

A Subclass of Bi-univalent Functions Defined by Srivastava-Owa Differential Operator and Gegenbauer Polynomials

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Abstract By taking Srivastava-Owa derivative operator in conjunction with Gegenbauer polynomials, we introduced a subclass of bi-univalent functions in this article. Some inequalities, such as coefficient bounds and the Fekete–Szegő functionals are accessed. This research extends the application of fractional differential operator to orthogonal polynomials.

Keywords: Analytic functions; Bi-univalent functions; Srivastava-Owa operator; Gegenbauer polynomial; Coefficient estimate; Fekete-Szegő.

Introduction

In the past decades, fractional calculus has become a vital subject in mathematical analysis. By studying various possibilities of defining powers of derivative and integral operators, fractional calculus appears as a generalisation of the usual calculus. This in turn allows its significant adaption in the theory of analytic functions. In fact, fractional differential and integral operators have been applied to investigate on several properties of univalent functions, such as coefficient bounds and distortion inequalities. Srivastava-Owa operator ([31], refer Definition 2.1 and 2.2 below) was introduced based on fractional calculus and is implemented to define various subclasses of complex functions, for instance, [6, 15, 22, 23].

On the other hand, ordinary differential equations (ODEs) is often used to model mathematical problems. In many cases, solving ODEs would mean dealing with orthogonal polynomials. Orthogonal polynomials have been applied substantially in numerical analysis, theory of differential and integral equations and probability theory. The appearance in wide range of fields has endorse the importance of orthogonal polynomials in physics and engineering. In theory, two polynomials P_s and P_t of order s and t are orthogonal, if

$$\int_{a_1}^{a_2} \mathfrak{B}(\xi) P_s(\xi) P_t(\xi) d\xi = 0 \text{ for } s \neq t,$$

where function $\mathfrak{B}(\xi) \in [0, \infty)$ for $\xi \in (a_1, a_2)$.

The focus of this paper, Gegenbauer polynomials are exceptional classification of orthogonal polynomials, which generalise Legendre polynomials and Chebyshev polynomials. As stated in [18], since the generating function of Gegenbauer polynomials has common algebraic expressions with the integral representation of typically real functions, many practical inequalities have been derived for Gegenbauer polynomials. This exaggerates their instrumental applications in various areas.

By associating Gegenbauer polynomials with certain bi-univalent functions defined by Srivastava-Owa differential operator, some properties of this subclass will be inspected in this paper.

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Preliminaries

Suppose that \mathcal{A} denotes the class of analytic functions expressed in terms of power series

$$\mathfrak{F}(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}, \tag{1.1}$$

with normalisation map $\mathfrak{F}(0) = 0 = \mathfrak{F}'(0) - 1$. Subclass of univalent functions in \mathcal{A} is indicated as \mathcal{S} . Besides that, a function ϕ is regarded to be in the class of functions with positive real part, \mathcal{P} , if it maps the unit disk \mathbb{U} to the right half complex plane, or algebraically if it satisfies

$$\Re(\phi(z)) > 0, \quad \phi(0) = 1.$$

An analytic function \mathfrak{F} is said as subordinate to another function \mathfrak{G} , if there exists a Schwarz function Φ which satisfies $\Phi(0) = 0, |\Phi(z)| < 1$ and $\mathfrak{F}(z) = \mathfrak{G}(\Phi(z))$ for all $z \in \mathbb{U}$. This relationship is denoted as $\mathfrak{F} < \mathfrak{G}$. For univalent \mathfrak{G} , the following equivalence is valid.

$$\mathfrak{F}(z) < \mathfrak{G}(z) \Leftrightarrow \mathfrak{F}(0) = \mathfrak{G}(0) \text{ and } \mathfrak{F}(\mathbb{U}) \subset \mathfrak{G}(\mathbb{U}).$$

Readers may refer to [21] for further details on subordination principles.

By Koebe's one-quarter theorem [13], if \mathfrak{F} is a biholomorphic analytic function from \mathbb{U} onto a domain in the complex plane, then its image must contain the disk $|w| < \frac{1}{4}$. In short, inverse map \mathfrak{F}^{-1} exists only when

$$\mathfrak{F}^{-1}(\mathfrak{F}(z)) = z (z \in \mathbb{U}) \text{ and } \mathfrak{F}(\mathfrak{F}^{-1}(w)) = w \left(|w| < \rho_0(\mathfrak{F}); \rho_0(\mathfrak{F}) \geq \frac{1}{4} \right).$$

For functions in the form (1.1), its inverse $\mathfrak{G} = \mathfrak{F}^{-1}$ can be represented as

$$\mathfrak{G}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

The class of bi-univalent functions (usually denoted as Σ) is the collection of all functions \mathfrak{F} in which both \mathfrak{F} and its inverse map \mathfrak{F}^{-1} are univalent in \mathbb{U} . Examples of bi-univalent functions are

$$\log \frac{1}{1-z}, \quad \frac{z}{1-z}, \quad \log \sqrt{\frac{1+z}{1-z}}, \quad \dots$$

In contrast, the famous Koebe function is an obvious example of non bi-univalent function, since there is no value of $z \in \mathbb{U}$ which is mapped onto the slit of real axis from $-\infty$ to $-\frac{1}{4}$.

Estimation of coefficient bounds for analytic functions in class Σ is a major theme in mathematical research. Early literature on this topic includes Lewin [19], Brannan and Clunie [11, 27] and Netanyahu [24]. Tan [32] in 1984 obtained the best non-sharp estimate $|a_2| < 1.485$. Nonetheless, the estimation of coefficients other than $|a_1|$ and $|a_2|$ is lacking due to complication of calculations.

When $\mathfrak{F} \in \mathcal{A}$ is defined in a simply connected region of the complex z -plane which contains the origin, we can interpret the fractional derivative and integral of order “ γ ” as follows.

Definition 2.1 [31] For $0 \leq \Upsilon < 1$, the fractional derivative of \mathfrak{E} is presumed as

$$D_z^\Upsilon \mathfrak{E}(z) = \frac{1}{\Gamma(1-\Upsilon)} \frac{d}{dz} \int_0^z \frac{\mathfrak{E}(\zeta)}{(z-\zeta)^\Upsilon} d\zeta,$$

where $\log(z-\zeta)$ must be real whenever $(z-\zeta) > 0$ to remove the multiplicity of $(z-\zeta)^{-\Upsilon}$.

Definition 2.2 For $0 \leq \Upsilon < 1$, the fractional integral of \mathfrak{E} is presumed as

$$I_z^\Upsilon \mathfrak{E}(z) = \frac{1}{\Gamma(\Upsilon)} \int_0^z \frac{\mathfrak{E}(\zeta)}{(z-\zeta)^{1-\Upsilon}} d\zeta,$$

where $\log(z-\zeta)$ must be real whenever $(z-\zeta) > 0$ to remove the multiplicity of $(z-\zeta)^{\Upsilon-1}$.

Based on Definition 2.1, Owa and Srivastava [25, 28, 30] established a fractional derivative operator (also known as Srivastava-Owa operator) as follows.

$$(\Omega_z^\Upsilon \mathfrak{E})(z) = \Gamma(2-\Upsilon) z^\Upsilon (D_z^\Upsilon \mathfrak{E})(z),$$

where $\Upsilon \neq 2, 3, 4, \dots$ and $z \in \mathbb{U}$.

In particular, we have $(\Omega_z^0 \mathfrak{E})(z) = \mathfrak{E}(z)$ and $(\Omega_z^1 \mathfrak{E})(z) = z\mathfrak{E}'(z)$.

Another subject of this paper, Gegenbauer polynomials can be determined using the recurrence relations [12, 20, 26]

$$C_n^\alpha(x) = \frac{2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-2)C_{n-2}^\alpha(x)}{n}.$$

This relation gives the initial values of Gegenbauer polynomials as follows.

$$\begin{aligned} C_0^\alpha(x) &= 1, \\ C_1^\alpha(x) &= 2\alpha x, \\ \text{and } C_2^\alpha(x) &= 2\alpha(1+\alpha)x^2 - \alpha. \end{aligned}$$

In addition, Gegenbauer polynomials can also be derived using a generating function

$$\mathfrak{S}_\alpha(x, z) = \frac{1}{(1-2xz+z^2)^\alpha},$$

where $z \in \mathbb{U}$, $\alpha \in \mathbb{R} - \{0\}$ and $x \in [-1, 1]$. It should be mentioned that \mathfrak{S}_0 generates nothing, and is taken as

$$\mathfrak{S}_0(x, z) = 1 - \log(1-2xz+z^2) = \sum_{n=0}^{\infty} C_n^0(x)z^n.$$

Since \mathfrak{S}_α is analytic in unit disk \mathbb{U} when x is fixed, it is possible to convey \mathfrak{S}_α as

$$\mathfrak{S}_\alpha(x, z) = \sum_{n=0}^{\infty} C_n^\alpha(x)z^n,$$

where $C_n^\alpha(x)$ is a Gegenbauer polynomial of degree n .

Deploying Srivastava-Owa operator and Gegenbauer polynomials, we define a new subclass of bi-univalent functions which satisfies the following subordinations.

Definition 2.3 A function $\mathfrak{F} \in \Sigma$ is said to be in class $\mathfrak{D}_{\Sigma}^{\Upsilon}(\mathfrak{S})$ if

$$\frac{(\Omega_z^{\Upsilon} \mathfrak{F})(z)}{z} \prec \mathfrak{S}(x, z)$$

and

$$\frac{(\Omega_w^{\Upsilon} \mathfrak{G})(w)}{w} \prec \mathfrak{S}(x, w)$$

for all $z, w \in \mathbb{U}$.

Thoroughly detailed in [17], the following lemmas are essential to produce our results.

Lemma 2.1 Let $\phi(z) = 1 + \sum_{n=1}^{\infty} \phi_n z^n$ be a function of class \mathcal{P} . Then $|\phi_n| \leq 2, \forall n \in \mathbb{Z}^+$.

According to [5], the following principle can be found in [17].

Lemma 2.2 Let $A_1, A_2 \in \mathbb{R}, B_1, B_2 \in \mathbb{C}$ and $|B_1| < r, |B_2| < r$. Then

$$|(A_1 + A_2)B_1 + (A_1 - A_2)B_2| \leq \begin{cases} 2r|A_1|, & \text{if } |A_1| \geq |A_2| \\ 2r|A_2|, & \text{if } |A_1| \leq |A_2| \end{cases}.$$

Recently, many researchers studied the bi-univalent functions associated with Gegenbauer polynomials. For example, [2–5, 9, 10, 16, 29] have done their studies on subclasses established by applying various operators to Gegenbauer polynomials. We have also seen many studies on subclasses derived using Srivastava-Owa operators, such as [1, 6–8, 23]. Since no such study has been proposed previously, we consider a new subclass of bi-univalent functions as defined by Definition 2.3. The aim of this paper is to tackle the problems of coefficient bounds and the Fekete-Szegő functional for this new subclass.

Coefficient Bounds for the Class $\mathfrak{D}_{\Sigma}^{\Upsilon}(\mathfrak{S})$

This section evaluates the bounds for coefficients $|a_2|$ and $|a_3|$.

Theorem 3.1 Let $\mathfrak{F} \in \mathfrak{D}_{\Sigma}^{\Upsilon}(\mathfrak{S})$ with $x \in (0,1], \alpha > 0$, then

$$|a_2| \leq \sqrt{\frac{4\alpha^2 x^3 (2 - \Upsilon)^2 (3 - \Upsilon)}{6\alpha x^2 (2 - \Upsilon) - [(2\alpha + 2)x^2 - 1 - 2x](3 - \Upsilon)'}}$$

and

$$|a_3| \leq \frac{\alpha x (3 - \Upsilon)(2 - \Upsilon)}{3} + \alpha^2 x^2 (2 - \Upsilon)^2.$$

Proof. Assume that two functions \mathfrak{F} and $\mathfrak{G} = \mathfrak{F}^{-1}$ established as in Definition 2.3. Then we can find two Schwarz functions u and v satisfying

$$\frac{(\Omega_z^{\Upsilon} \mathfrak{F})(z)}{z} = \mathfrak{S}_{\alpha}(x, u(z)), \tag{3.1}$$

and

$$\frac{(\Omega_w^\gamma \mathcal{G})(w)}{w} = \mathfrak{S}_\alpha(x, v(w)). \tag{3.2}$$

Using u and v , two new Caratheodory analytic functions $\phi(z)$ and $\psi(w)$ are derived, which can be expressed as follows

$$\phi(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \phi_1 z + \phi_2 z^2 + \dots,$$

and

$$\psi(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + \psi_1 w + \psi_2 w^2 + \dots.$$

with $\phi(0) = 1 = \psi(0)$, and constraints $|\phi_i| \leq 2, |\psi_i| \leq 2, \forall i \in \mathbb{Z}^+$.

We can express u and v as

$$u(z) = \frac{\phi(z) - 1}{\phi(z) + 1} = \frac{\phi_1}{2} z + \left(\frac{\phi_2}{2} - \frac{\phi_1^2}{4}\right) z^2 + \dots,$$

and

$$v(w) = \frac{\psi(w) - 1}{\psi(w) + 1} = \frac{\psi_1}{2} w + \left(\frac{\psi_2}{2} - \frac{\psi_1^2}{4}\right) w^2 + \dots.$$

Therefore, we have

$$\mathfrak{S}_\alpha(x, u(z)) = 1 + \frac{\phi_1}{2} C_1^\alpha(x) z + \left[\left(\frac{\phi_2}{2} - \frac{\phi_1^2}{4}\right) C_1^\alpha(x) + \frac{\phi_1^2}{4} C_2^\alpha(x)\right] z^2 + \dots,$$

and

$$\mathfrak{S}_\alpha(x, v(w)) = 1 + \frac{\psi_1}{2} C_1^\alpha(x) w + \left[\left(\frac{\psi_2}{2} - \frac{\psi_1^2}{4}\right) C_1^\alpha(x) + \frac{\psi_1^2}{4} C_2^\alpha(x)\right] w^2 + \dots.$$

Compare the coefficients of (3.1) and (3.2), we acquire these four equations.

$$\frac{2}{2 - \gamma} a_2 = \frac{\phi_1}{2} C_1^\alpha(x), \tag{3.3}$$

$$\frac{6}{(3 - \gamma)(2 - \gamma)} a_3 = \left(\frac{\phi_2}{2} - \frac{\phi_1^2}{4}\right) C_1^\alpha(x) + \frac{\phi_1^2}{4} C_2^\alpha(x) \tag{3.4}$$

$$\frac{-2}{2 - \gamma} a_2 = \frac{\psi_1}{2} C_1^\alpha(x), \tag{3.5}$$

and

$$\frac{6}{(3 - \gamma)(2 - \gamma)} (2a_2^2 - a_3) = \left(\frac{\psi_2}{2} - \frac{\psi_1^2}{4}\right) C_1^\alpha(x) + \frac{\psi_1^2}{4} C_2^\alpha(x). \tag{3.6}$$

It can be easily derived from (3.3) and (3.5) that

$$\phi_1 = -\psi_1. \tag{3.7}$$

Taking the sum of squares of (3.3) and (3.5), we achieve

$$\frac{8}{(2 - \gamma)^2} a_2^2 = \frac{[C_1^\alpha(x)]^2}{4} (\phi_1^2 + \psi_1^2). \tag{3.8}$$

In addition, summing (3.4) and (3.6), we have

$$\begin{aligned} \frac{12}{(3-\gamma)(2-\gamma)} a_2^2 &= C_1^\alpha(x) \left(\frac{\phi_2 + \psi_2}{2} - \frac{\phi_1^2 + \psi_1^2}{4} \right) + C_2^\alpha(x) \left(\frac{\phi_1^2 + \psi_1^2}{4} \right) \\ &= C_1^\alpha(x) \left(\frac{\phi_2 + \psi_2}{2} \right) + [C_2^\alpha(x) - C_1^\alpha(x)] \left(\frac{\phi_1^2 + \psi_1^2}{4} \right). \end{aligned} \tag{3.9}$$

Substitute (3.8) into (3.9), eliminate the denominator and express in terms of a_2^2 , we get

$$\begin{aligned} \frac{12}{(3-\gamma)(2-\gamma)} a_2^2 &= C_1^\alpha(x) \left(\frac{\phi_2 + \psi_2}{2} \right) + [C_2^\alpha(x) - C_1^\alpha(x)] \left(\frac{8}{[C_1^\alpha(x)]^2(2-\gamma)^2} \right) a_2^2, \\ 24[C_1^\alpha(x)]^2(2-\gamma)a_2^2 &= [C_1^\alpha(x)]^3(2-\gamma)^2(3-\gamma)(\phi_2 + \psi_2) + 16[C_2^\alpha(x) - C_1^\alpha(x)](3-\gamma)a_2^2, \end{aligned}$$

and therefore

$$a_2^2 = \frac{[C_1^\alpha(x)]^3(2-\gamma)^2(3-\gamma)(\phi_2 + \psi_2)}{24[C_1^\alpha(x)]^2(2-\gamma) - 16[C_2^\alpha(x) - C_1^\alpha(x)](3-\gamma)}. \tag{3.10}$$

Since $|\phi_i| \leq 2$ and $|\psi_i| \leq 2$, we have

$$|a_2|^2 \leq \frac{[C_1^\alpha(x)]^3(2-\gamma)^2(3-\gamma)}{3[C_1^\alpha(x)]^2(2-\gamma) - 2[C_2^\alpha(x) - C_1^\alpha(x)](3-\gamma)}. \tag{3.11}$$

Substitute the values of $C_n^\alpha(x)$, (3.11) can be simplified as

$$\begin{aligned} |a_2|^2 &\leq \frac{8\alpha^3 x^3(2-\gamma)^2(3-\gamma)}{12\alpha^2 x^2(2-\gamma) - 2[2\alpha(\alpha+1)x^2 - \alpha - 2\alpha x](3-\gamma)}, \\ &= \frac{4\alpha^2 x^3(2-\gamma)^2(3-\gamma)}{6\alpha x^2(2-\gamma) - [(2\alpha+2)x^2 - 1 - 2x](3-\gamma)}. \end{aligned}$$

Then our result is acquired.

Note that the expression in the denominator of $|a_2|$ is always positive within the admissible range. The proof is given as follows: Let $f(\alpha, x, \gamma) = 6\alpha x^2(2-\gamma) - [(2\alpha+2)x^2 - 1 - 2x](3-\gamma)$.

$$\frac{\partial f}{\partial \alpha} = 6x^2(2-\gamma) - 2x^2(3-\gamma) = (6-4\gamma)x^2$$

which is positive for $x \in (0,1]$ and $\gamma \in [0,1)$, so f is minimum when $\alpha \rightarrow 0$. We acquire

$$f > \lim_{\alpha \rightarrow 0^+} f = (-2x^2 + 2x + 1)(3-\gamma).$$

The expression $-2x^2 + 2x + 1$ is a quadratic function which concave downwards, with line of symmetry $x = \frac{1}{2}$. Therefore, the minimum value must occur when $x \rightarrow 0$ or $x = 1$. This deduces that $f \geq (3-\gamma) > 0$.

Subsequently, we estimate the bounds of coefficient $|a_3|$. By subtracting (3.6) from (3.4),

$$\frac{6}{(3-\gamma)(2-\gamma)} (2a_3 - 2a_2^2) = \left(\frac{\phi_2 - \psi_2}{2} - \frac{\phi_1^2 - \psi_1^2}{4} \right) C_1^\alpha(x) + \left(\frac{\phi_1^2 - \psi_1^2}{4} \right) C_2^\alpha(x).$$

Then (3.7) implies that $\phi_1^2 - \psi_1^2 = 0$, hence

$$\frac{6}{(3-\gamma)(2-\gamma)} (2a_3 - 2a_2^2) = \left(\frac{\phi_2 - \psi_2}{2} \right) C_1^\alpha(x).$$

Express the equation in terms of a_3 , we obtain

$$a_3 = \frac{C_1^\alpha(x)(3 - \Upsilon)(2 - \Upsilon)(\phi_2 - \psi_2)}{24} + a_2^2. \tag{3.12}$$

By substituting (3.8) into (3.12),

$$a_3 = \frac{C_1^\alpha(x)(3 - \Upsilon)(2 - \Upsilon)(\phi_2 - \psi_2)}{24} + \frac{[C_1^\alpha(x)]^2(2 - \Upsilon)^2(\phi_1^2 + \psi_1^2)}{32}.$$

Lemma 2.1 gives the constraints $|\phi_2| \leq 2$ and $|\psi_2| \leq 2$ and by applying triangle inequality, we obtain

$$|a_3| \leq \frac{C_1^\alpha(x)(3 - \Upsilon)(2 - \Upsilon)}{6} + \frac{[C_1^\alpha(x)]^2(2 - \Upsilon)^2}{4}.$$

Replacing the expression of $C_1^\alpha(x)$, we arrive at

$$|a_3| \leq \frac{\alpha x(3 - \Upsilon)(2 - \Upsilon)}{3} + \alpha^2 x^2(2 - \Upsilon)^2,$$

which is our desired result.

Fekete-Szegő Inequality for the Class $\mathfrak{D}_\Sigma^\Upsilon(\xi)$

The bounds of Fekete-Szegő functional for the class $\mathfrak{D}_\Sigma^\Upsilon(\xi)$ would be accessed in this section. Initially introduced in [14], Fekete and Szegő specifies that for univalent functions \mathfrak{E} , the inequality

$$|a_3 - \eta a_2^2| \leq 1 + 2e^{-2\eta/(1-\eta)}$$

is sharp when $\eta \in \mathbb{R}, 0 \leq \eta < 1$.

Theorem 4.1 Let $\mathfrak{E} \in \mathfrak{D}_\Sigma^\Upsilon(\xi)$, then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha||x|(3 - \Upsilon)(2 - \Upsilon)}{3}, & \text{if } \eta \in [\eta_1, \eta_2] \\ \frac{2\alpha^2 x^3(1 - \eta)(3 - \Upsilon)(2 - \Upsilon)^2}{6\alpha x^2(2 - \Upsilon) - [(2\alpha + 2)x^2 - 1 - 2x](3 - \Upsilon)}, & \text{if } \eta \notin [\eta_1, \eta_2] \end{cases}.$$

where

$$\eta_1 = \frac{[(2\alpha + 2)x^2 - 1 - 2x](3 - \Upsilon)}{6\alpha x^2(2 - \Upsilon)},$$

and

$$\eta_2 = 2 - \frac{[(2\alpha + 2)x^2 - 1 - 2x](3 - \Upsilon)}{6\alpha x^2(2 - \Upsilon)}.$$

Proof. Let $\mathfrak{E} \in \mathfrak{D}_\Sigma^\Upsilon(\xi)$. From (3.10) and (3.12), there exists some real number η such that

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{C_1^\alpha(x)(3 - \Upsilon)(2 - \Upsilon)(\phi_2 - \psi_2)}{24} + a_2^2 - \eta a_2^2 \\ &= \frac{\alpha x(3 - \Upsilon)(2 - \Upsilon)(\phi_2 - \psi_2)}{12} + \frac{\alpha^2 x^3(1 - \eta)(2 - \Upsilon)^2(3 - \Upsilon)(\phi_2 + \psi_2)}{12\alpha x^2(2 - \Upsilon) - 2[(2\alpha + 2)x^2 - 1 - 2x](3 - \Upsilon)} \\ &= |\alpha||x|(3 - \Upsilon)(2 - \Upsilon) \left[\frac{\phi_2 - \psi_2}{12} + \frac{\alpha x^2(1 - \eta)(2 - \Upsilon)(\phi_2 + \psi_2)}{12\alpha x^2(2 - \Upsilon) - 2[(2\alpha + 2)x^2 - 1 - 2x](3 - \Upsilon)} \right] \\ &= |\alpha||x|(3 - \Upsilon)(2 - \Upsilon) \left[\left(\kappa + \frac{1}{12}\right)\phi_2 + \left(\kappa - \frac{1}{12}\right)\psi_2 \right], \end{aligned}$$

where

$$\kappa = \frac{\alpha x^2(1-\eta)(2-\Upsilon)}{12\alpha x^2(2-\Upsilon) - 2[(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)}.$$

Since $|\phi_2| < 2$, $|\psi_2| < 2$, by applying Lemma 2.2, we acquire

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha||x|(3-\Upsilon)(2-\Upsilon)}{3}, & \text{if } |\kappa| \leq \frac{1}{12} \\ \frac{2\alpha^2 x^3(1-\eta)(3-\Upsilon)(2-\Upsilon)^2}{6\alpha x^2(2-\Upsilon) - [(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)}, & \text{if } |\kappa| \geq \frac{1}{12} \end{cases}.$$

On the other hand, from $|\kappa| \leq \frac{1}{12}$, we get

$$\left| \frac{\alpha x^2(1-\eta)(2-\Upsilon)}{12\alpha x^2(2-\Upsilon) - 2[(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)} \right| \leq \frac{1}{12}.$$

After some calculations, we obtain

$$|1-\eta| \leq 1 - \frac{[(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)}{6\alpha x^2(2-\Upsilon)}$$

and hence

$$\frac{[(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)}{6\alpha x^2(2-\Upsilon)} \leq \eta \leq 2 - \frac{[(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)}{6\alpha x^2(2-\Upsilon)}.$$

Hence,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha||x|(3-\Upsilon)(2-\Upsilon)}{3}, & \text{if } \eta \in [\eta_1, \eta_2] \\ \frac{2\alpha^2 x^3(1-\eta)(3-\Upsilon)(2-\Upsilon)^2}{6\alpha x^2(2-\Upsilon) - [(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)}, & \text{if } \eta \notin [\eta_1, \eta_2] \end{cases}.$$

where

$$\eta_1 = \frac{[(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)}{6\alpha x^2(2-\Upsilon)},$$

and

$$\eta_2 = 2 - \frac{[(2\alpha+2)x^2 - 1 - 2x](3-\Upsilon)}{6\alpha x^2(2-\Upsilon)}.$$

The proof is now complete.

Corollary 4.1 Assume that $\mathfrak{G} \in \mathcal{D}_{\Sigma}^{\Upsilon}(\xi)$, then

$$|a_3 - a_2^2| \leq \frac{|\alpha||x|(3-\Upsilon)(2-\Upsilon)}{3}.$$

Proof. Consider $\eta = 1$ in Theorem 4.1.

Conclusions

Using the Srivastava-Owa differential operator, this research establishes a new subclass of bi-univalent functions affiliated with Gegenbauer polynomials. For this subclass, bounds for some coefficients are estimated. Using these bounds, the Fekete-Szegő inequality is evaluated. The findings are expected to provide deeper insight into the geometric properties of the class.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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