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RESEARCH ARTICLE

The application of fuzzy logistic equations in population growth with parameter estimation via minimization

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Abstract

This paper presents a numerical solution for the first order fuzzy logistic equations by extended Runge-Kutta fourth order method with estimated parameters. The parameters are estimated by minimization technique using conjugate gradient approach. Then, the fuzzy logistic model with the estimated parameters is used to fit the population growth in Malaysia. Numerical example is given to show the efficiency of the proposed model.

Keywords: Fuzzy logistic equations, Population growth, Parameter estimation, Minimization technique

INTRODUCTION

Relating population growth models statistically to data is central to answering many important questions in ecology. Ecologists are often faced with the problem of predicting the future of populations from periodic counts of abundance. Therefore, ecologists rely on mathematical models to understand ecological systems and to predict future system behaviour.

One of the fundamental tasks of engineering and science, and indeed of mankind in general, is the extraction of information from data. Parameter estimation is a discipline that provides tools for the efficient use of data in the estimation of constants appearing in mathematical models and for aiding in modeling of phenomena.

Parameter estimation is needed in the modern world for the solution of the many diverse problems related to the space program, investigation of the atom, and modeling of ecology. Example and application in this paper, however, are directed to estimation problems occuring in the engineering and science in which fuzzy differential equations as well as ordinary differential and logistic equations are used to population growth model.

Fortunately, simultaneous with the development of increased need of parameter estimation, computers have been built that make parameter estimation practicable for a great array of applications. It should be noted that both digital computational and data acquisition facilities are practical necessities in parameter estimation. Both these facilities have been readily available only since the late 1950s or early 1960s, whereas estimation was first extensively discussed in reference (Legendre, 1806) and (Gauss, 1809). In Gauss's classic paper, he claimed usage of the method of least squares as early as 1795 in connection with the orbit determination of minor planets. For this reason, Gauss is recognized as being the first to use this important tool of parameter estimation.

In this paper, we presented approaches for incorporating parameter estimation techniques in fuzzy logistic equations through the use of the minimization technique via center difference. This paper aims to discuss the fuzzy logistic equation subject to uncertainties in parameter intrinsic growth rate, G and initial population growth, y_0. We test a method of extended Rung-Kutta fourth order method to fit population models to data of a country population growth.

MODEL DISCRIPTION

The logistic model was introduced to describe population growth considering a self-limitation term that corrects the unlimited growth of the Malthusian model (Kot, 2001). The logistic model attempts to model real-world population dynamics by adding a carrying capacity, denoted by K, which provides a theoretical maximum limit value for the population.

The logistic model is derived by modifying growth and death rates to vary in proportion to the size of the population. A relatively large population has to fight over scarcer resources, while a small population can grow more rapidly. Generally, the greater the population the lower the birth rate, and greater the death rate. By adding in variable birth and death rates, the logistic population model is derived:

$$d + ay = \text{death rate},\ b - cy = \text{birth rate}$$
$$y'(t) = [(b - cy) - (d + ay)]y = [b - d - (a + c)y]y.$$

This is simplified by letting the natural reproductive rate, $G_0 = b - d$, and then $a + c = \frac{G_0}{K}$ where $K = \frac{G_0}{(a+c)}$. These get plugged into the question:

$$y' = \left(G_0 - \frac{G_0 \cdot y}{K}\right)y = G_0\left(1 - \frac{y}{K}\right)y$$

where $y(t)$ is the population of the organism. The parameter G is referred to as the rate constant, and if $G > 0$, we call it a growth constant, while if $G < 0$, we call it a decay constant. Since $y'(t) = 0$

when $y = K$, K is called the carrying capacity of the population. In population biology, K and G_0 form the basis of $G = K$ selection theory. This logistic equation is autonomous, and separable. It can be written as:

$$\frac{Kdy}{y(K-y)} = Gdt$$
$$\left(\frac{1}{y} + \frac{1}{K-y}\right)dy = Gdt.$$

After integrating both sides:

$$\ln|y| - \ln|K-y| = Gt + C,$$
$$\ln\left|\frac{K-y}{y}\right| = -Gt - C,$$
$$\left|\frac{K-y}{y}\right| = e^{-Gt-C},$$
$$\frac{K-y}{y} = Ae^{-Gt}.$$

From here we get:

$$y(t) = \frac{K}{1 + Ae^{-Gt}}.$$

The Eq. (1) is referred to as the logistic model:

$$y' = Gy(t)\left(\frac{1-y(t)}{K}\right), \qquad y(0) = y_0. \tag{1}$$

To better represent the imprecise nature of the problem, we focus to introduce fuzzy into initial value problem, y_0. The function $y'(t)$ is a fuzzy derivative of $y(t)$ and y_0 is a fuzzy number. We denote the fuzzy function $y(t)$ by $y(t) = [\underline{y}(t), \overline{y}(t)]$. The $\alpha-$cuts of $y(t)$ are follows:

$$y^i_\alpha(t) = [\underline{y}^i_\alpha(t), \overline{y}^i_\alpha(t)], \qquad i = 1, 2, \ldots, n.$$

Parameter estimation

The Eq. (1) contain parameters, G and K. Parameter estimation involves gradient- based the minimization technique incorporated into the objective function to the model equations. Parameter values G and K will fitting as closely as possible to the data of a country population growth. The parameters, G and K were estimated from the minimum of the objective error using conjugate gradient technique. The initial value of the critical points was based on disjoint domain of objective surface. Usually, the range for the parameters was calculated based on these critical points, within an order of magnitude.

In real population, the growth rate of population in certain country which is denoted as parameter G in Eq. (1) is normally imprecise due to the implicit lack of information and the mistakes in the measurement process present in related problems. Therefore, in this paper we introduce fuzzy parameter to design meaningful and realistic models and parameter K is a number which is constant. The parameters will be applied to the model to make predictions of the country population.

NUMERICAL SOLUTION OF THE MODEL

The basis of all Runge-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} = y_n + \sum_{i=0}^{m} w_i k_i \tag{2}$$

where w_i are constant for all i and $k_i = hf(t_n + a_i h, y_n + \sum_j^i c_{ij} k_j)$.

Many efforts have been made to improve the order of Runge–Kutta methods by means of increasing the numerical terms in Taylor series expansion. This increases the number of function evaluations accordingly (Chakrivat *et al.*, 1983; Enright, 1974; Gear, 1971; Hairer *et al.*, 1991; Rosenbrock, 1963). Recently, in reference (Goeken *et al*., 2000) and (Wu *et al*., 2006) proposed a class of Runge–Kutta methods using higher derivatives and presented new third, ourth and fifth order numerical methods. Specifically, f' is embedded in f i.e. f' is approximated by a difference quotient of past and current evaluations of f, $f'(y_n) \approx \frac{(f(y_n) - f(y_{n-1}))}{h}$. This motivates a family of extended Runge–Kutta-like formulae of the form

$$y_{n+1} = y_n + h\sum_{j=1}^{m} b_j k_j^{(1)} + h^2 \sum_{j=1}^{m} c_j k_j^{(2)} \tag{3}$$

where

$$\left.\begin{aligned} k_j^{(1)} &= f\left(y_n + h\sum_{s=1}^{j-1} a_{js} k_s^{(1)}\right) \\ k_j^{(2)} &= f'\left(y_n + h\sum_{s=1}^{j-1} b_{js} k_s^{(1)}\right) \end{aligned}\right\}, \quad j = 1, 2, \ldots, m. \tag{4}$$

Obviously, with $c_j = 0, j = 1, 2, \cdots, m$ in Eq. (3), the methods reduce to classical Runge–Kutta methods

$$y_{n+1} = y_n + h\sum_{j=1}^{m} b_j k_j, \tag{5}$$

where

$$k_j = f\left(y_n + h\sum_{s=1}^{j-1} a_{js} k_s\right), \; j = 1, 2, \ldots, m.$$

Fourth order formula

Extended Runge–Kutta-like methods Eq. (3) and Eq. (4) with $m = 3$ are of the following form:

$$y_{n+1} = y_n + h\left(b_1 k_1^{(1)} + b_2 k_2^{(1)} + b_3 k_3^{(1)}\right) + h^2\left(c_1 k_1^{(2)} + c_2 k_2^{(2)} + c_3 k_3^{(2)}\right) \tag{6}$$

where

$$\begin{aligned} k_1^{(1)} &= f(y_n), & k_1^{(2)} &= f'(y_n), \\ k_2^{(1)} &= f\left(y_n + ha_{21}k_1^{(1)}\right), & k_2^{(2)} &= f'\left(y_n + hb_{21}k_1^{(1)}\right), \\ k_3^{(1)} &= f\left(y_n + ha_{31}k_1^{(1)} + ha_{32}k_2^{(1)}\right), & k_3^{(2)} &= f'\left(y_n + hb_{31}k_1^{(1)} + hb_{32}k_2^{(1)}\right). \end{aligned}$$

Specific nonzero constants, in the extended Runge–Kutta-like formulae of order 4 (ERK4) (Mirza *et al*., 2002) are:

$$b_1 = 1, c_1 = \frac{1}{6}, c_3 = \frac{1}{3}, a_{21} = \frac{1}{4}, b_{32} = \frac{1}{2}. \tag{7}$$

NUMERICAL EXAMPLE

In this section, the fuzzy logistic equation (1) with fuzzy initial condition is given:

$$\begin{cases} \dfrac{dy}{dt}(t) = Gy(t)\left(1 - \dfrac{y(t)}{K}\right), t \in [0,1] \\ y(0) = (0.2827473 + 0.005r, 0.2827473 - 0.1r), \;\; 0 < r \le 1. \end{cases}$$

We want to estimate the parameters, G and K. By plotting the objective function, we can get the range of estimation parameter. Hence, G and K can be estimated by minimizing the objective function using conjugate gradient to get the new value for parameter G and K.

In this work with the fuzzy logistic model, we shall use actual data presented in Table 1 from website of indexmundi.com as state at references, with reduced unit. The results from 2010 to 2014 is generated from fuzzy logistic equation using extended Runge-Kutta fourth order method are shown in (Fig. 2(a)) and (Fig. 2(b)).

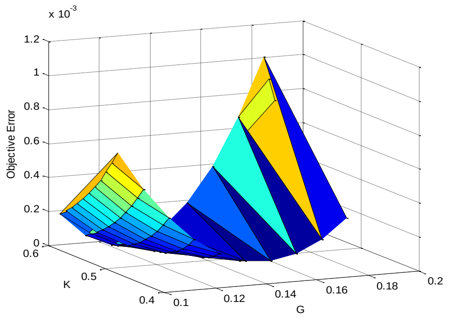
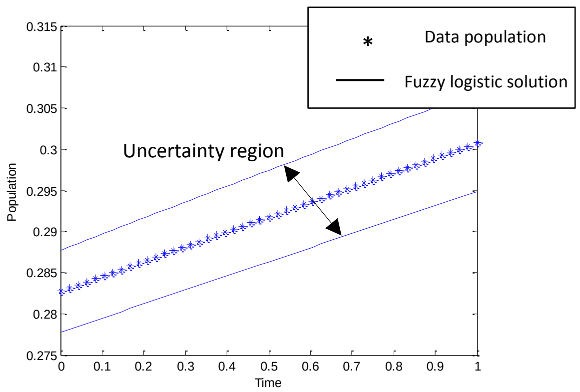
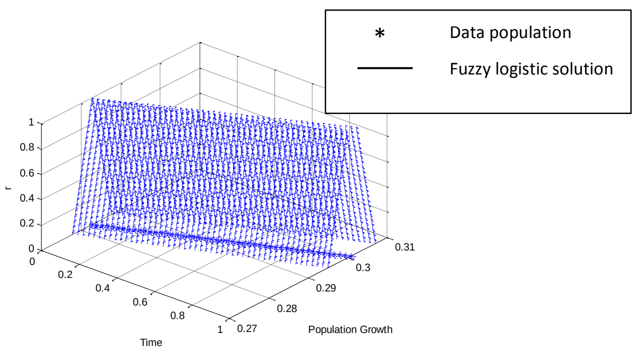


Fig. 1 Objective error for parameter G and K.



(a)



(b)

Fig. 2 Result of (a) 2D Fuzzy Logistic Equation and (b) 3D Fuzzy Logistic Equation.

Table 1 Population data of Malaysia.

Time	Population
0	0.282747
0.02075	0.283126
0.04175	0.283504
0.0625	0.283882
0.08325	0.28426
0.10425	0.284638
0.125	0.285017
0.14575	0.285395
0.16675	0.285773
0.1875	0.286151
0.20825	0.28653
0.22925	0.286908
0.25	0.287286
0.27075	0.287662
0.29175	0.288038
0.3125	0.288414
0.33325	0.288791
0.35425	0.289167
0.375	0.289543
0.39575	0.289919
0.41675	0.290295
0.4375	0.290671
0.45825	0.291047
0.47925	0.291423
0.5	0.2918
0.52075	0.292173
0.54175	0.292547
0.5625	0.292921
0.58325	0.293294
0.60425	0.293668
0.625	0.294042
0.64575	0.294415
0.66675	0.294789
0.6875	0.295163
0.70825	0.295537
0.72925	0.29591
0.75	0.296284
0.77075	0.296655
0.79175	0.297026
0.8125	0.297396
0.83325	0.297767
0.85425	0.298138
0.875	0.298509
0.89575	0.29888
0.91675	0.29925
0.9375	0.299621
0.95825	0.299992
0.97925	0.300363
1	0.300734

CONCLUSION

We have shown how the study of fuzzy differential equations can be motivated using modelling of populations. In addition, this approach permits parameter estimation studies and a completion of the modelling cycle involving data and realistic situations. The parameter estimation using minimization technique gives the closer solution to the population data. The expected result is bounded inside the region of lower and upper solution.

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