common method in estimating model parameters. An alternative method that is the expectation-maximization (EM) algorithm is presented in this paper to estimate the two- and three-parameter Burr Type XII distributions in the presence of complete and censored data. Furthermore, simulation study is conducted to compare the efficiency and accuracy of MLE and EM algorithm approaches. The result indicates that EM algorithm is more efficient and accurate than those estimates obtained via MLE approach.

Keywords: Burr Type XII Distribution, Maximum Likelihood Estimation, EM Algorithm, Censored Data

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1. INTRODUCTION

The word ‘Burr’ was introduced by [1] in 1942 when a few forms of cumulative distribution function were suggested to fit the data. Burr Type XII was used in many fields because of the potentiality in practical situations. In his guide to the Dagum distribution (2007), [2] stated that in economics, the Burr Type XII distribution is known as the Singh-Maddala distribution. Burr Type XII distribution has at least two unknown parameter. [3] derived the probability density function of a six-parameter generalized Burr Type XII distribution and obtained cumulative distribution function meanwhile [4] introduced properties of seven parameters Burr Type XII distribution.

In statistics, estimation process is very important to find the approximate value of unknown parameters. Several methods have been used to estimate the parameters of the Burr Type XII distribution such as Maximum Likelihood Estimation (MLE), Least Squares Estimation (LSE) and Bayesian Estimation. The parameter estimation process involves the presence of complete and censored data.

Nowadays, researchers used censored data to estimate the parameter. Different mechanisms can lead to different type of censored data such as right-censored, left-censored and randomly censored. The usage of censored data gives the possibility to compute the estimation method to fit a model to censored data.

This paper aims to estimate the parameters*c* and *k* for 2-parameter and *c*, k and *s* for 3-parameter of Burr Type XII distribution with complete and censored data using two methods which include MLE and EM algorithm approaches. Then, the estimated parameters from both methods were compared based on bias and mean square error (MSE).

2. ESTIMATION OF PARAMETER IN BURR TYPE XII DISTRIBUTION

2.1 2-Parameter Burr Type XII distribution

The probability density function (*pdf*) of the standard Burr Type XII distribution for 2-parameter is written in the form of

$$f(t;c,k) = ckt^{c-1}(1+t^c)^{-(k+1)} \tag{1}$$

and cumulative distribution function (*cdf*)

$$F(t;c,k) = 1-(1+t^c)^{-k} \tag{2}$$

for $t \geq 0$ with both *c* and *k* are shape parameters.

2.1.1 Maximum likelihood estimation (MLE)

MLE is the most popular method of parameter estimation. The likelihood function of the censored data is given by

$$L=\prod_{i=1}^{r} f(t_i)\prod_{j=r+1}^{n}[1-F(t_j)]=\prod_{i=1}^{r}\frac{ckt_i^{c-1}}{(1+t_i^{\,c})^{k+1}}\prod_{j=r+1}^{n}\frac{1}{(1+t_j^{\,c})^{k}} \quad (3)$$

There are r failures at times $t_1, t_2, \ldots, t_r$.
With $f(t)$ and $F(t)$ given by equations (1) and (2) respectively, the logarithm of the likelihood function becomes

$$\ln L = r\ln c + r\ln k + (c-1)\sum_{i=1}^{r}\ln t_i - (k+1)\sum_{i=1}^{r}\ln(1+t_i^{\,c}) - k\sum_{j=r+1}^{n}\ln(1+t_j^{\,c}) \quad (4)$$

Differentiate equation (4) with respect to c and k and equate each result to zero, then

$$\frac{d\ln L}{dc}=\frac{r}{c}+\sum_{i=1}^{r}\ln t_i-(k+1)\sum_{i=1}^{r}\frac{t_i^{\,c}\ln t_i}{1+t_i^{\,c}}-k\sum_{j=r+1}^{n}\frac{t_j^{\,c}\ln t_j}{1+t_j^{\,c}}=0 \quad (5)$$

$$\frac{d\ln L}{dk}=\frac{r}{k}-\sum_{i=1}^{r}\ln(1+t_i^{\,c})-\sum_{j=r+1}^{n}\ln(1+t_j^{\,c})=0 \quad (6)$$

Two equations are solved simultaneously to obtain the estimates of c and k.

$$\sum_{i=1}^{r}\ln t_i=\sum_{i=1}^{r}\frac{t_i^{\,c}\ln t_i}{1+t_i^{\,c}}+\left[\left(\frac{r}{\sum_{i=1}^{r}\ln(1+t_i^{\,c})+\sum_{j=r+1}^{n}\ln(1+t_j^{\,c})}\right)\sum_{i=1}^{r}\frac{t_i^{\,c}\ln t_i}{1+t_i^{\,c}}\right]+\left[\left(\frac{r}{\sum_{i=1}^{r}\ln(1+t_i^{\,c})+\sum_{j=r+1}^{n}\ln(1+t_j^{\,c})}\right)\sum_{j=r+1}^{n}\frac{t_j^{\,c}\ln t_j}{1+t_j^{\,c}}\right]-\frac{r}{c} \quad (7)$$

$$\hat{k}=\frac{r}{\sum_{i=1}^{r}\ln(1+t_i^{\,c})+\sum_{j=r+1}^{n}\ln(1+t_j^{\,c})} \quad (8)$$

Equation (7) exhibits no explicit solutions to solve the equations analytically, and the maximization is performed through the mathematical approach that is the Newton-Raphson method to obtain the approximate solutions. According to [6], the definition is given by

$$S_1=\sum_{i=1}^{r}\ln t_i \qquad S_5=\sum_{j=r+1}^{n}\ln(1+t_j^{\,c})$$

$$S_2=\sum_{i=1}^{r}\ln(1+t_i^{\,c}) \qquad S_6=\sum_{j=r+1}^{n}\frac{t_j^{\,c}\ln t_j}{1+t_j^{\,c}}$$

$$S_3=\sum_{i=1}^{r}\frac{t_i^{\,c}\ln t_i}{1+t_i^{\,c}} \qquad S_7=\sum_{j=r+1}^{n}\frac{(t_j^{\,c}\ln^2 t_j)(1+t_j^{\,c})-(t_j^{\,c}\ln t_j)^2}{(1+t_j^{\,c})^2}$$

$$S_4=\sum_{i=1}^{r}\frac{(t_i^{\,c}\ln^2 t_i)(1+t_i^{\,c})-(t_i^{\,c}\ln t_i)^2}{(1+t_i^{\,c})^2}$$

Then, equation (7) and (8) are solved as follows.

$$\hat{c}_{i+1}=c_i+\left[\frac{\left(\frac{r}{c_i}+S_1-S_3-\frac{r}{(S_2+S_5)}(S_3+S_6)\right)}{\left(\frac{r}{c_i^{2}}+S_4-\frac{r}{(S_2+S_5)^2*(S_3+S_6)^2}+\frac{r}{(S_2+S_5)}(S_4+S_7)\right)}\right] \quad (9)$$

$$\hat{k}=\frac{r}{S_2+S_5} \quad (10)$$

2.1.2 EM Algorithm

According to [5], let $y=(y_1^{\,T},\ldots,y_n^{\,T})^T$ denotes the observed data where $y_i=(d_i,\delta_i)^T$ and $\delta_i=0$ for censored data or 1 for failure (observed) data. T_i is censored or uncensored at d_i $(i = 1,\ldots, n)$. Then, the probability density function of 2-parameter Burr Type XII distribution when given $T>d_j$ is calculated as follows

$$f(t\,|\,t>d_j)=\frac{f(t)}{1-F(d_j)}=ck(1+d_j^{\,c})^k\frac{t^{c-1}}{(1+t^c)^{k+1}},\ t>d_j \quad (11)$$

From $f(t)$ in equation (1), the complete data log-likelihood function of the Burr Type XII distribution is expressed as

$$\log(L_c(c,k))=\sum_{i=1}^{n}\log[f_c(t_i;c,k)$$
$$=n\log k+n\log c+(c-1)\sum_{i=1}^{n}(\log t_i)-(k+1)\sum_{i=1}^{n}[\log(1+t_i^{\,c})] \quad (12)$$

The Q-function of 2-parameter Burr Type XII distribution of censored data is obtained as

$$Q(\theta,\theta^{(m)})=E_{\theta^{(m)}}[\log L_c(c,k)]$$
$$=n\log k+n\log c+(c-1)\sum_{i=1}^{r}\log(d_i)-(k+1)\sum_{i=1}^{r}\log(1+d_i^{\,c})+$$
$$(c-1)\sum_{j=r+1}^{n}E_{\theta^{(m)}}(\log T_j\,|\,T_j>d_j)-(k+1)\sum_{j=r+1}^{n}E_{\theta^{(m)}}[\log(1+T_j^{\,c})\,|\,T_j>d_j] \quad (13)$$

Using numerical integral and apply Taylor series, equation (13) is solved as follows.

$$E_{\theta^{(m)}}(\log T_j\,|\,T_j>d_j)=\int_{d_i}^{\infty}\log(t_j)ck(1+d_j^{\,c})^k t_j^{\,c-1}(1+t_j^{\,c})^{(-k-1)}dt_j$$

$$= ck(1+d_j^{\,c})^k \int_{d_i}^{\infty} \frac{(\log t_j)(t_j^{\,c-1})}{(1+t_j^{\,c})^{k+1}} dt_j$$

$$E_{\theta^{(m)}}[\log(1+T_j^{\,c}) \mid T_j > d_j] \cong E_{\theta^{(m)}}[\log(1+T_j^{\,c}) \mid T_j > d_j] +$$
$$c - c^{(m)} E_{\theta^{(m)}}[\frac{T_j^{\,c}}{1+T_j^{\,c}} \log(T_j) \mid T_j > d_j] +$$
$$\frac{1}{2}(c-c^{(m)})^2 E_{\theta^{(m)}}[\frac{T_j^c}{(1+T_j^c)^2} (\log(T_j))^2 \mid T_j > d_j]$$

$$E_{\theta^{(m)}}[\log(1+T_j^{\,c}) \mid T_j > d_j] = \int_{d_j}^{\infty} \log(1+t_j^{\,c})$$
$$ck(1+d_j^{\,c}) t_j^{\,c-1}(1+t_j^{\,c})^{(-k-1)}\, dt_j$$
$$= ck(1+d_j^{\,c})^k \int_{d_j}^{\infty} \frac{[\log(1+t_j^{\,c})](t_j^{\,c-1})}{(1+t_j^{\,c})^{k+1}} dt_j$$

$$E_{\theta^{(m)}}[\frac{T_j^c}{1+T_j^c} \log(T_j) \mid T_j > d_j] = \int_{d_j}^{\infty} \frac{t_j^{\,c}}{1+t_j^{\,c}} \log(t_j)$$
$$ck(1+d_j^{\,c}) t_j^{\,c-1}(1+t_j^{\,c})^{(-k-1)}\, dt_j$$
$$= ck(1+d_j^{\,c})^k \int_{d_j}^{\infty} \frac{[\log(t_j)](t_j^{\,2c-1})}{(1+t_j^{\,c})^{k+2}} dt_j$$

$$E_{\theta^{(m)}}[\frac{T_j^c}{(1+T_j^c)^2} (\log(t_i))^2 \mid T_j > d_j] = \int_{d_j}^{\infty} \frac{t_j^{\,c}}{(1+t_j^{\,c})^2}$$
$$[\log(t_j)]^2 ck(1+d_j^{\,c}) t_j^{\,c-1}(1+t_j^{\,c})^{(-k-1)} dt_j$$
$$= ck(1+d_j^{\,c})^k \int_{d_j}^{\infty} \frac{[\log(t_j)]^2 (t_j^{\,2c-1})}{(1+t_j^{\,c})^{k+3}} dt_j$$

2.2 3-Parameter Burr Type XII distribution

The probability density function (*pdf*) of the standard Burr Type XII distribution for 3-parameter is written in the form of

$$f(t;c,k,s) = \frac{ckt^{c-1}}{s^c}[1+(\frac{t}{s})^c]^{-(k+1)} \tag{14}$$

and cumulative distribution function (*cdf*)

$$F(t;c,k,s) = 1-[1+(\frac{t}{s})^c]^{-k} \tag{15}$$

for $t \geq 0$ with both *c* and *k* are shape parameters and *s* is scale parameter
.

2.2.1 Maximum Likelihood Estimation (MLE)

The likelihood function for the censored data is given by

$$L = \prod_{i=1}^{r} f(t_i) \prod_{j=r+1}^{n} [1-F(t_j)] = \prod_{i=1}^{r} \frac{ck(\frac{t_i}{s})^{c-1}}{[1+(\frac{t_i}{s})^c]^{k+1}} \prod_{j=r+1}^{n} \frac{1}{[1+(\frac{t_j}{s})^c]^k} \tag{16}$$

There are *r* failures at times $t_1, t_2, \ldots, t_r$.
Equations (12) and (13) become

$$\ln L = r \ln ck - r \ln s + (c-1)\sum_{i=1}^{r} \ln(\frac{t_i}{s}) -$$
$$(k+1)\sum_{i=1}^{r} \ln[1+(\frac{t_i}{s})^c] - k\sum_{j=r+1}^{n} \ln[1+(\frac{t_j}{s})^c] \tag{17}$$

Differentiate equation (17) with respect to *c*, *k* and *s* and equate each result to zero, the equation becomes

$$\frac{d\ln L}{dc} = \frac{r}{c} + \sum_{i=1}^{r} \ln(\frac{t_i}{s}) - (k+1)\sum_{i=1}^{r} \frac{(\frac{t_i}{s})^c \ln(\frac{t_i}{s})}{1+(\frac{t_i}{s})^c} - k\sum_{j=r+1}^{n} \frac{(\frac{t_j}{s})^c \ln(\frac{t_j}{s})}{1+(\frac{t_j}{s})^c} = 0 \tag{18}$$

$$\frac{d\ln L}{dk} = \frac{r}{k} - \sum_{i=1}^{r} \ln[1+(\frac{t_i}{s})^c] - \sum_{j=r+1}^{n} \ln[1+(\frac{t_j}{s})^c] = 0 \tag{19}$$

$$\frac{d\ln L}{ds} = -\frac{r}{s} + (c-1)(-\frac{1}{s}) - (k+1)\sum_{i=1}^{r} \frac{c(\frac{t_i}{s})^{c-1}(-\frac{t_i}{s^c})}{[1+(\frac{t_i}{s})^c]} - k\sum_{j=r+1}^{n} \frac{c(\frac{t_j}{s})^{c-1}(-\frac{t_j}{s^c})}{[1+(\frac{t_j}{s})^c]}$$
$$= -\frac{r+c-1}{s} - (k+1)\sum_{i=1}^{r} \frac{c(\frac{t_i}{s})^{c-1}(-\frac{t_i}{s^c})}{[1+(\frac{t_i}{s})^c]} - k\sum_{j=r+1}^{n} \frac{c(\frac{t_j}{s})^{c-1}(-\frac{t_j}{s^c})}{[1+(\frac{t_j}{s})^c]} \tag{20}$$

Three equations are solved simultaneously to obtain the estimates of *c*, *k* and *s*.

$$\sum_{i=1}^{r} \ln(\frac{t_i}{s}) = \sum_{i=1}^{r} \frac{(\frac{t_i}{s})^c \ln(\frac{t_i}{s})}{1+(\frac{t_i}{s})^c} + [(\frac{r}{\sum_{i=1}^{r}\ln[1+(\frac{t_i}{s})^c] + \sum_{j=r+1}^{n}\ln[1+(\frac{t_j}{s})^c]}) \sum_{i=1}^{r} \frac{(\frac{t_i}{s})^c \ln(\frac{t_i}{s})}{1+(\frac{t_i}{s})^c}]$$
$$+[(\frac{r}{\sum_{i=1}^{r}\ln[1+(\frac{t_i}{s})^c] + \sum_{j=r+1}^{n}\ln[1+(\frac{t_j}{s})^c]}) \sum_{j=r+1}^{n} \frac{(\frac{t_j}{s})^c \ln(\frac{t_j}{s})}{1+(\frac{t_j}{s})^c}] - \frac{r}{c} \tag{21}$$

$$k = \frac{r}{\sum_{i=1}^{r}\ln[1+(\frac{t_i}{s})^c] + \sum_{j=r+1}^{n}\ln[1+(\frac{t_j}{s})^c]} \tag{22}$$

$$s = \frac{r+c-1}{(k+1)\sum_{i=1}^{r} \frac{c(\frac{t_i}{s})^{c-1}(-\frac{t_i}{s^c})}{[1+(\frac{t_i}{s})^c]} + k\sum_{j=r+1}^{n} \frac{c(\frac{t_j}{s})^{c-1}(-\frac{t_j}{s^c})}{[1+(\frac{t_j}{s})^c]}} \tag{23}$$

The Newton-Raphson method is used to obtain the approximate solutions for equation (19) as follows.

$$S_1 = \sum_{i=1}^{r} \ln(\frac{t_i}{s}) \qquad S_5 = \sum_{j=r+1}^{n} \ln[1+(\frac{t_j}{s})^c]$$

$$S_2 = \sum_{i=1}^{r} \ln[1+(\frac{t_i}{s})^c] \qquad S_6 = \sum_{j=r+1}^{n} \frac{(\frac{t_j}{s})^c \ln(\frac{t_j}{s})}{[1+(\frac{t_j}{s})^c]}$$

$$S_3 = \sum_{i=1}^{r} \frac{(\frac{t_i}{s})^c \ln(\frac{t_i}{s})}{1+(\frac{t_i}{s})^c}$$

$$S_4 = \sum_{i=1}^{r} \frac{[(\frac{t_i}{s})^c \ln^2(\frac{t_i}{s})][1+(\frac{t_i}{s})^c] - [(\frac{t_i}{s})^c \ln(\frac{t_i}{s})]^2}{[1+(\frac{t_i}{s})^c]^2}$$

$$S_7 = \sum_{j=r+1}^{n} \frac{[(\frac{t_j}{s})^c \ln^2(\frac{t_j}{s})][1+(\frac{t_j}{s})^c] - [(\frac{t_j}{s})^c \ln(\frac{t_j}{s})]^2}{[1+(\frac{t_j}{s})^c]^2}$$

Equation (21) and (22) are solved as follows

$$\hat{c}_{i+1} = c_i + [\frac{(\frac{r}{c_i} + S_1 - S_3 - \frac{n}{(S_2+S_5)}(S_3+S_6))}{(\frac{r}{c_i^2} + S_4 - \frac{n}{(S_2+S_5)^2 * (S_3+S_6)^2} + \frac{r}{(S_2+S_5)}(S_4+S_7))}] \quad (24)$$

$$\hat{k} = \frac{r}{S_2 + S_5} \quad (25)$$

2.2.2 EM Algorithm

Let $y = (y_1^T, ..., y_n^T)^T$ denote the observed data where $y_i = (d_i, \delta_i)^T$ and $\delta_i = 0$ for censored data or 1 for failure (observe) data. T_i is censored or uncensored at $d_i (i = 1, ..., n)$. Then, the probability density function of 3-parameter Burr Type XII distribution when given $T > d_j$ is calculated as follows

$$f(t \mid t > d_j) = \frac{f(t)}{1 - F(d_j)} = \frac{ck}{s}[1+(\frac{d_j}{s})^c]^k \frac{(\frac{t}{s})^{c-1}}{[1+(\frac{d_j}{s})^c]^{k+1}}, t > d_j \quad (26)$$

f (t) as given in equation (11) can be expressed as

$$\log(L_c(c,k,s)) = \sum_{i=1}^{n} \log[f_c(t_i; c,k,s)]$$
$$= n \ln ck - n \ln s + (c-1)\sum_{i=1}^{n} \ln(\frac{t_i}{s}) - (k+1)\sum_{i=1}^{n} \ln[1+(\frac{t_i}{s})^c] \quad (27)$$

The Q-function of 3-parameter Burr Type XII distribution for multiple censored data is obtained as

$$Q(\theta, \theta^{(m)}) = E_{\theta^{(m)}}[\log \ L_c(c,k,s)]$$
$$= n\log ck - n\log s + (c-1)\sum_{i=1}^{r} \log(\frac{d_i}{s}) - (k+1)\sum_{i=1}^{r} \log(1+(\frac{d_i}{s})^c) +$$
$$(c-1)\sum_{j=r+1}^{n} E_{\theta^{(m)}}[\log(\frac{T_j}{s}) \mid T_j > d_j] - (k+1)\sum_{j=r+1}^{n} E_{\theta^{(m)}}[\log(1+(\frac{T_j}{s})^c) \mid T_j > d_j] \quad (28)$$

Equation (28) is solved using numerical integral and applies Taylor series as follows.

$$E_{\theta^{(m)}}[\log(\frac{T_j}{s}) \mid T_j > d_j] = \int_{d_i}^{\infty} \log(\frac{t_j}{s})$$
$$ck[1+(\frac{d_j}{s})^c]^k (\frac{t_j}{s})^{c-1}[1+(\frac{t_j}{s})^c]^{(-k-1)} dt_j$$
$$= ck[1+(\frac{d_j}{s})^c]^k \int_{d_i}^{\infty} \frac{[\log(\frac{t_j}{s})][(\frac{t_j}{s})^{c-1}]}{[1+(\frac{t_j}{s})^c]^{k+1}} dt_j$$

$$E_{\theta^{(m)}}[\log(1+(\frac{T_j}{s})^c) \mid T_j > d_j] \cong E_{\theta^{(m)}}[\log(1+(\frac{T_j}{s})^c) \mid T_j > d_j] +$$
$$c - c^{(m)} E_{\theta^{(m)}}[\frac{(\frac{T_j}{s})^c}{1+(\frac{T_j}{s})^c} \log(\frac{T_j}{s}) \mid T_j > d_j] +$$

$$\frac{1}{2}(c - c^{(m)})^2 E_{\theta^{(m)}}[\frac{(\frac{T_j}{s})^c}{(1+(\frac{T_j}{s})^c)^2} (\log(\frac{T_i}{s}))^2 \mid T_j > d_j]$$

$$E_{\theta^{(m)}}[\log(1+(\frac{T_j}{s})^c) \mid T_j > d_j] = \int_{d_j}^{\infty} \log[1+(\frac{t_j}{s})^c]$$
$$ck[1+(\frac{d_j}{s})^c](\frac{t_j}{s})^{c-1}[1+(\frac{t_j}{s})^c]^{(-k-1)} dt_j$$
$$= ck[1+(\frac{d_j}{s})^c] \int_{d_j}^{\infty} \frac{[\log(1+(\frac{t_j}{s})^c)][(\frac{t_j}{s})^{c-1}]}{[1+(\frac{t_j}{s})^c]^{k+1}} dt_j$$

$$E_{\theta^{(m)}}[\frac{(\frac{T_j}{s})^c}{1+(\frac{T_j}{s})^c} \log(\frac{T_j}{s}) \mid T_j > d_j] = \int_{d_j}^{\infty} \frac{(\frac{t_j}{s})^c}{[1+(\frac{t_j}{s})^c]} \log(\frac{t_j}{s})$$

$$ck[1+(\frac{d_j}{s})^c](\frac{t_j}{s})^{c-1}[1+(\frac{t_j}{s})^c]^{(-k-1)}dt_j$$

$$= ck[1+(\frac{d_j}{s})^c]\int_{d_j}^{\infty}\frac{[\log(\frac{t_j}{s})][(\frac{t_j}{s})^{2c-1}]}{[1+(\frac{t_j}{s})^c]^{k+2}}dt_j$$

$$E_{\theta^{(m)}}[\frac{(\frac{T_j}{s})^c}{(1+(\frac{T_j}{s})^c)^2}(\log(\frac{T_j}{s}))^2 \,|\, T_j > d_j] = \int_{d_j}^{\infty}\frac{(\frac{t_j}{s})^c}{[1+(\frac{t_j}{s})^c]^2}[\log(\frac{t_j}{s})]^2 \times$$

$$ck[1+(\frac{d_j}{s})^c](\frac{t_j}{s})^{c-1}[1+(\frac{t_j}{s})^c]^{(-k-1)}dt_j$$

$$= ck[1+(\frac{d_j}{s})^c]\int_{d_j}^{\infty}\frac{[\log(\frac{t_j}{s})]^2[(\frac{t_j}{s})^{2c-1}]}{[1+(\frac{t_j}{s})^c]^{k+3}}dt_j$$

2.3 Efficiency and accuracy

The bias is calculated by the difference between the expected value of an estimator and the true value of the estimator in order to ascertain the accuracy of the estimators in the model. Let $\hat{\theta}$ is the estimator of the parameter, and then biasedness is calculated as

$$Bias\ (\hat{\theta}) = E(\hat{\theta}) - \theta \tag{29}$$

Mean square error (MSE) is a way of measurement through the average of the square error.

$$\begin{aligned} MSE &= E[(\hat{\theta}-\theta)^2] \\ MSE &= Var\ (\hat{\theta}) + [Bias\ (\hat{\theta})]^2 \end{aligned} \tag{30}$$

It provides better quality measurement for the estimator since it accesses on both the variances and bias term.

3.0 SIMULATIONS AND RESULTS

1000 numbers of complete sample sizes were generated and the curve showed that the simulated data were well fitted to the estimated 2-parameter Burr Type XII survival function in Figure (a). Figure (b) and (c) show that the complete simulated data undergone some amount of censoring and each of them contains 10% and 30% of the censored data respectively.

The Burr Type XII distribution will be written as Burr_{XII} (*c*, *k*) for 2-parameter and Burr_{XII} (*c*, *k*, *s*) for 3-parameter. The estimated parameter of 2-parameter Burr Type XII distribution is investigated by varying the *k* values in Table 1 and *c* values in Table 2. The discussion of the estimated parameter will be given to Burr_{XII} (1, 2) from Table 1 since the explanations were the same for other parameter values. The sample size is standardized to 200 and the result for the MLE and EM algorithm showed the estimated parameter for *c* and *k* are approximately close to the true parameter of 1 and 2 for uncensored data. With respect to the bias and MSE of parameter *c* and *k*, the EM algorithm outperforms the MLE with bias is -0.0764 for *c* parameter and 0.0317 for the *k* parameter; with MSE is 0.0058 and 0.0010 for *c* and *k* parameter respectively which is smaller than the bias and MSE of MLE with bias 0.1241 and -0.0702 for *c* and *k* parameter respectively and MSE is 0.0154 for *c* and 0.0049 for *k* parameter.

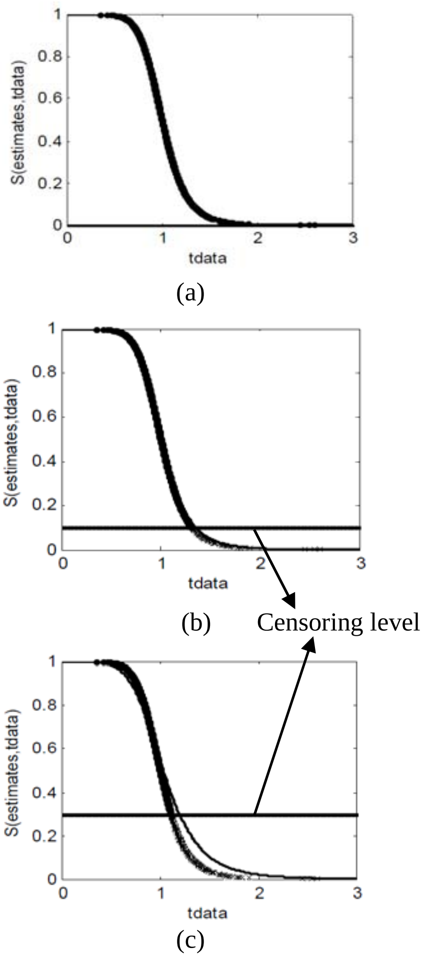

Figure 1 Curve fitting comparison of 2-parameter Burr Type XII with (a) uncensored, (b) 10% censoring and (c) 30% censoring

In the presence of 30% censoring level, the accuracy of $\hat{c}$ and $\hat{k}$ dropped with the value 1.1816 and 1.7132 respectively. Table 1 showed that when the percentage of incomplete data (censored) is increased, the value of estimated parameter *c* and *k* is also increased. However, the EM algorithm still gives the smaller values of bias and MSE than the MLE. This is also same for several values of *c* and *k*.

The estimated parameter of Burr Type XII distribution will be discussed only to the example of Burr_{XII}

(2, 5, 1) from Table 4 since the explanations were the same for other parameter values by varying the *s* values in Table 3, *k* values in Table 4 and *c* values in Table 5 for 3-parameter. The estimated parameter for *c*, *k* and *s* for complete data are approximately close to the true parameter of 2, 5 and 1 respectively. The bias of the EM algorithm is smaller than the MLE method with 0.0194 for *c* parameter, 0.0526 for *k* parameter and 0.0025 for the *s* parameter and the MSEs 0.0061, 0.0045 and 0.0001 for *c*, *k* and *s* parameters respectively. The estimated values of *s* parameter for 10% and 30% censoring level are very far away from the true value for both approaches. This is because *s* is the scale parameter and since we used random censoring, the scale parameter change and different from the true value. However, the EM algorithm still outperforms MLE with respect to the bias and MSE (see in Appendix A).

4.0 CONCLUSION

MLE and EM algorithm approaches are presented in this research to estimate the 2- and 3-parameter Burr Type XII distribution of complete and censored data. The estimated parameters of the Burr Type XII distribution are away from the true value as the percentage of censoring level increase. The model with the less censoring is performed better than the great censoring. Based on bias and MSE of 2- and 3-parameter Burr Type XII distribution for both approaches, the estimated parameters using EM algorithm gives the smaller value than the estimated parameter using MLE estimates for complete and censored data. EM algorithm has the advantage and is stable in numerical computation because of its robustness against the initial value. Lastly, for further research, EM algorithm can be used to estimate the parameter for another type of Burr distribution and compare with other estimation approaches such as Bayesian method.

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Appendix A

Table 1 Comparison of the estimators, bias, MSE (parentheses) and negative log-likelihood for multiple data sets of 2-parameter Burr Type XII distribution with true value of c = 3 and k = 2, 5, 8

k	CL	MLE		Bias (MSE)		-ln L	EM		Bias (MSE)		- ln L
		c	*k*	*c*	*k*		*c*	*k*	*c*	*k*	
	0%	2.9911	2.0565	-0.0089 (0.0001)	0.0565 (0.0032)	72.3900	2.9989	2.0495	-0.0011 (0.0000)	0.0495 (0.0025)	72.3923
2	10%	3.0324	1.9862	0.0324 (0.0010)	-0.0138 (0.0002)	72.7297	3.0166	2.0807	0.0166 (0.0003)	0.0807 (0.0065)	72.9342
	30%	3.0648	1.8877	0.0648 (0.0042)	-0.1123 (0.0126)	67.1335	2.9794	2.0925	-0.0206 (0.0004)	0.0925 (0.0086)	68.0573
	0%	3.0695	5.3724	0.0695 (0.0048)	0.3724 (0.1387)	-32.8630	3.0655	5.1685	0.0655 (0.0043)	0.1685 (0.0284)	-32.7243
5	10%	3.2108	5.4182	0.2108 (0.0444)	0.4182 (0.1749)	-28.5975	3.0995	5.2493	0.0995 (0.0099)	0.2493 (0.0622)	-28.3740
	30%	3.2707	4.8173	0.2707 (0.0733)	-0.1827 (0.0334)	-8.5840	3.1085	5.2698	0.1085 (0.0118)	0.2698 (0.0728)	-6.7639
	0%	3.0457	7.8380	0.0457 (0.0021)	-0.1620 (0.0262)	-65.0026	3.0156	8.0526	0.0156 (0.0002)	0.0526 (0.0028)	-64.8149
8	10%	3.0562	7.7978	0.0562 (0.0032)	-0.2022 (0.0409)	-55.2219	3.0330	7.8265	0.0330 (0.0011)	-0.1735 (0.0301)	-55.1928
	30%	3.1769	7.8810	0.1769 (0.0313)	-0.1190 (0.0142)	-33.5429	3.0586	8.1907	0.0586 (0.0034)	0.1907 (0.0364)	-32.6455

Table 2 Comparison of the estimators, bias, MSE (parentheses) and negative log-likelihood for multiple data sets of 2-parameter Burr Type XII distribution with true value of c = 1, 2, 4 and k = 2

c	CL	MLE		Bias (MSE)		-ln L	EM		Bias (MSE)		-ln L
		c	*k*	*c*	*k*		*c*	*k*	*c*	*k*	
	0%	1.1241	1.9298	0.1241 (0.0154)	-0.0702 (0.0049)	170.2058	0.9236	2.0317	-0.0764 (0.0058)	0.0317 (0.0010)	176.1415
1	10%	1.1251	1.8185	0.1251 (0.0157)	-0.1815 (0.0329)	164.1454	1.1032	1.8489	0.1032 (0.0107)	-0.1511 (0.0228)	164.2259
	30%	1.1830	1.5363	0.1830 (0.0335)	-0.4637 (0.2150)	152.5279	1.1816	1.7132	0.1816 (0.0330)	-0.2868 (0.0823)	153.3904
	0%	1.9928	2.0311	-0.0072 (0.0001)	0.0311 (0.0010)	120.5805	1.9991	2.0070	-0.0009 (0.0000)	0.0070 (0.0000)	120.5964
2	10%	2.0184	1.9062	0.0184 (0.0003)	-0.0938 (0.0088)	120.0149	1.9915	1.9266	-0.0085 (0.0001)	-0.0734 (0.0054)	120.0529
	30%	2.0573	1.6798	0.0573 (0.0033)	-0.3202 (0.1025)	111.8820	2.0433	1.8722	0.0433 (0.0019)	-0.1278 (0.0163)	112.7460
	0%	4.0774	1.9177	0.0774 (0.0060)	-0.0823 (0.0068)	34.5685	3.9688	2.0516	-0.0312 (0.0010)	0.0516 (0.0027)	35.1391
4	10%	4.0592	1.8471	0.0592 (0.0035)	-0.1529 (0.0234)	39.3504	4.0420	1.9394	0.0420 (0.0018)	-0.0606 (0.0037)	39.5695
	30%	4.2276	1.8175	0.2276 (0.0518)	-0.1825 (0.0333)	34.6019	4.1415	2.1559	0.1415 (0.0200)	0.1559 (0.0243)	36.8425

Table 3 Comparison of the estimators, bias, MSE (parentheses) and negative log-likelihood for multiple data sets of 3-parameter Burr Type XII distribution with true value of $c = 2$, $k = 3$ and $s = 1, 3$

s	CL	MLE			Bias (MSE)			-ln L	MLE			Bias (MSE)			-ln L
		c	*k*	*s*	*c*	*k*	*s*		*c*	*k*	*s*	*c*	*k*	*s*	
	0%	2.0278	2.9791	0.9938	0.0278	-0.0209	-0.0062	59.7459	2.0167	2.9990	0.9967	0.0167	-0.0010	-0.0033	58.6142
					(0.0008)	(0.0004)	(0.0000)					(0.0003)	(0.0000)	(0.0000)	
1	10%	2.0331	2.9300	3.3024	0.0331	-0.0700	2.3024	-146.6664	1.9731	3.0431	3.0786	-0.0269	0.0431	2.0786	-160.8956
					(0.0011)	(0.0049)	(5.3010)					(0.0007)	(0.0019)	(4.3206)	
	30%	2.0488	2.5393	12.9523	0.0488	-0.4607	11.9523	-103.2675	2.0431	2.6270	12.8095	0.0431	-0.3730	11.8095	-110.2365
					(0.0024)	(0.2122)	(142.8575)					(0.0019)	(0.1391)	(139.4643)	
	0%	2.0402	3.0488	3.0074	0.0402	0.0488	0.0074	-168.1272	2.0346	3.0051	2.9945	0.0346	0.0051	-0.0055	-166.4025
					(0.0016)	(0.0024)	(0.0001)					(0.0012)	(0.0000)	(0.0000)	
3	10%	2.0603	2.9432	4.2020	0.0603	-0.0568	1.2020	-240.1818	2.0514	3.0343	4.0423	0.0514	0.0343	1.0423	-233.6244
					(0.0036)	(0.0032)	(1.4448)					(0.0026)	(0.0012)	(1.0864)	
	30%	2.1626	2.7475	6.7819	0.1626	-0.2525	3.7819	-223.6393	2.0851	3.1046	6.5125	0.0851	0.1046	3.5125	-241.6338
					(0.0264)	(0.0638)	(14.3028)					(0.0072)	(0.0109)	(12.3377)	

Table 4 Comparison of the estimators, bias, MSE (parentheses) and negative log-likelihood for multiple data sets of 3-parameter Burr Type XII distribution with true value of $c = 2$, $k = 1, 5$ and $s = 1$

k	CL	MLE			Bias (MSE)			-ln L	MLE			Bias (MSE)			-ln L
		c	*k*	*s*	*c*	*k*	*s*		*c*	*k*	*s*	*c*	*k*	*s*	
	0%	1.8754	1.1811	0.7012	-0.1246	0.1811	-0.2988	450.0444	2.0779	1.0672	1.0112	0.0779	0.0672	0.0112	268.2439
					(0.0155)	(0.0328)	(0.0893)					(0.0061)	(0.0045)	(0.0001)	
1	10%	2.2237	0.7437	16.2757	0.2237	-0.2563	15.2757	8.6902	1.9155	1.1748	16.2427	-0.0845	0.1748	15.2427	-193.4323
					(0.0500)	(0.0657)	(233.3470)					(0.0071)	(0.0306)	(232.3399)	
	30%	2.4473	0.6476	44.7391	0.4473	-0.3524	43.7391	155.4795	2.1003	1.2531	44.3236	0.1003	0.2531	43.3236	-95.4526
					(0.2001)	(0.1242)	(1913.1089)					(0.0101)	(0.0641)	(1876.9343)	
	0%	2.0403	5.1358	1.0106	0.0403	0.1358	0.0106	-21.9258	2.0194	5.0526	1.0025	0.0194	0.0526	0.0025	-18.7267
					(0.0016)	(0.0184)	(0.0001)					(0.0004)	(0.0028)	(0.0000)	
5	10%	2.0685	5.0572	3.1564	0.0685	0.0572	2.1564	-187.3184	2.0245	5.0613	3.0281	0.0245	0.0613	2.0281	-197.0788
					(0.0047)	(0.0033)	(4.6501)					(0.0006)	(0.0038)	(4.1132)	
	30%	2.1601	4.8090	8.1730	0.1601	-0.1910	7.1730	-112.5038	1.8489	5.1537	8.0241	-0.1511	0.1537	7.0241	-232.9054
					(0.0256)	(0.0365)	(51.4519)					(0.0228)	(0.0236)	(49.3380)	

Table 5 Comparison of the estimators, bias, MSE (parentheses) and negative log-likelihood for multiple data sets of 3-parameter Burr Type XII distribution with true value of $c = 0.5, 2$, $k = 2$ and $s = 2$

c	CL	MLE			Bias (MSE)			-ln L	MLE			Bias (MSE)			-ln L
		c	*k*	*s*	*c*	*k*	*s*		*c*	*k*	*s*	*c*	*k*	*s*	
	0%	0.5099	1.9041	2.0245	0.0099	-0.0959	0.0245	-29.8771	0.4966	2.0135	2.1089	-0.0034	0.0135	0.1089	-46.3002
					(0.0001)	(0.0092)	(0.0006)					(0.0000)	(0.0002)	(0.0119)	
0.5	10%	0.5213	1.8274	6.7165	0.0213	-0.1726	4.7165	-417.8265	0.4835	2.1217	6.3874	-0.0165	0.1217	4.3874	-411.1649
					(0.0005)	(0.0298)	(22.2454)					(0.0003)	(0.0148)	(19.2493)	
	30%	0.5289	1.7233	11.1714	0.0289	-0.2767	9.1714	-423.4693	0.4723	2.2153	10.982	-0.0277	0.2153	8.9820	-432.5536
					(0.0008)	(0.0766)	(84.1146)					(0.0008)	(0.0464)	(80.6763)	
	0%	1.9076	2.0659	1.9957	-0.0924	0.0659	-0.0043	-16.5046	2.0207	2.0595	2.0042	0.0207	0.0595	0.0042	-17.6473
					(0.0085)	(0.0043)	(0.0000)					(0.0004)	(0.0035)	(0.0000)	
2	10%	1.8888	2.2623	3.9560	-0.1112	0.2623	1.9560	-202.1237	1.9005	1.9227	3.8156	-0.0995	-0.0773	1.8156	-177.0412
					(0.0124)	(0.0688)	(3.8259)					(0.0099)	(0.0060)	(3.2964)	
	30%	1.8342	2.2730	7.8225	-0.1658	0.2730	5.8225	-232.4619	1.8374	2.0933	7.6836	-0.1626	0.0933	5.6836	-220.9551
					(0.0275)	(0.0745)	(33.9015)					(0.0264)	(0.0087)	(32.3033)	