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RESEARCH ARTICLE

The independence polynomial of *n*-th central graph of dihedral groups

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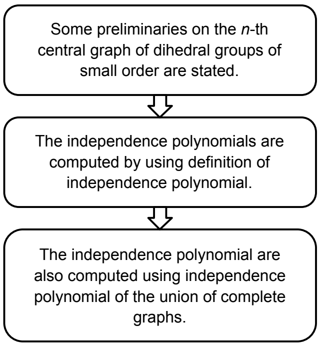
Graphical abstract



Abstract

An independent set of a graph is a set of pairwise non-adjacent vertices while the independence number is the maximum cardinality of an independent set in the graph. The independence polynomial of a graph is defined as a polynomial in which the coefficient is the number of the independent set in the graph. Meanwhile, a graph of a group G is called n-th central if the vertices are elements of G and two distinct vertices are adjacent if they are elements in the n-th term of the upper central series of G. In this research, the independence polynomial of the n-th central graph is found for some dihedral groups. The results are computed by using the definition of independence polynomial and also by using the independence polynomial of the union of complete graphs.

Keywords: Independence polynomial, n-th central graph, dihedral group

INTRODUCTION

Throughout this paper, only simple graph is considered and will be referred as graph. A graph is simple if it has no loops and no multiple edges. A graph is a pair $\Gamma = (V, E)$ where V is the vertex set and E, the edge set, is the unordered pair of elements of V. Let $u, v \in V$, vertices u and v are adjacent to each other in Γ if and only if there is an edge between u and v, i.e. $(u, v) \in E$. An edge $e = (x, y)$ is said to be incident with each one of its end vertices, x and y.

Many researches have been done to study the algebraic properties of groups using the properties of graphs. Some examples of graphs that were associated to groups are the conjugate graph (Erfanian and Tolue, 2012), non-commuting graph (Abdollahi *et. al.*, 2006), center graph (Balakrishnan *et. al.*, 2011), and n-th central graph (Karimi *et. al.*, 2016).

The combinatorial information of a graph is stored in the coefficients of a specific graph polynomial, such as the independence polynomial. Hoede and Li (1994) had proved a few useful tools to calculate the independence polynomial. Ferrin (2014) had presented a few independence polynomials of some common graphs including the complete graph, complete bipartite graph, and path graph. Besides the common graphs, the independence polynomial can also be determined for graphs which are associated to groups.

This paper is structured into three parts. The first part is the introduction. The second part includes the preliminaries, namely the theorems that act as tools in this research. The last part is the main result. We will compute the independence polynomial of the n-th central graph of the dihedral groups G_1 and G_2, given in the following:

(i) $G_1 = \langle a, b : a^3 = b^2 = 1, bab = a^{-1} \rangle$

(ii) $G_2 = \langle a, b : a^4 = b^2 = 1, bab = a^{-1} \rangle$

PRELIMINARIES

Some basic concepts that are needed in order to compute the independence polynomial of a graph associated to a group are included in this section. Below are some definitions from graph theory.

Definition 2.1 (Rosen, 2013) Independent Set

An independent set is a set of vertices of a graph in which no two vertices in the set are adjacent.

Definition 2.2 (Rosen, 2013) Independence Number

The independence number of a graph Γ, denoted as $\alpha(\Gamma)$ is the maximum number of vertices in an independent set of vertices for the graph.

Definition 2.3 (Balakrishnan and Ranganathan, 2012) Neighborhood, Closed Neighborhood, Empty Graphy, Null Graph

Let (u,v) be an edge of graph Γ. Then the vertex u is called the neighbor of the vertex v. Open neighborhood (or just neighborhood), of v is the set of all vertices adjacent to v, denoted as follows :

$$N(v)=\{u \in V \mid (u,v) \in E,\, u \neq v\}.$$

Closed neighborhood of v in Γ is the set $N[v]=N(v) \cup \{v\}$.

If the neighborhood of every vertex is empty, then there is no edges in the graph. Such graph is called an empty graph, denoted by E_n. If $n=0$, then the graph is called null graph, denoted by $E_0 := \varnothing$.

The following are some basic properties regarding the concepts of independence polynomial that will be used in the computation of the independence polynomial of graph throughout this paper.

Definition 2.4 (Hoede and Li, 1994) Independence Polynomial

The independence polynomial of a graph Γ is the polynomial whose coefficient on x^k is given by the number of independent sets of size k in Γ. This is denoted by $I(\Gamma;x)$ as follows:

$$I(\Gamma;x)=\sum_{k=0}^{\alpha(\Gamma)} c_k x^k,$$

where c_k is the number of independent sets of size k in Γ and $\alpha(\Gamma)$ is the independence number of graph Γ.

However, it is difficult to compute the independence polynomial of a large graph only by using the definition. Hence, other researcher established some methods in finding the independence polynomial by reducing the calculations to recursively smaller graphs. The theorem below is the first method where it uses the relation of independence polynomial of disjoint graphs.

Theorem 2.1 (Hoede and Li, 1994)

Let Γ_1 and Γ_2 be two disjoint graphs. Then we have the independence polynomial of the union of two graphs as follows:

$$I(\Gamma_1 \cup \Gamma_2;x)=I(\Gamma_1;x)\cdot I(\Gamma_2;x).$$

Next theorem gives the method of decomposing the graph vertex by vertex. The vertex of the graph is removed one by one causing the graph to be separated into connected components and later by Theorem 2.1, the independence polynomial can be computed.

Theorem 2.2 (Hoede and Li, 1994)

Let Γ be a simple graph and $v \in V$. Then we obtain the independence polynomial of Γ by removing its vertex, as follows:

$$I(\Gamma;x)=I(\Gamma-v:x)+xI(\Gamma-N[v];x).$$

Another method to compute the independence polynomial of a graph is by removing the edges of the graph as stated in the next theorem. Ferrin (2014) mentioned that decomposing the graph by removing edges will yield a more obvious reccurrence.

Theorem 2.3 (Hoede and Li, 1994)

Let Γ be a simple graph and $e=(u,v) \in E$. Then the independence polynomial of Γ is obtained by removing its edge, as follows:

$$I(\Gamma;x)=I(\Gamma \setminus e:x)-x^2 I(\Gamma-(N[u]\cup N[v]);x).$$

Ferrin (2014) had presented three propositions below to compute the independence polynomial of graphs along with the Theorem 2.1, 2.2 and 2.3.

Proposition 2.1 (Ferrin, 2014)

The independence polynomial of an empty graph Γ of order n is $I(\Gamma;x)=(1+x)^n$.

Proposition 2.2 (Ferrin, 2014)

The independence polynomial of complete graph, K_n is $I(K_n;x)=nx+1$.

Proposition 2.3 (Ferrin, 2014)

Let G_1 and G_2 be two distinct groups with graph Γ_{G_1} and Γ_{G_2} associated to both groups respectively. If $G_1 \cong G_2$, then $\Gamma_{G_1} \cong \Gamma_{G_2}$, hence $I(\Gamma_{G_1};x) \cong I(\Gamma_{G_2};x)$.

Next, we will state some basic concepts from group theory and graph theory that mostly are related to n–th central graph that will be used in this paper.

Definition 2.5 (Fraleigh, 2003) Center

The center $Z(G)$ of a group G is defined by $Z(G)=\{a \in G \mid ag=ga \text{ for all } g \in G\}$.

Definition 2.6 (Balakrishnan et. al., 2011) Center Graph

Let $Z(G)$ be the center of a group G. The center graph $\Gamma_Z(G)$ of G is a graph with vertex set containing all the elements of G and two distinct vertices x and y are adjacent if and only if $xy \in Z(G)$.

Definition 2.7 (Fraleigh, 2003) Upper Central Series

The chain of normal subgroups

$$Z_0(G)=\{e\} \leq Z_1(G)=Z(G) \leq Z_2(G) \leq \ldots$$

is the upper central series of the group G where

$$Z_{i+1}(G)/Z_i(G)=Z\left(G/Z_i(G)\right) \text{ for } i \geq 0.$$

Definition 2.8 (Karimi *et. al.,* 2016) *n*-th Central Graph

Let G be a group and $Z_n(G)$ the n-th term of upper central series of G. The n-th central graph of G, denoted by $\Gamma_Z^n(G)$ is a graph with vertex set containing all elements of G and two vertices x and y are adjacent if and only if $xy \in Z_n(G)$. If $n=1$, then $\Gamma_Z^1(G)$ is the center graph.

Proposition 2.4 (Karimi *et. al.,* 2016)

Let G_1 be a dihedral group of order 6, $G_1=\langle a,b : a^3=b^2=1,\, bab=a^{-1}\rangle$. Then the n-th central graph of G_1, $\Gamma_Z^n(G_1)$, is a graph with four isolated vertices and one edge as follows:

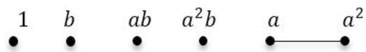


Fig. 1 The n-th Central Graph of G_1, $\Gamma_Z^n(G_1)$

Proposition 2.5 (Balakrishnan *et. al.,* 2016)

Let G_1 be a dihedral group of order 6, $G_1 = \langle a,b : a^3 = b^2 = 1, bab = a^{-1} \rangle$. The center graph of G_1, $\Gamma_Z^1(G_1)$, is a graph with four isolated vertices and one edge as in Figure 1.

Proposition 2.6 (Karimi *et. al.,* 2016)

Let G_2 be a dihedral group of order 8, $G_2 = \langle a,b : a^4 = b^2 = 1, bab = a^{-1} \rangle$. Then the n-th central graph of G_2, $\Gamma_Z^n(G_2)$, is a complete graph with 8 vertices as shown in Figure 2:

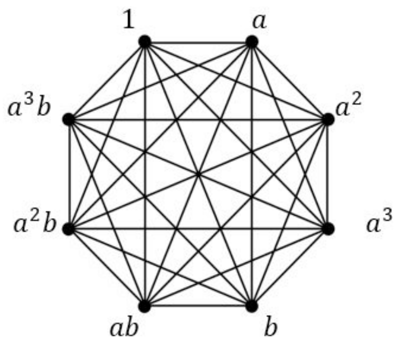


Fig. 2 The n-th Central Graph of G_2, $\Gamma_Z^n(G_2)$

Proposition 2.7 (Karimi *et. al.,* 2016)

Let G_2 be a dihedral group of order 8, $G_2 = \langle a,b : a^4 = b^2 = 1, bab = a^{-1} \rangle$. The center graph of G_2, $\Gamma_Z^1(G_2)$, is a graph with four edges as shown in Figure 3.

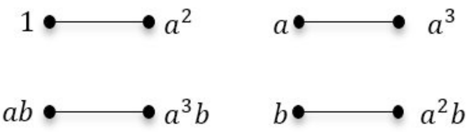


Fig. 3 The Center Graph of G_2, $\Gamma_Z^1(G_2)$

The aim of this paper is to compute the independence polynomial of four graphs: $\Gamma_Z^n(G_1)$, $\Gamma_Z^1(G_1)$, $\Gamma_Z^n(G_2)$ and $\Gamma_Z^1(G_2)$.

MAIN RESULTS

This section presents the results of this research. The independence polynomial of the graphs $\Gamma_Z^n(G_1)$, $\Gamma_Z^1(G_1)$, $\Gamma_Z^n(G_2)$ and $\Gamma_Z^1(G_2)$ are computed and presented.

The Independence Polynomial of the *n*-th Central Graph of Some Dihedral Groups

Based on the definitions, theorems and propositions mentioned in the previous section, the following results are obtained. The first theorem presents the independence polynomial of the n-th central graph of G_1.

Theorem 3.1

Let G_1 be a dihedral group of order 6, $G_1 = \langle a,b : a^3 = b^2 = 1, bab = a^{-1} \rangle$. Then the independence polynomial of the n-th central graph of G_1, $\Gamma_Z^n(G_1)$, is $I(\Gamma_Z^n(G_1);x) = 1 + 6x + 14x^2 + 16x^3 + 9x^4 + 2x^5$.

Proof The graph $\Gamma_Z^n(G_1)$ has the independence number $\alpha(\Gamma_Z^n(G_1)) = 5$. Based on the graph in Figure 1, there are two vertex sets of size 5 which are $\{1, a, b, ab, a^2b\}$ and $\{1, a^2, b, ab, a^2b\}$. The vertex sets of size 4 are $\{1, b, ab, a^2b\}$, $\{1, a, b, ab\}$, $\{1, a^2, b, ab\}$, $\{1, a, ab, a^2b\}$, $\{1, a^2, ab, a^2b\}$, $\{1, a, b, a^2b\}$. $\{1, a^2, b, a^2b\}$, $\{b, a, ab, a^2b\}$ and $\{b, a^2, ab, a^2b\}$. There are 16 vertex sets of size 3 and 14 vertex sets of size 2. And the vertex sets of size one are each set containing each vertex of G_1 denoted as 1, $\{a\}$, $\{a^2\}$, $\{b\}$, $\{ab\}$ and $\{a^2b\}$. Hence by Definition 2.4,

$$\begin{aligned} I(\Gamma_Z^n(G_1); \mathrm{x}) &= \sum_{k=0}^{5} c_k x^k \\ &= c_0x^0 + c_1x^1 + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 \\ &= 1 + 6x + 14x^2 + 16x^3 + 9x^4 + 2x^5. \end{aligned}$$

□

Next theorem shows the independence polynomial of the center graph of G_1.

Theorem 3.2

Let G_1 be a dihedral group of order 6, $G_1 = \langle a,b : a^3 = b^2 = 1, bab = a^{-1} \rangle$. Then the independence polynomial of the center graph of G_1, $\Gamma_Z^1(G_1)$, is $I(\Gamma_Z^1(G_1);x) = 1 + 6x + 14x^2 + 16x^3 + 9x^4 + 2x^5$.

Proof By Proposition 2.5, the center graph of G_1 is the same as the n-th central graph of G_1. Hence, the proof follows from the proof of Theorem 3.1. □

Then, the following theorem is the independence polynomial of the n-th central graph of G_2.

Theorem 3.3

Let G_2 be a dihedral group of order 8, $G_2 = \langle a,b : a^4 = b^2 = 1, bab = a^{-1} \rangle$. Then the independence polynomial of the n-th central graph of G_2, $\Gamma_Z^n(G_2)$, is $I(\Gamma_Z^n(G_2);x) = 1 + 8x$. □

Proof Since $\Gamma_Z^n(G_2) \cong K_8$, we have $I(\Gamma_Z^n(G_2);x) \cong I(K_8;x)$. Hence, by Proposition 2.2, we obtain $I(\Gamma_Z^n(G_2);x) = 1+8x$. □

And lastly, the theorem below is the independence polynomial of the center graph of G_2.

Theorem 3.4

Let G_2 be a dihedral group of order 8, $G_2 = \langle a,b : a^4 = b^2 = 1,\ bab = a^{-1} \rangle$. Then the independence polynomial of the center graph of G_2, $\Gamma_Z^1(G_2)$, is $I(\Gamma_Z^1(G_2);x) = 1+8x+24x^2+32x^3+16x^4$.

Proof Note that $\Gamma_Z^1(G_2) \cong K_2 \cup K_2 \cup K_2 \cup K_2$. From Proposition 2.2, we have that $I(K_2;x) = 1+2x$. Therefore by Theorem 2.1,

$I(\Gamma_Z^1(G_2);x) = (1+2x)^4 = 1+8x+24x^2+32x^3+16x^4$. □

CONCLUSION

In this paper, the independence polynomial of the n-th central and center graph of some dihedral groups are computed. For the group G_1, the independence polynomial of its n-th central graph is $I(\Gamma_Z^n(G_1);x) = 1+6x+14x^2+16x^3+9x^4+2x^5$ and when $n=1$, the independence polynomial of its center graph is also $I(\Gamma_Z^1(G_1);x) = 1+6x+14x^2+16x^3+9x^4+2x^5$. For the group G_2, the independence polynomial of its n-th central graph is $I(\Gamma_Z^n(G_2);x) = 1+8x$ since it is a complete graph with 8 vertices, K_8, and when $n=1$, the independence polynomial of its center graph is $I(\Gamma_Z^1(G_2);x) = 1+8x+24x^2+32x^3+16x^4$.

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