MJFAS MALAYSIAN JOURNAL OF FUNDAMENTAL AND APPLIED SCIENCES
PRINT ISSN: 2289-5981 | ONLINE ISSN: 2289-599X

RESEARCH ARTICLE

On the generalized conjugacy class graph of some dihedral groups

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Article history
Received 5 February 2017
Accepted 20 Mac 2017

Abstract

A graph is a mathematical structure which consists of vertices and edges that is used to model relations between object. In this research, the generalized conjugacy class graph is constructed for some dihedral groups to show the relation between orbits and their cardinalities. In order to construct the graph, the probability that an element of the dihedral groups fixes a set must first be obtained. The set under this study is the set of all pairs of commuting elements in the form of (a, b) where a and b are elements of the dihedral groups and the lowest common multiple of the order of the elements is two. The orbits of the set are then computed using conjugation action. Based on the results obtained, the generalized conjugacy class graph is constructed and some graph properties are also found.

***Keywords*:** Graph theory, conjugacy class, dihedral group, commutativity degree

INTRODUCTION

A graph is an object used to model the connection between two objects in a graphical manner. There are many fields that used graph as a representation. For example, in chemistry, graph is used to represent atoms and its bonds. In mathematics, mainly in the field of group theory, graph is also used to represent the behaviour of some groups.

Many studies have been done in order to show the relation between group theory and graph theory. For example, Neumann (1967) introduced non-commuting graph that shows the relation between non-central elements of a group and the commutativity of the elements. Then, Tolue and Erfanian (2013) generalized the non-commuting graph to the relative non-commuting graph from the relative commutativity degree of a group. The vertices of the graph are the elements in the group excluding the centralizer of the subgroup and the vertices are connected if their commutator is not equal to one. Besides that, a graph related to conjugacy classes is introduced by Bertram (1990). The vertices of the graph is the non-central conjugacy classes and the vertices are adjacent if the cardinalities of the conjugacy classes are not coprime. The graphs mentioned have been applied by various researchers from different type of groups. Recently, Omer *et al.* (2015) introduced the generalized conjugacy class graph whose the vertices are the non-central orbits under group action on a set and the edges are formed if the cardinalities of the orbits are not coprime.

In this study, the generalized conjugacy class graph is constructed to show the behaviour of the orbits of dihedral groups of order 16, 18 and 20. The set under this study is the set of all pairs of commuting elements in the form of (a, b) where a and b are the elements of the dihedral groups and the lowest common multiple of the order of the elements is two. In mathematical symbols, we can write it as the following:

$$\Omega = \{(a, b) \in G \times G \mid ab = ba, a \neq b, lcm\,(|a|, |b|) = 2\}.$$

In order to obtain the generalized conjugacy class graph, the orbits of the group need to be found first. From the same orbits, the probability that an element of the dihedral groups fixes the set Ω can also be calculated. Hence, in this study the probability of dihedral group of order 16, 18 and 20 fixes the set Ω are also calculated.

PRELIMINARIES

In this section, some definitions and properties related to this study are given.

Definition 1 (Goodman, 2003) Orbit

Suppose G is a finite group that acts on a set Ω and $\omega \in \Omega$. The orbit of ω, denoted by $O(\omega)$, is the subset $O(\omega) = \{g\omega \mid g \in G, \omega \in \Omega\}$. In this research, the group action is conjugation action, hence the orbit is written as

$$O(\omega) = \{g\omega g^{-1} \mid g \in G, \omega \in \Omega\}.$$

The probability that two random elements commute is known as commutativity degree of a group. The commutativity degree of a group is used to determine the abelianness of a group. This concept, introduced by Miller in 1944, is the probability that a pair of elements x and y selected randomly from a group commute. It is also defined as following:

Definition 2 (Miller, 1944) Commutativity degree

The probability that two random element (x,y) in a group G commute is called the commutativity degree and is defined as:

$$P(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2}.$$

This concept has been used in many studies for some finite groups Tarnauceanu (2009). Erdos and Turan (1968) applied this concept for symmetric group, S_n. Later in 1973, another method of computing commutativity degree of a group was introduced by Gustafson. The concept uses the number of conjugacy classes of a group and is defined as in the following:

Definition 3 (Gustafson, 1973) Commutativity degree

Let G be a finite group and $K(G)$ is the number of conjugacy classes of the group. Then the commutativity degree of a group, denoted by $P(G)$, is given as:

$$P(G) = \frac{K(G)}{|G|}.$$

Omer *et al*. (2013) extended the concept of commutativity degree by introducing the probability that an element fixes a set in which some group actions on a set are used to determine the probability. The probability that an element of a group fixes a set is defined as below:

Definition 4 (Omer *et al*., 2013) Probability that an element of a group fixes a set

Let G be a group. Let S be the set of all subsets of commuting elements of size two in G, where G acts on S by conjugation. Then the commutativity degree of an element of a group fixes a set is given as follows:

$$P_G(S) = \frac{|\{(g,s) \mid gS=S \; for \; g\in G \; and \; s\in S\}|}{|G||S|}.$$

In term of the number of conjugacy classes, the probability that an element of a group fixes a set is given by the following definition:

Definition 5 (Omer *et al*., 2013) Probability that an element of a group fixes a set

Let G be a finite group and let X be a set of elements of G of size two in the form of (a, b) where a and b commute. Let S be the set of all commutating elements of G of size two and G acts on S by conjugation. Then the probability that an element of a group fixes a set is given by:

$$P_G(S) = \frac{K}{|S|},$$

where K is the number of conjugacy classes of S in G.

Many real-life problems have been described in term of diagram. This includes some concepts in group theory that are applied in various graphs. In this study, the results obtained from the orbits will be applied into graph theory, namely the generalized conjugacy class graph. The definition of the graph is given below:

Definition 6 (Omer *et al*., 2015) Generalized conjugacy class graph, $\Gamma_G^{\Omega_c}$

Let G be a finite non-abelian group and let Ω be a set of G. If G acts on the set Ω, the vertices of generalized conjugacy class graph are $K(\Omega) - |A|$, where $K(\Omega)$ is non-central conjugacy classes under group action on Ω and $A = \{\omega \in \Omega, \omega g = g\omega, g \in G\}$. Two vertices of $\Gamma_G^{\Omega_c}$ are connected by an edge if their cardinalities are not set-wise relatively prime.

After obtaining the generalized conjugacy class graph, some graph properties can also be analysed. In this study, the chromatic number and the clique number of the graphs are found.

Definition 7 (Erfanian and Tolue, 2012) Chromatic number, χ

Let $k > 0$ be an integer. A k-vertex coloring of a graph Γ is an assignment of k colors to the vertices such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(\Gamma)$ of a graph Γ is the minimum k for which Γ has a k-vertex coloring.

Definition 8 (Erfanian and Tolue, 2012) Clique number, ω

A subset C of vertices of Γ is called a clique if the induced subgraph on C is a complete graph. The maximum size of a clique is called clique number of Γ denoted by $\omega(\Gamma)$.

MAIN RESULTS

In this section, the probability that an element of a group fixes a set Ω is calculated for dihedral groups of order 16, 18 and 20. Then, based on the results, the generalized conjugacy class graph and its properties are found. The set used throughout this study is defined as follows:

Definition 8: The set Ω

The set under this study is the set of all pairs of commuting elements in the form of (a,b) where a and b is the elements of the dihedral groups and the lowest common multiple of the order of the elements is two. In mathematical symbols, we can write it as:

$$\Omega = \{(a,b) \in G \times G \mid ab = ba, a \neq b, lcm\,(|a|,|b|) = 2\}.$$

The results on the probability that an element of dihedral groups of order 16, 18 and 20 fixes the set Ω are given in the following propositions:

Proposition 1

Let G be the dihedral group of order 16, D_8 and G acts on Ω by conjugation. Then the number of orbits of Ω is $K(\Omega) = 12$ and the probability that an element of D_8 fixes the set Ω is $P_{D_8}(\Omega) = \frac{12}{42} = \frac{2}{7}$.

Proof:

By using Definition 8, we found that there are 42 elements in the set Ω which are

$$\Omega = \{(e,a^4),(e,b),(e,ab),(e,a^2b),(e,a^3b),(e,a^4b),(e,a^5b),(e,a^6b),(e,a^7b),(b,e),(b,a^4b),(b,a^4),(ab,e),(ab,a^5b),(ab,a^4),(a^2b,e),(a^2b,a^6b),(a^2b,a^4),(a^3b,e),(a^3b,a^7b),(a^3b,a^4),(a^4b,e),(a^4b,b),(a^4b,a^4),(a^5b,e),(a^5b,ab),(a^5b,a^4),(a^6b,e),(a^6b,a^2b),(a^6b,a^4),(a^7b,e),(a^7b,a^3b),(a^7b,a^4),(a^4,e),(a^4,b),(a^4,ab),(a^4,a^2b),(a^4,a^3b),(a^4,a^4b),(a^4,a^5b),(a^4,a^6b),(a^4,a^7b)\}.$$

Suppose D_8 acts on Ω by conjugation, the orbits calculated are listed as follows:

(i) $\omega_1 = O(e,b) = \{(e,b),(e,a^2b),(e,a^4b),(e,a^6b)\}$
(ii) $\omega_2 = O(b,e) = \{(b,e),(a^2b,e),(a^4b,e),(a^6b,e)\}$
(iii) $\omega_3 = O(e,ab) = \{(e,ab),(e,a^3b),(e,a^5b),(e,a^7b)\}$
(iv) $\omega_4 = O(ab,e) = \{(ab,e),(a^3b,e),(a^5b,e),(a^7b,e)\}$
(v) $\omega_5 = O(e,a^4) = \{(e,a^4)\}$
(vi) $\omega_6 = O(a^4,e) = \{(a^4,e)\}$
(vii) $\omega_7 = O(b,a^4b) = \{(b,a^4b),(a^4b,b),(a^2b,a^6b),(a^6b,a^2b)\}$
(viii) $\omega_8 = O(b,a^4) = \{(b,a^4),(a^2b,a^4),(a^4b,a^4),(a^6b,a^4)\}$
(ix) $\omega_9 = O(a^4,b) = \{(a^4,b),(a^4,a^2b),(a^4,a^4b),(a^4,a^6b)\}$
(x) $\omega_{10} = O(ab,a^4) = \{(ab,a^4),(a^3b,a^4),(a^5b,a^4),(a^7b,a^4)\}$
(xi) $\omega_{11} = O(a^4,ab) = \{(a^4,ab),(a^4,a^3b),(a^4,a^5b),(a^4,a^7b)\}$
(xii) $\omega_{12} = O(ab,a^5b) = \{(ab,a^5b),(a^5b,ab),(a^3b,a^7b),(a^7b,a^3b)\}$

Therefore, it is shown that the number of orbits, $K(\Omega) = 12$. By Definition 5, the probability that an element of D_8 fixes the set Ω is $P_{D_8}(\Omega) = \frac{12}{42} = \frac{2}{7}$. ■

Proposition 2

Let G be the dihedral group of order 18, D_9, and D_9 acts on the set Ω by conjugation action. Then the number of orbits $K(\Omega) = 2$ and the probability that an element of D_9 fixes the set Ω is $P_{D_9}(\Omega) = \frac{2}{18} = \frac{1}{9}$.

Proof:
The set Ω consists of 18 elements, which are
$\Omega = \{(e,b),(e,ab),(e,a^2b),(e,a^3b),(e,a^4b),(e,a^5b),(e,a^6b),(e,a^7b),(e,a^8b),(b,e),(ab,e),(a^2b,e),(a^3b,e),(a^4b,e),(a^5b,e),(a^6b,e),(a^7b,e),(a^8b,e)\}$.

Suppose that D_9 acts on Ω by conjugation action, there are two orbits found, which are:

(i) $\omega_1 = O(e,b) = \{(e,b),(e,ab),(e,a^2b),(e,a^3b),(e,a^4b),(e,a^5b),(e,a^6b),(e,a^7b),(e,a^8b)\}$
(ii) $\omega_2 = O(b,e) = \{(b,e),(ab,e),(a^2b,e),(a^3b,e),(a^4b,e),(a^5b,e),(a^6b,e),(a^7b,e),(a^8b,e)\}$

Hence, it is shown that the number of orbits, $K(\Omega) = 2$. By Definition 5, the probability that an element of D_9 fixes the set Ω is $P_{D_9}(\Omega) = \frac{2}{18} = \frac{1}{9}$. ■

Proposition 3
Let G be the dihedral group of order 20, D_{10} and G acts on Ω by conjugation. Then the number of orbits of Ω is $K(\Omega) = 12$ and the probability that an element of D_{10} fixes the set Ω is $P_{D_{10}}(\Omega) = \frac{12}{52} = \frac{3}{13}$.

Proof:
By using Definition 8, we found that there are 52 elements in the set Ω which are

$\Omega = \{(e,a^5),(e,ab),(e,b),(e,a^2b),(e,a^3b),(e,a^4b),(e,a^5b),(e,a^6b),(e,a^7b),(e,a^8b),(e,a^9b),(a^5,e),(a^5,ab),(a^5,b),(a^5,a^2b),(a^5,a^3b),(a^5,a^4b),(a^5,a^5b),(a^5,a^6b),(a^5,a^7b),(a^5,a^8b),(a^5,a^9b),(b,e),(b,a^5),(b,a^5b),(ab,e),(ab,a^5),(ab,a^6b),(a^2b,e),(a^2b,a^5),(a^2b,a^7b),(a^3b,e),(a^3b,a^5),(a^3b,a^8b),(a^4b,e),(a^4b,a^5),(a^4b,a^9b),(a^5b,e),(a^5b,a^5),(a^5b,b),(a^6b,e),(a^6b,a^5),(a^6b,ab),(a^7b,e),(a^7b,a^5),(a^7b,a^2b),(a^8b,e),(a^8b,a^5),(a^8b,a^3b),(a^9b,e),(a^9b,a^5),(a^9b,a^4b)\}$.

Suppose D_8 acts on Ω by conjugation, the orbits calculated are listed as follows:

(i) $\omega_1 = O(e,b) = \{(e,b),(e,a^2b),(e,a^4b),(e,a^6b),(e,a^8b)\}$
(ii) $\omega_2 = O(b,e) = \{(b,e),(a^2b,e),(a^4b,e),(a^6b,e),(a^8b,e)\}$
(iii) $\omega_3 = O(e,ab) = \{(e,ab),(e,a^3b),(e,a^5b),(e,a^7b),(e,a^9b)\}$
(iv) $\omega_4 = O(ab,e) = \{(ab,e),(a^3b,e),(a^5b,e),(a^7b,e),(a^9b,e)\}$
(v) $\omega_5 = O(e,a^5) = \{(e,a^5)\}$
(vi) $\omega_6 = O(a^5,e) = \{(a^5,e)\}$
(vii) $\omega_7 = O(b,a^5b) = \{(b,a^5b),(a^2b,a^7b),(a^4b,a^9b),(a^6b,ab),(a^8b,\ a^3b)\}$
(viii) $\omega_8 = O(b,a^5) = \{(b,a^5),(a^2b,a^5),(a^4b,a^5),(a^6b,a^5),(a^8b,a^5)\}$
(ix) $\omega_9 = O(a^5,b) = \{(a^5,b),(a^5,a^2b),(a^5,a^4b),(a^5,a^6b),(a^5,a^8b)\}$
(x) $\omega_{10} = O(ab,a^5) = \{(ab,a^5),(a^3b,a^5),(a^5b,a^5),(a^7b,a^5),(a^9b,a^5)\}$
(xi) $\omega_{11} = O(a^5,ab) = \{(a^5,ab),(a^5,a^3b),(a^5,a^5b),(a^5,a^7b),(a^5,a^9b)\}$
(xii) $\omega_{12} = O(a^5b,b) = \{(a^5b,b),(a^7b,a^2b),(a^9b,a^4b),(ab,a^6b),(a^3b,a^8b)\}$

Therefore, it is shown that the number of orbits, $K(\Omega) = 12$. By Definition 5, the probability that an element of D_{10} fixes the set Ω is $P_{D_{10}}(\Omega) = \frac{13}{52} = \frac{3}{13}$. ■
Based on the results of the orbits, the generalized conjugacy class graph and its properties are also found for D_8, D_9 and D_{10}. The results are given in the following propositions:

Proposition 4
Let G be the dihedral group of order 16 and 20, D_8 and D_{10} respectively. If G acts on Ω by conjugation, then the generalized conjugacy class graph, $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ are complete graphs with ten vertices, K_{10}.

Proof:
Based on Proposition 1 and Proposition 3, the number of orbits for D_8 and D_{10} are $K(\Omega) = 12$. Two of the orbits, which are ω_5 and ω_6 are central orbits. From Definition 6, the vertices of the generalized conjugacy class graph $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ are the non-central orbits. Hence, the number of vertices in $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$, $|V(\Gamma_{D_8}^{\Omega_c})| = |V(\Gamma_{D_{10}}^{\Omega_c})| = 12 - 2 = 10$. Since all of the non-central orbits in D_8 and D_{10} have four and five elements respectively, the cardinalities are not coprime. Therefore, all the vertices are adjacent to each other. Hence, $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ are complete graphs with ten vertices, K_{10}. ■

Figure 1 shows the complete graph of ten vertices, K_{10} of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$.

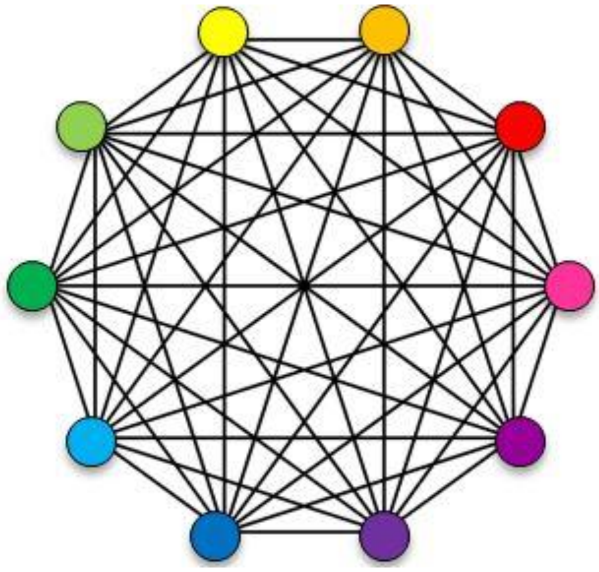

Figure 1 Complete graph of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$, K_{10}.

Proposition 5
The chromatic number of the generalized conjugacy class graph of the dihedral group of order 16 and 20, $\chi(\Gamma_{D_8}^{\Omega_c}) = \chi(\Gamma_{D_{10}}^{\Omega_c}) = 10$.

Proof:
Based on Figure 1, there are ten colours that can be applied to the vertices of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ because all of its vertices are connected to each other. Hence, the chromatic number of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$, $\chi(\Gamma_{D_8}^{\Omega_c}) = \chi(\Gamma_{D_{10}}^{\Omega_c}) = 10$. ■

Proposition 6
The clique number for the generalized conjugacy class graph of the dihedral group of order 16 and 20, $\omega(\Gamma_{D_8}^{\Omega_c}) = \omega(\Gamma_{D_{10}}^{\Omega_c}) = 9$.

Proof:
The largest complete subgraph of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ is the complete graph K_9 with nine vertices. Hence, the clique number of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$, $\omega(\Gamma_{D_8}^{\Omega_c}) = \omega(\Gamma_{D_{10}}^{\Omega_c}) = 9$. ■

Proposition 7

Let G be the dihedral group of order 18, D_9 and G acts on Ω by conjugation. Then, the generalized conjugacy class graph of D_9, $\Gamma_{D_9}^{\Omega_c}$ is a complete graph with two vertices, K_2.

Proof:

Based on Proposition 2, the number of orbits $K(\Omega) = 2$. Since both of the orbits are non-central, the number of vertices of $\Gamma_{D_9}^{\Omega_c}$, $|V(\Gamma_{D_9}^{\Omega_c})| = 2 - 0 = 2$. Both ω_1 and ω_2 has nine elements, making them not coprime with each other. Hence, the vertices are connected and therefore the generalized conjugacy class graph $\Gamma_{D_9}^{\Omega_c}$ is a complete graph of two vertices, K_2. ∎

Figure 2 shows the complete graph of two vertices, K_2 of $\Gamma_{D_9}^{\Omega_c}$.

Figure 2 Complete graph of $\Gamma_{D_9}^{\Omega_c}$, K_2.

Proposition 8

The chromatic number of the generalized conjugacy class graph of the dihedral group of order 18, $\chi(\Gamma_{D_9}^{\Omega_c}) = 2$.

Proof:

Based on Figure 2, there are two colours that can be applied to the vertices of $\Gamma_{D_9}^{\Omega_c}$ because both of its vertices are connected to each other. Hence, the chromatic number of $\Gamma_{D_9}^{\Omega_c}$, $\chi(\Gamma_{D_9}^{\Omega_c}) = 2$. ∎

Proposition 9

The clique number for the generalized conjugacy class graph of the dihedral group of order 18, $\omega(\Gamma_{D_9}^{\Omega_c}) = 0$.

Proof:

Since $\Gamma_{D_9}^{\Omega_c}$ has only two vertices, there is no complete subgraph in $\Gamma_{D_9}^{\Omega_c}$. Hence, the clique number of $\Gamma_{D_9}^{\Omega_c}$, $\omega(\Gamma_{D_9}^{\Omega_c}) = 0$.

CONCLUSION

In this study, the orbits of some dihedral groups, namely dihedral groups of order 16, 18 and 20 are determined. These orbits are then used to find the probability that the group element fixes the set Ω as well as the generalized conjugacy class graph. The results are summarized in the following table:

Table 1 Summary of the main results

Dihedral Group	Number of orbits	Probability that an element fixes Ω	Generalized conjugacy class graph
D_8	12	$\frac{2}{7}$	Complete graph, K_{10}
D_9	2	$\frac{1}{9}$	Complete graph, K_2
D_{10}	12	$\frac{3}{13}$	Complete graph, K_{10}

ACKNOWLEDGEMENT

The authors would like to acknowledge Universiti Teknologi Malaysia (UTM) and Ministry of Higher Education Malaysia (MOHE) for the financial funding through the Research University Grant Scheme (GUP) vote no. 11J96 and 13H79.

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