

Proof:

The set Ω consists of 18 elements, which are

$$\Omega = \{(e, b), (e, ab), (e, a^2b), (e, a^3b), (e, a^4b), (e, a^5b), (e, a^6b), (e, a^7b), (e, a^8b), (b, e), (ab, e), (a^2b, e), (a^3b, e), (a^4b, e), (a^5b, e), (a^6b, e), (a^7b, e), (a^8b, e)\}.$$

Suppose that D_9 acts on Ω by conjugation action, there are two orbits found, which are:

- (i) $\omega_1 = O(e, b) = \{(e, b), (e, ab), (e, a^2b), (e, a^3b), (e, a^4b), (e, a^5b), (e, a^6b), (e, a^7b), (e, a^8b)\}$
- (ii) $\omega_2 = O(b, e) = \{(b, e), (ab, e), (a^2b, e), (a^3b, e), (a^4b, e), (a^5b, e), (a^6b, e), (a^7b, e), (a^8b, e)\}$

Hence, it is shown that the number of orbits, $K(\Omega) = 2$. By Definition 5, the probability that an element of D_9 fixes the set Ω is $P_{D_9}(\Omega) = \frac{2}{18} = \frac{1}{9}$. ■

Proposition 3

Let G be the dihedral group of order 20, D_{10} and G acts on Ω by conjugation. Then the number of orbits of Ω is $K(\Omega) = 12$ and the probability that an element of D_{10} fixes the set Ω is $P_{D_{10}}(\Omega) = \frac{12}{52} = \frac{3}{13}$.

Proof:

By using Definition 8, we found that there are 52 elements in the set Ω which are

$$\Omega = \{(e, a^5), (e, ab), (e, b), (e, a^2b), (e, a^3b), (e, a^4b), (e, a^5b), (e, a^6b), (e, a^7b), (e, a^8b), (e, a^9b), (a^5, e), (a^5, ab), (a^5, b), (a^5, a^2b), (a^5, a^3b), (a^5, a^4b), (a^5, a^5b), (a^5, a^6b), (a^5, a^7b), (a^5, a^8b), (a^5, a^9b), (b, e), (b, a^5), (b, a^5b), (ab, e), (ab, a^5), (ab, a^6b), (a^2b, e), (a^2b, a^5), (a^2b, a^7b), (a^3b, e), (a^3b, a^5), (a^3b, a^8b), (a^4b, e), (a^4b, a^5), (a^4b, a^9b), (a^5b, e), (a^5b, a^5), (a^5b, b), (a^6b, e), (a^6b, a^5), (a^6b, ab), (a^7b, e), (a^7b, a^5), (a^7b, a^2b), (a^8b, e), (a^8b, a^5), (a^8b, a^3b), (a^9b, e), (a^9b, a^5), (a^9b, a^4b)\}.$$

Suppose D_8 acts on Ω by conjugation, the orbits calculated are listed as follows:

- (i) $\omega_1 = O(e, b) = \{(e, b), (e, a^2b), (e, a^4b), (e, a^6b), (e, a^8b)\}$
- (ii) $\omega_2 = O(b, e) = \{(b, e), (a^2b, e), (a^4b, e), (a^6b, e), (a^8b, e)\}$
- (iii) $\omega_3 = O(e, ab) = \{(e, ab), (e, a^3b), (e, a^5b), (e, a^7b), (e, a^9b)\}$
- (iv) $\omega_4 = O(ab, e) = \{(ab, e), (a^3b, e), (a^5b, e), (a^7b, e), (a^9b, e)\}$
- (v) $\omega_5 = O(e, a^5) = \{(e, a^5)\}$
- (vi) $\omega_6 = O(a^5, e) = \{(a^5, e)\}$
- (vii) $\omega_7 = O(b, a^5b) = \{(b, a^5b), (a^2b, a^7b), (a^4b, a^9b), (a^6b, ab), (a^8b, a^3b)\}$
- (viii) $\omega_8 = O(b, a^5) = \{(b, a^5), (a^2b, a^5), (a^4b, a^5), (a^6b, a^5), (a^8b, a^5)\}$
- (ix) $\omega_9 = O(a^5, b) = \{(a^5, b), (a^5, a^2b), (a^5, a^4b), (a^5, a^6b), (a^5, a^8b)\}$
- (x) $\omega_{10} = O(ab, a^5) = \{(ab, a^5), (a^3b, a^5), (a^5b, a^5), (a^7b, a^5), (a^9b, a^5)\}$
- (xi) $\omega_{11} = O(a^5, ab) = \{(a^5, ab), (a^5, a^3b), (a^5, a^5b), (a^5, a^7b), (a^5, a^9b)\}$
- (xii) $\omega_{12} = O(a^5b, b) = \{(a^5b, b), (a^7b, a^2b), (a^9b, a^4b), (ab, a^6b), (a^3b, a^8b)\}$

Therefore, it is shown that the number of orbits, $K(\Omega) = 12$. By Definition 5, the probability that an element of D_{10} fixes the set Ω is $P_{D_{10}}(\Omega) = \frac{13}{52} = \frac{3}{13}$. ■

Based on the results of the orbits, the generalized conjugacy class graph and its properties are also found for D_8, D_9 and D_{10} . The results are given in the following propositions:

Proposition 4

Let G be the dihedral group of order 16 and 20, D_8 and D_{10} respectively. If G acts on Ω by conjugation, then the generalized conjugacy class graph, $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ are complete graphs with ten vertices, K_{10} .

Proof:

Based on Proposition 1 and Proposition 3, the number of orbits for D_8 and D_{10} are $K(\Omega) = 12$. Two of the orbits, which are ω_5 and ω_6 are central orbits. From Definition 6, the vertices of the generalized conjugacy class graph $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ are the non-central orbits. Hence, the number of vertices in $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$, $|V(\Gamma_{D_8}^{\Omega_c})| = |V(\Gamma_{D_{10}}^{\Omega_c})| = 12 - 2 = 10$. Since all of the non-central orbits in D_8 and D_{10} have four and five elements respectively, the cardinalities are not coprime. Therefore, all the vertices are adjacent to each other. Hence, $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ are complete graphs with ten vertices, K_{10} . ■

Figure 1 shows the complete graph of ten vertices, K_{10} of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$.

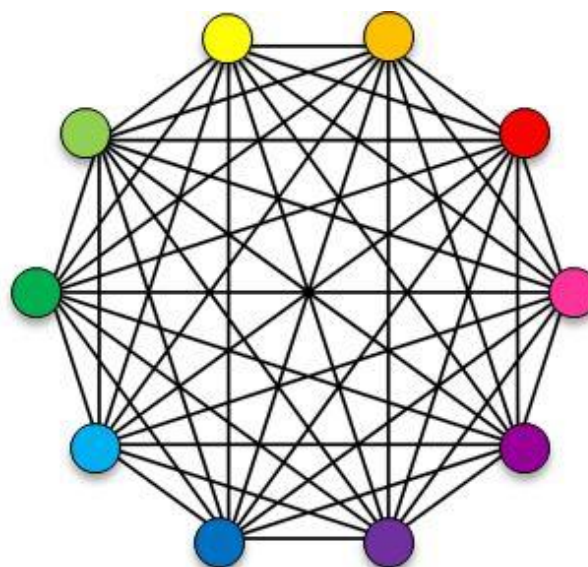


Figure 1 Complete graph of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$, K_{10} .

Proposition 5

The chromatic number of the generalized conjugacy class graph of the dihedral group of order 16 and 20, $\chi(\Gamma_{D_8}^{\Omega_c}) = \chi(\Gamma_{D_{10}}^{\Omega_c}) = 10$.

Proof:

Based on Figure 1, there are ten colours that can be applied to the vertices of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ because all of its vertices are connected to each other. Hence, the chromatic number of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$, $\chi(\Gamma_{D_8}^{\Omega_c}) = \chi(\Gamma_{D_{10}}^{\Omega_c}) = 10$. ■

Proposition 6

The clique number for the generalized conjugacy class graph of the dihedral group of order 16 and 20, $\omega(\Gamma_{D_8}^{\Omega_c}) = \omega(\Gamma_{D_{10}}^{\Omega_c}) = 9$.

Proof:

The largest complete subgraph of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$ is the complete graph K_9 with nine vertices. Hence, the clique number of $\Gamma_{D_8}^{\Omega_c}$ and $\Gamma_{D_{10}}^{\Omega_c}$, $\omega(\Gamma_{D_8}^{\Omega_c}) = \omega(\Gamma_{D_{10}}^{\Omega_c}) = 9$. ■

Proposition 7

Let G be the dihedral group of order 18, D_9 and G acts on Ω by conjugation. Then, the generalized conjugacy class graph of D_9 , $\Gamma_{D_9}^{\Omega_c}$ is a complete graph with two vertices, K_2 .

Proof:

Based on Proposition 2, the number of orbits $K(\Omega) = 2$. Since both of the orbits are non-central, the number of vertices of $\Gamma_{D_9}^{\Omega_c}$, $|V(\Gamma_{D_9}^{\Omega_c})| = 2 - 0 = 2$. Both ω_1 and ω_2 has nine elements, making them not coprime with each other. Hence, the vertices are connected and therefore the generalized conjugacy class graph $\Gamma_{D_9}^{\Omega_c}$ is a complete graph of two vertices, K_2 . ■

Figure 2 shows the complete graph of two vertices, K_2 of $\Gamma_{D_9}^{\Omega_c}$.



Figure 2 Complete graph of $\Gamma_{D_9}^{\Omega_c}$, K_2 .

Proposition 8

The chromatic number of the generalized conjugacy class graph of the dihedral group of order 18, $\chi(\Gamma_{D_9}^{\Omega_c}) = 2$.

Proof:

Based on Figure 2, there are two colours that can be applied to the vertices of $\Gamma_{D_9}^{\Omega_c}$ because both of its vertices are connected to each other. Hence, the chromatic number of $\Gamma_{D_9}^{\Omega_c}$, $\chi(\Gamma_{D_9}^{\Omega_c}) = 2$. ■

Proposition 9

The clique number for the generalized conjugacy class graph of the dihedral group of order 18, $\omega(\Gamma_{D_9}^{\Omega_c}) = 0$.

Proof:

Since $\Gamma_{D_9}^{\Omega_c}$ has only two vertices, there is no complete subgraph in $\Gamma_{D_9}^{\Omega_c}$. Hence, the clique number of $\Gamma_{D_9}^{\Omega_c}$, $\omega(\Gamma_{D_9}^{\Omega_c}) = 0$.

CONCLUSION

In this study, the orbits of some dihedral groups, namely dihedral groups of order 16, 18 and 20 are determined. These orbits are then used to find the probability that the group element fixes the set Ω as well as the generalized conjugacy class graph. The results are summarized in the following table:

Table 1 Summary of the main results

Dihedral Group	Number of orbits	Probability that an element fixes Ω	Generalized conjugacy class graph
D_8	12	$\frac{2}{7}$	Complete graph, K_{10}
D_9	2	$\frac{1}{9}$	Complete graph, K_2
D_{10}	12	$\frac{3}{13}$	Complete graph, K_{10}

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