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RESEARCH ARTICLE

Laplace Variational Iteration Method for Solving Conformable Fractional Order Integro-Differential Equations

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Abstract. This paper aims to integrate the Laplace transformation method with the variational iteration method to deliver an analytical approximate solution for fractional-order integro-differential equations, where the fractional-order derivative and integration are defined in the conformable sense. The iterative solution sequence is obtained using the Laplace variational iteration method, and the convergence of this sequence of approximate solutions to the exact solution is established and demonstrated. First, we shall study the approximate solution of a linear fractional integro-differential equation, and secondly, solve the nonlinear fractional integro-differential equations modeled using conformable differointegration. Some illustrative examples are considered to verify the validity and accuracy of the proposed technique, in which approximate solutions are compared with the exact solutions if they exist. Through the comparison, we conclude that the present hybrid approach is very effective for solving this type of problem.

Keywords: Integro-differential equations, Laplace variational iteration method, conformable fractional-order.

Introduction

Numerous scientific and technical phenomena, including heat conduction in memory-containing materials, diffusion processes, and others, have been effectively modeled by fractional order integro-differential equations (FIDEs). [1]. Also, in recent years, the study of FIDEs as a basic theoretical part of some applications has been investigated and has attracted more interest from many authors. [2].

Differentiation and integration of fractional order are introduced through different definitions, such as Riemann-Liouville, Caputo, Grünwald, etc. In addition, a new concept of definition of fractional derivative and integral was given by the researcher Khalil *et al*. in 2014 [3,4], which is called the conformable fractional derivatives and integrals. It has interested many researchers because it includes several properties that correlate to the classical derivatives and integrals, especially the Leibniz laws. Its description is more precise and welcoming than other fractional definitions since it has received more extensive coverage of usual differointegration properties. Conformable fractional derivatives are usable to describe many different applications and phenomena, and they provide several intriguing advantages. For instance, a local derivative can be used in place of the basic derivative because it is based on the formulation's limit. [5]. As research progresses, the conformable derivative is anticipated to remain a crucial factor in broadening our comprehension across various scientific fields. The theory of curves involves analyzing the movement of a point within a plane or space, employing methods from linear algebra and calculus. Over the past decade, notable advancements in the literature reveal that fractional calculus has begun to be applied to curves and surfaces within differential geometry. [6] The pioneering work in this area was carried out by T. Yajima and K. Kamasaki, who studied surfaces using fractional calculus. [7] Subsequently, T. Yajima and colleagues derived Frenet formulas incorporating fractional derivatives. [8] Another significant contribution came from K.A. Lazopoulos and A.K. Lazopoulos, who

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explored fractional differentiable manifolds.[9] Further research by M.E. Aydın and collaborators examined plane curves within equiaffine geometry using fractional orders,[10] while U. Gozutok and colleagues analyzed essential concepts relating to curves and the Frenet frame in fractional order through the conformable local fractional derivative. [11] In a similar vein, A. Has and B. Yılmaz focused on specific curves and curve pairs in fractional order utilizing the conformable Frenet frame [12,13]; they also investigated electromagnetic fields and magnetic curves under the framework of conformable derivatives [14-16]. Numerous other studies continue to build on this evolving area of research, contributing valuable insights and reinforcing its importance in applied mathematics and geometry. [17-19].

An important class of FIDEs of the following form is examined in this work:

$$\mathcal{T}_\alpha^t\, \mathcal{U}(t) = g(t) + \mathcal{J}_\beta^t\, k\mathcal{U}(t),\ \ \mathcal{U}(0) = \mathcal{U}_0,\ \ t \geq 0 \tag{1}$$

where the operator $\mathcal{J}_\beta^t$ is the conformable fractional integral of order β, and the operator $\mathcal{T}_\alpha^t$ is the conformable fractional derivative of order $\alpha, 0 < \beta \leq \alpha \leq 1$, $g(t)$ is a known analytic function, k is the kernel of integral equation, $\mathcal{U}$ is a continuous function, and $\mathcal{U}_0 \in \mathbb{R}$, and eq. (1) may be written equivalently in the following explicit form as conformable fractional order integro-differential equations.

$$\mathcal{T}_\alpha^t\, \mathcal{U}(t) = g(t) + \int_a^t v^{\alpha-1}\, \mathcal{U}(v)dv$$

The researcher Ji-Huan He first proposed this technique, the variational iteration method (VIM), in 1999. [20] and some a rudimentary introduction to VIM and its recent advances, along with fresh interpretations, is available in [21,22]. This strategy has been effectively utilized for addressing several types of non-linear problems [23,24]. In 2009, Wen-Hua Wang employed the VIM to address specific categories of FIDEs [25]. In 2011, the investigators Muhammet and Adin employed the VIM to address the issue of non-linear FIDEs [26]. There are many types of mixed approximation methods solution to solving Eq. (1), like Laplace-Adomian decomposition method, modified Laplace-decomposition method, Laplace-homotopy analysis method, Laplace-homotopy perturbation method, Variational Iteration Least Square Methods [27] etc. In summary, the homotopy analysis method facilitates the establishment of a continuous mapping from an initial guess approximation to the exact solution of the given equation, offering considerable flexibility in selecting starting approximations and auxiliary linear operators [28].

A combination of the VIM and Laplace transformation method is used by Hesameddini and Latifizadeh in 2009 [29]. Some drawbacks of the VIM and how to overcome them using the Laplace transform are discussed by Wn in 2012 [30]. Laplace VIM with non-singular kernels for modified fractional derivatives is used by Martinez and Gomez-Agnilar in 2020 [31]. A new method of Laplace transformation and variational iteration technique has been proposed by Elzaki in 2018 for solving nonlinear partial differential equations [32]. A new Laplace VIM has also been proposed for solving the three-dimensional Schrödinger equation, the two-dimensional telegraph equation, the three-dimensional telegraph equation, and the 3-dimensional diffusion and wave equations as they are discussed [33,34]. It is particularly noteworthy that this method cleverly combines the Laplace transformation method with VIM. The approximate solution iteration is one of the main advantages of the proposed VIM method of this work. The main advantage of using the Laplace transform in combination with VIM yields more precise results than conventional methods and articulates the answer through quick iterations using sophisticated computational terminology.

Preliminaries:

In this section, we shall give some of the most fundamental definitions and basic concepts concerning fractional calculus, which might be important for the present work, so that other related concepts can be found in the related literature.

Definition 1, [35,3]**.** In this definition, we will review the conformable fractional derivative of $\mathcal{U}$ of order α, when $0 < \alpha \leq 1$, and we give a function $\mathcal{U} :\ [0, \infty) \to \mathbb{R}$, as follows:

$$\mathcal{T}_\alpha \mathcal{U}(t) = \lim_{\theta \to 0} \frac{\mathcal{U}(t+\theta(t)^{1-\alpha}) - \mathcal{U}(t)}{\theta}\ , \text{for all } t \geq 0. \tag{2}$$

Any real function that satisfies Eq. (2) and possesses a corresponding limit is termed an $\alpha-$differentiable function. Furthermore, if $\mathcal{U}$ is $\alpha-$differentiable at some $t \in (0, b)$ and $\lim_{t\to 0^+} \mathcal{T}_\alpha \mathcal{U}(t)$ exists, then we delineate:

$$\mathcal{T}_\alpha \mathcal{U}(0) = \lim_{t\to 0^+} \mathcal{T}_\alpha \mathcal{U}(t).$$

This is the representation of the correlation between the first derivative and conformable derivative:

$$\mathcal{T}_\alpha \mathcal{U}(t) = t^{1-\alpha} \mathcal{U}^{\prime}(t), \mathcal{U} \in C^1(0, \infty) \tag{3}$$

Examine the limit when α approaches 1 from the left. For $t > 0$, We obtain the conventional definition of a function's derivative, $\mathcal{T}_1 \mathcal{U}(t) = \mathcal{U}^{\prime}(t)$. This signifies that the conformable derivative extends the concept of the integer-order derivative. Moreover, the physics interpretation of the conformable derivative signifies a modification of the classical derivative in terms of both magnitude and direction. [36].

$$ {}_{\alpha}^{b}\mathcal{T}\mathcal{U}(b) = \lim_{t\to b^-} {}^{b}\mathcal{T}_{\alpha}\mathcal{U}(t) $$

Definition 2, [35,4]. This is the definition of the left conformable fractional integral of order α for a continuous function $\mathcal{U}: [0,\infty) \to \mathbb{R}$.

$$ \mathcal{I}_{\alpha}\mathcal{U}(t) = \int_a^t \mathcal{U}(v)d\alpha(v,a) = \int_a^t \frac{\mathcal{U}(v)}{v^{1-\alpha}}dv $$
$$ = \int_a^t v^{\alpha-1}\mathcal{U}(v)dv = \mathcal{I}_1 t^{\alpha-1}\mathcal{U}(t), \quad t \geq 0, $$

Definition 3, [35,4]. At every time $t \geq 0$, the following is the definition of the conformable fractional exponential function:

$$ E_{\alpha}(\lambda,t) = e^{\left(\lambda\frac{t^{\alpha}}{\alpha}\right)} $$

And λ is a constant, $\lambda \in \mathbb{R}$ where $\alpha \in (0,1]$.
Derived from Eq. (3):

$$ \mathcal{T}_{\alpha}E_{\alpha}(\lambda,t) = \lambda E_{\alpha}(\lambda,t), $$

The famous stretched exponential function, the eigen-function $E_{\alpha}(-1,t)$ [37] of $\mathcal{T}_{\alpha}$ has an eigen-value of one. A conformable fractional exponential function can also be the short-time limit of a one-parameter Mittag–Leffler function [38-40].

In the following, various properties of the fractional derivative of order α it is useful, where $\mathcal{U}$ and z are constant functions and $\mathcal{T}_{\alpha}$ is related to fractional conformable differentiation [3]:

1. $\mathcal{T}_{\alpha}(c_1\mathcal{U} + c_2 z) = c_1\mathcal{T}_{\alpha}(\mathcal{U}) + c_2\mathcal{T}_{\alpha}(z), \forall\, c_1, c_2 \in \mathbb{R}$.
2. $\mathcal{T}_{\alpha}(\mathcal{U}z) = \mathcal{U}\mathcal{T}_{\alpha}(z) + z\mathcal{T}_{\alpha}(\mathcal{U})$.
3. $\mathcal{T}_{\alpha}\left(\frac{\mathcal{U}}{z}\right) = \frac{z\mathcal{T}^{\alpha}(\mathcal{U}) - \mathcal{U}\mathcal{T}^{\alpha}(z)}{z^2}, z \neq 0$.
4. $\mathcal{T}_{\alpha}(t^p) = pt^{p-\alpha}, \forall p \in \mathbb{R}$.
5. $\mathcal{T}_{\alpha}(c) = 0, \forall c \in \mathbb{R}$.
6. If in addition, $\mathcal{U}$ is differentiable, then $\mathcal{T}_{\alpha}\mathcal{U}(t) = t^{1-\alpha}\frac{d\mathcal{U}}{dt}(t)$, when $\alpha \in (0,1], t > 0$.

Remark 1, [4].

1. Let $\mathcal{U} : [a,\infty) \to \mathbb{R}$ be a continuous function and $\alpha \in (0,1]$. And $\mathcal{I}_{\alpha}$ is related to fractional conformable integration of order α Then for all $t > a$, we have
$$ \mathcal{T}_{\alpha}^{a}\mathcal{I}_{\alpha}^{a}\mathcal{U}(t) = \mathcal{U}(t). $$
2. Let $\mathcal{U} : (-\infty, b] \to \mathbb{R}$ be a continuous function and $\alpha \in (0,1]$. Then for all $t < b$, we have
$$ {}^{b}\mathcal{T}_{\alpha}{}^{b}\mathcal{I}_{\alpha}\mathcal{U}(t) = \mathcal{U}(t). $$
3. Let $\mathcal{U} : (a,b) \to \mathbb{R}$ be a differentiable function and $\alpha \in (0,1]$. Then for all $t > a$, we have
$$ \mathcal{I}_{\alpha}^{a}\mathcal{T}_{\alpha}^{a}\mathcal{U}(t) = \mathcal{U}(t) - \mathcal{U}(a). $$
And the right case, for all $t < b$ we have:
$$ {}^{b}\mathcal{I}_{\alpha}{}^{b}\mathcal{T}_{\alpha}\mathcal{U}(t) = \mathcal{U}(t) - \mathcal{U}(b). $$

Remark 2. Recall the conformable FIDE given by equation (1), then in comparison with the property that $\mathcal{I}^0\mathcal{U}(t) = \mathcal{T}^0\mathcal{U}(t) = \mathcal{U}(t)$, the following special cases may arise from conformable FIDE:

1. If $\alpha = 0$, then Eq. (1) will be reduced to:

$\mathcal{U}(t) = g(t) + \mathcal{I}_{\beta}^{t}\, k\mathcal{U}(t)$
which is known as the conformable integral equation.

2. If $\beta = 0$, then Eq. (1) will be reduced to:

$\mathcal{T}_{\alpha}^{t}\,\mathcal{U}(t) = g(t) + k\mathcal{U}(t)$
which is known as a conformable differential equation.

3. If $\alpha = 1, \beta = 1$, then Eq. (1) will be reduced to:

$\mathcal{U}'(t) = g(t) + \int_a^t k\mathcal{U}(t)dt$
which is known as an integro-differential equation.

4. If $\alpha = 0, \beta = 1$, then Eq. (1) will be reduced to:

$\mathcal{U}(t) = g(t) + \int_a^t k\mathcal{U}(t)dt$
which is the usual Volterra integral equation.

5. If $\alpha = 1, \beta = 0$, then Eq. (1) will be reduced to

$\mathcal{U}'(t) = g(t) + k\mathcal{U}(t)$
which is a first-order ordinary differential equation.

Definition 4, [35]. Assume that $0 < \alpha \leq 1$ and $\mathcal{U}: [0, \infty) \to \mathbb{R}$ is a real-valued function. The fractional Laplace transform of order α of $\mathcal{U}$ is defined as follows:

$$\mathcal{L}_\alpha\{\mathcal{U}(t)\}(s) = \int_0^\infty exp\left(-s\frac{t^\alpha}{\alpha}\right) E_\alpha(-s,t)\mathcal{U}(t)d_\alpha t, s > 0$$

The conformable fractional-order derivative's Laplace transform can be evaluated to be:

$$\mathcal{L}_\alpha\{T_\alpha \mathcal{U}(t)\} = s\mathcal{L}_\alpha\{\mathcal{U}(t)\} - \mathcal{U}(0)$$

The following theorem establishes the connection between the standard and fractional Laplace transforms:

Theorem 1, [41] Provided that the function $\mathcal{U}(t)$ is of exponential order i.e. For a piecewise conformable continuous function $\mathcal{U}(t)$ on the interval $(0, \infty]$, the conformable Laplace transform $\mathcal{L}_\alpha\{\mathcal{U}(t)\}$, of its fractional derivative exists. This specifically requires that $|\mathcal{U}(t)| \leq \mathcal{M}e^{\eta\frac{t^\alpha}{\alpha}}$ for some constants $\mathcal{M}, \eta, \alpha > 0$. The transform is defined as $\int_0^\infty exp\left(-s\frac{t^\alpha}{\alpha}\right)\mathcal{U}(t)t^{\alpha-1}dt$.

Theorem 2, [35]. Let $\mathcal{U}: [0, \infty) \to \mathbb{R}$ be a function such that $\mathcal{L}_\alpha\{\mathcal{U}(t)\}(s) = F_\alpha(s)$ exists. Then: $F_\alpha(s) = \mathcal{L}\left\{\mathcal{U}(\alpha t)^{1/\alpha}\right\}(s)$, where $\mathcal{L}\{j(t)\}(s) = \int_0^\infty e^{-st}j(t)dt$.

Remark 3. The existence condition of the conformable Laplace transforms for a real number $s > \eta$ the transformation exists as long as the function $\mathcal{U}(t)$ is continuous or piecewise continuous its growth rate is no faster than that of the conformable exponential function (approximately $\mathcal{M}e^{\eta\frac{t^\alpha}{\alpha}}$)

Theorem 3, [35]. If there is $s > 0$, such that $F_\alpha(s) = \mathcal{L}\{\mathcal{U}(t)\}$ then;

i. If $\wp$ is a constant, then $\mathcal{L}\{\wp\} = \frac{\wp}{s}$.

ii. If q be a constant, then $\mathcal{L}_\alpha\{t^q\}(s) = \alpha^{q/\alpha}\frac{\Gamma\left(1+\frac{q}{\alpha}\right)}{s^{1+\frac{q}{\alpha}}}$.

iii. If $\wp, q$ are an arbitrary constant, then $\mathcal{L}_\alpha\left\{t^q \exp\left(\wp\frac{t^\alpha}{\alpha}\right)\right\}(s) = \alpha^{q/\alpha}\frac{\Gamma\left(1+\frac{q}{\alpha}\right)}{(s-\wp)^{1+\frac{q}{\alpha}}}$.

Analysis of the Method:

This study presents the Laplace-VIM for addressing conformable fractional order integro-differential equations. FIDE of the form FIDE:

$$\mathcal{T}_\alpha^t\,\mathcal{U}(t) = g(t) + \mathcal{J}_\beta^t\,k\mathcal{U}(t), \text{ and } 0 \leq t \leq 1 \tag{4}$$

assuming that $\mathcal{U}(0) = \mathcal{U}_0$, is the initial condition. To simplify the problem, our presumption is that T_α^t the conformable derivative, I_β^t the conformable integral, $\alpha \in (0,1]$, $0 < \beta \leq \alpha \leq 1$, k is the kernel of the FIDE (4), and $g(t) \in L^2[0,1)$.

To demonstrate the fundamental concept of the Laplace-VIM for FIDEs, we examine the FIDE (4). The fundamental stages are outlined as follows: first, utilize the fractional Laplace transformation on both sides of Eq. (4) and utilize the convolution principle, resulting in

$$\begin{aligned}\mathcal{L}_\alpha\{\mathcal{U}_{n+1}(t)\} &= \mathcal{L}_\alpha\{\mathcal{U}_n(t)\} + \mathcal{L}_\alpha\{\lambda(t) * [\mathcal{T}_\alpha^t\,\mathcal{U}_n(t) - \mathcal{J}_\beta^t\,k\mathcal{U}_n(t) - g(t)]\}\\ &= \mathcal{L}_\alpha\{\mathcal{U}_n(t)\} + \mathcal{L}_\alpha\{\lambda(t)\}\mathcal{L}_\alpha\{\mathcal{T}_\alpha^t\,\mathcal{U}_n(t) - \mathcal{J}_\beta^t\,k\tilde{\mathcal{U}}_n(t) - g(t)\}\end{aligned} \tag{5}$$

Then Eq. (5) becomes:

$$\begin{aligned}\mathcal{L}_\alpha\{\mathcal{U}_{n+1}(t)\} &= \mathcal{L}_\alpha\{\mathcal{U}_n(t)\} + \Lambda(s)\left\{s\mathcal{L}_\alpha\{\mathcal{U}_n(t)\} - \mathcal{U}_n(0) + \mathcal{L}_\alpha\{-\mathcal{J}_\beta^t\,k\tilde{\mathcal{U}}_n(t) - g(t)\}\right\}\\ &= \mathcal{L}_\alpha\{\mathcal{U}_n(t)\} + \Lambda(s)\{s\mathcal{L}_\alpha\{\mathcal{U}_n(t)\} - \mathcal{U}_n(0)\} + \Lambda(s)\mathcal{L}_\alpha\{-\mathcal{J}_\beta^t\,k\tilde{\mathcal{U}}_n(t) - g(t)\}\end{aligned} \tag{6}$$

where $\Lambda(s) = \mathcal{L}_\alpha\{\lambda(t)\}$ and the kernel is of the difference kernel. The iteration formula of Eq. (6) will be employed to propose the primary framework utilizing the general Lagrange multiplier.

Consider the term $\tilde{\mathcal{U}}_n$ as a restricted variation, which will make Eq. (6) stationary with respect to $\mathcal{U}_n$, and hence:

$$\mathcal{L}_\alpha\{\delta\mathcal{U}_{n+1}(t)\} = \mathcal{L}_\alpha\{\delta\mathcal{U}_n(t)\} + \Lambda(s)\{s\mathcal{L}_\alpha\{\delta\mathcal{U}_n(t)\}\} \tag{7}$$

The extremum optimality condition is $\mathcal{L}_\alpha\{\delta\mathcal{U}_{n+1}(t)\} = 0$ and hence Eq. (7) leads to:

$\Lambda(s) = -\frac{1}{s}, s > 0$

The recursive iterated approximations are acquired by applying the inverse Laplace ($\mathcal{L}_\alpha^{-1}$) transform to Eq. (6) after substituting $\Lambda(s)$ yields to:

$$\mathcal{U}_{n+1}(t) = \mathcal{U}_n(t) - \mathcal{L}_\alpha^{-1}\left\{\frac{1}{s}[s\mathcal{L}_\alpha\{\mathcal{U}_n(t)\} - \mathcal{U}_n(0)] + \frac{1}{s}\mathcal{L}_\alpha\{-\mathcal{J}_\beta^t\,k\mathcal{U}_n(t) - g(t)\}\right\} \tag{8}$$

$$= \mathcal{U}_n(t) - \mathcal{L}_\alpha^{-1}\left\{\left[\mathcal{W}_n(t) - \frac{\mathcal{U}_n(0)}{s}\right] + \frac{1}{s}\mathcal{L}_\alpha\{-\mathcal{J}_\beta^t\,k\mathcal{U}_n(t) - g(t)\}\right\} \tag{9}$$

where $\mathcal{W}_n(t) = \mathcal{L}_\alpha\{\mathcal{U}_n(t)\}$.Rearranging Eq. (9) will imply:

$$\mathcal{U}_{n+1}(t) = \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_n(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{\mathcal{J}_\beta^t k\mathcal{U}_n(t) + g(t)\}\right\} \quad (10)$$

With an initial approximation to equation (7), the following equation can be considered:

$$\mathcal{U}_0(t) = \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}(0)}{s}\right\} \quad (11)$$

Finally, the solution of equation (1) shall be $\mathcal{U}(t) = \lim_{n\to\infty} \mathcal{U}_n(t)$.

Convergence analysis of the fractional Laplace-VIM:

Investigate the fractional order conformable integro-differential equation

$\mathcal{T}_\alpha^t\, \mathcal{U}(t) = g(t) + \mathcal{J}_\beta^t\, k\mathcal{U}(t)$

with the initial

$\mathcal{U}(0) = \mathcal{U}_0$

The iteration formula of the Laplace-VIM is

$$\mathcal{U}_{n+1}(t) = \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_n(0)}{s} + \mathcal{L}_\alpha^{-1}\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{\mathcal{J}_\beta^t k\mathcal{U}_n(t) + g(t)\}\right\} \quad (12)$$

Let the operator $\mathcal{A}[\mathcal{U}]$ be defined as,

$$\mathcal{A}[\mathcal{U}] = \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_n(0)}{s} + \mathcal{L}_\alpha^{-1}\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{\mathcal{J}_\beta^t k\mathcal{U}_n(t) + g(t)\}\right\} \quad (13)$$

where the components τ_ε and, $\varepsilon = 0,1,2,3\cdots$, are defined as:

$$\left.\begin{aligned} \tau_0 &= \mathcal{U}_0, \\ \tau_1 &= \mathcal{A}[\tau_1], \\ \tau_2 &= \mathcal{A}[\tau_2], \\ \tau_3 &= \mathcal{A}[\tau_3], \\ &\vdots \\ \tau_{\varepsilon+1} &= \mathcal{A}[\tau_\varepsilon] \end{aligned}\right\} \quad (14)$$

Hence,

$\mathcal{U}_\varepsilon(t) = \lim_{\varepsilon\to\infty} \mathcal{U}_\varepsilon(t)$

To analyze the convergence of the new Laplace-VIM, we will analyze the subsequent theorem.

Theorem. Consider $\mathcal{A}$, as specified in (14), be an operator mapping the Hilbert space $\mathcal{H}$ onto itself; the solution, described in (15), is guaranteed to converge provided there exists a constant γ in the range $0 < \gamma < 1$. $\|\mathcal{A}[\tau_{\varepsilon+1}]\| \le \gamma\|\mathcal{A}[\tau_\varepsilon]\| \quad (i.e\|[\tau_{\varepsilon+1}]\| \le \gamma\|[\tau_\varepsilon]\|),\ \forall\, \varepsilon \in \mathbb{N}\cup\{0\}$.

Proof. let the sequence $\{V_n\}_{n=0}^{\infty}$ be defined as follows,

$$\left.\begin{aligned} \mathcal{V}_0 &= g(t) + \tau_0, \\ \mathcal{V}_1 &= g(t) + \tau_1, \\ \mathcal{V}_2 &= g(t) + \tau_2, \\ \mathcal{V}_3 &= g(t) + \tau_3, \\ &\vdots \\ \mathcal{V}_\varepsilon &= g(t) + \tau_n \end{aligned}\right\} \quad (15)$$

the sequence $\{\mathcal{V}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$. and we will show that it is so. For this objective, consider:

$\|\mathcal{V}_{\varepsilon+1} - \mathcal{V}_\varepsilon\| = \|\tau_{\varepsilon+1} - \tau_\varepsilon\| \le \gamma\|\tau_\varepsilon\| \le \gamma^2\|\tau_{\varepsilon-1}\| \le \cdots \le \gamma^{\varepsilon+1}\|\tau_0\|$.

also, for all $\varepsilon, n \in \mathbb{N}, \varepsilon \ge n$, we have

$$\begin{aligned} \|\mathcal{V}_\varepsilon - \mathcal{V}_n\| &= \|(\mathcal{V}_\varepsilon - \mathcal{V}_{\varepsilon-1}) + (\mathcal{V}_{\varepsilon-1} - \mathcal{V}_{\varepsilon-2}) + \cdots + (\mathcal{V}_{n+1} - \mathcal{V}_n)\| \\ &\le \|\mathcal{V}_\varepsilon - \mathcal{V}_{\varepsilon-1}\| + \|\mathcal{V}_{\varepsilon-1} - \mathcal{V}_{\varepsilon-2}\| + \cdots + \|\mathcal{V}_{n+1} - \mathcal{V}_n\| \\ &\le \gamma^\varepsilon\|\tau_0\| + \gamma^{\varepsilon-1}\|\tau_0\| + \cdots + \gamma^{n+1}\|\tau_0\|. \\ &= \frac{1-\gamma^{\varepsilon-n}}{1-\gamma}\gamma^{n+1}\|\tau_0\|. \end{aligned}$$

Since $0 < \gamma < 1$, therefore,

$$\lim_{\varepsilon,n\to\infty} \|\mathcal{V}_\varepsilon - \mathcal{V}_n\| = 0. \quad (16)$$

Consequently, $\{\mathcal{V}_n\}_{n=0}^{\infty}$ constitutes a Cauchy sequence within the Hilbert space $\mathcal{H}$, so signifying the convergence of the series solution (15).

Illustrative Examples:

In this section, we demonstrate the correctness and usefulness of the method presented in this research for solving a select number of the conformable of FIDEs by applying the Laplace-VIM

.

Example 1. Consider the problem of solving the conformable fractional order linear integro-differential equation:

$$\mathcal{T}_\alpha^t\,\mathcal{U}(t) = 1 - \mathcal{I}_\beta^t\,\mathcal{U}(t) \tag{17}$$

$$\mathcal{U}(0) = 0$$

The exact solution is $\mathcal{U}(t) = \sin(t)$, when $\alpha = \beta = 1$ and $t \in [0,1]$. See citied [42]

First, the Laplace-VIM is used to convert both sides of Eq. (17), which implies the following in light of definition (4):

$$\mathcal{U}_{n+1}(t) = \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_n(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{1 - \mathcal{I}_\beta^t\,\mathcal{U}_n(t)\}\right\} \tag{18}$$

Now, if $n = 0$, and using the initial iteration $\mathcal{U}_0(t) = 0$. Therefore:

$$\begin{aligned}
\mathcal{U}_1(t) &= \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_0(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{1 - \mathcal{I}_\beta^t\,\mathcal{U}_0(t)\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_0(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{1 - \int_0^t v^{\beta-1}\mathcal{U}_0(v)dv\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\left(\frac{1}{s}\right)\mathcal{L}_\alpha\{1\}\right\}\\
&= t
\end{aligned}$$

If $n = 1$, then:

$$\begin{aligned}
\mathcal{U}_2(t) &= \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_1(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{1 - \mathcal{I}_\beta^t\,\mathcal{U}_1(t)\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_1(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{1 - \int_0^t v^{\beta-1}\,\mathcal{U}_1(v)\,dv\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{1 - \int_0^t v^{\beta-1}\,v\,dv\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\left(\frac{1}{s}\right)\mathcal{L}_\alpha\left\{1 - \frac{t^{\beta+1}}{\beta+1}\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\left(\frac{1}{s}\right)\left[\frac{1}{s}\right] - \frac{1}{\beta+1}\mathcal{L}_\alpha\left\{t^{\beta+1}\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\left(\frac{1}{s^2}\right) - \frac{1}{\beta+1}\cdot\frac{\alpha^{\frac{\beta+1}{\alpha}}\Gamma\left(1+\frac{\beta+1}{\alpha}\right)}{s^{2+\frac{\beta+1}{\alpha}}}\right\}\\
&= t - \frac{t^{\frac{\beta+2}{\alpha}}}{3(\beta+1)}
\end{aligned}$$

If $n = 2$, then:

$$\begin{aligned}
\mathcal{U}_3(t) &= \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_2(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{1 - \mathcal{I}_\beta^t\,\mathcal{U}_2(t)\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_1(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{1 - \int_0^t v^{\beta-1}\,\mathcal{U}_2(v)dv\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{1 - \int_0^t v^{\beta-1}\left(v - \frac{v^{\frac{\beta+2}{\alpha}}}{3(\beta+1)}\right)dv\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{1 - \int_0^t v^{\beta} - \left(\frac{v^{\frac{\beta-\alpha+\alpha\beta+2}{\alpha}}}{3(\beta+1)}\right)dv\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{1 - \frac{t^{\beta+1}}{\beta+1} + \frac{t^{\frac{\beta-\alpha+\alpha\beta+2}{\alpha}+1}}{3(\beta+1)*\frac{\beta-\alpha+\alpha\beta+2}{\alpha}+1}\right\}\right\}\\
&= \mathcal{L}_\alpha^{-1}\left\{\frac{1}{s^2} - \frac{\alpha^{\frac{\beta+1}{\alpha}}\Gamma\left(1+\frac{\beta+1}{\alpha}\right)}{(\beta+1)s^{2+\frac{\beta+1}{\alpha}}} + \frac{\alpha}{3(\beta+1)(\beta+\alpha\beta+2)}\frac{\alpha^{\frac{\beta+\alpha\beta+2}{\alpha^2}}\Gamma\left(1+\frac{\beta+\alpha\beta+2}{\alpha^2}\right)}{s^{2+\frac{\beta+\alpha\beta+2}{\alpha^2}}}\right\}\\
&= t - \frac{t^{\frac{\beta+2}{\alpha}}}{3(\beta+1)} + \frac{\alpha^3 t^{\frac{\beta+\alpha\beta+3}{\alpha^2}}}{3(\beta+1)(\beta+\alpha\beta+2)(\beta+\alpha\beta+3)}
\end{aligned}$$

and so on for further approximate solutions. The software tools of either Maple or Mathematica can also be utilized to graph figures and to calculate the other parts of the iteration formula components (17). The above obtained approximate solution u_3 is compared with above exact solution for different values of alpha and beta including $\alpha = \beta = 1$ (see Figure 1) as well as compared with numerical results given in [42] also with $\alpha = 1$ (see Table 1).

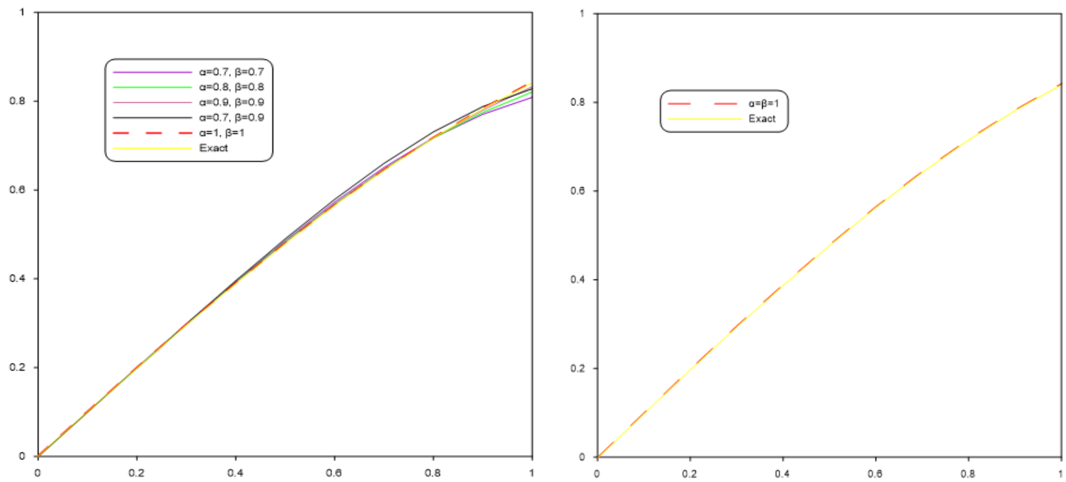

Figure 1. The exact solution when $\alpha = \beta = 1$ is contrasted with the approximate solution of Example 1 for various values of α, β

Table 1. Comparison of the approximate solution among the Laplace-VIM and the technique [42] for problem (17), and the absolute error when $\alpha = \beta = 1$, with the exact solution

t	$\alpha = 1$ method[42]	$\alpha = \beta = 1$	Exact solution	Absolute error
0	0	0	0	0
0.1	0.099833481645214	0.099833416666667	0.099833416646828	1.984×10^{-11}
0.2	0.198669447416521	0.198669333333333	0.198669330795061	2.538×10^{-9}
0.3	0.295520379224619	0.29552025	0.29552020666134	4.334×10^{-8}
0.4	0.389418569280661	0.389418666666667	0.389418342308651	3.244×10^{-7}
0.5	0.479425817990868	0.479427083333333	0.479425538604203	1.545×10^{-6}
0.6	0.564642802749863	0.564648	0.564642473395035	5.527×10^{-6}
0.7	0.644218063680569	0.644233916666667	0.644217687237691	1.623×10^{-5}
0.8	0.717356511141392	0.717397333333333	0.717356090899523	4.124×10^{-5}
0.9	0.783327369998728	0.78342075	0.783326909627483	9.384×10^{-5}
1	0.841471481288698	0.841666666666667	0.841470984807897	1.957×10^{-4}

Example 2. Consider the problem of solving the fractional order conformable, nonlinear integro-differential equation:

$$\mathcal{T}_\alpha^t\,\mathcal{U}(t) = -1 + \mathcal{J}_\beta^t\,\mathcal{U}(t)^2 \tag{18}$$

$$\mathcal{U}(0) = 0$$

The exact solution is $\mathcal{U}(t) = \left(\frac{1}{28}t^4 - t\right)\left(\frac{1}{12}t^3 + 1\right)^{-1}$ and $t \in [0,1]$, when $\alpha = \beta = 1$. See citied [42]

First, the Laplace-VIM is used to convert on both sides of Eq. (18), which implies the following in light of definition (4):

$$\mathcal{U}_{n+1}(t) = \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_n(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{-1 + \mathcal{J}_\beta^t\,\mathcal{U}_n(t)^2\}\right\} \tag{19}$$

Now, if $n = 0$, and using the initial iteration $y_0(t) = 0$. Therefore:

$$\mathcal{U}_1(t) = \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_0(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{-1 + \mathcal{J}_\beta^t\,\mathcal{U}_0(t)^2\}\right\}$$
$$= \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_0(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{-1 + \int_0^t v^{\beta-1}\mathcal{U}_0(v)^2 dv\right\}\right\}$$
$$= -t$$

If $n = 1$, then:

$$\mathcal{U}_2(t) = \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_1(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\{-1 + \mathcal{J}_\beta^t\,\mathcal{U}_1(t)^2\}\right\}$$
$$= \mathcal{L}_\alpha^{-1}\left\{\frac{\mathcal{U}_1(0)}{s}\right\} + \mathcal{L}_\alpha^{-1}\left\{\left\{\frac{1}{s}\right\}\mathcal{L}_\alpha\left\{-1 + \int_0^t v^{\beta-1}\,\mathcal{U}_1(v)^2 dv\right\}\right\}$$
$$= -t + \frac{t^{\frac{\beta+3}{\alpha}}}{4(\beta+2)}$$

and so on for further approximate solutions. The software tools of either Maple or Mathematica can also be utilized to graph figures and to calculate the other parts of the iteration formula components (18). The above obtained approximate solution u_2 is compared with above exact solution for different values of alpha and beta including $\alpha = \beta = 1$ (Figure 2) as well as compared with numerical results given in [42] also with $\alpha = 1$ (see Table 2).

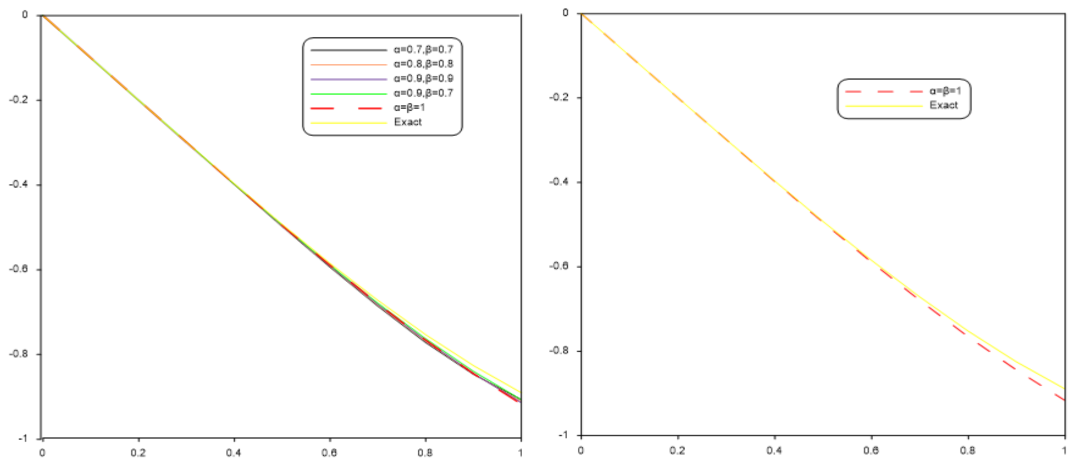

Figure 2. The exact solution when $\alpha = \beta = 1$ is contrasted with the approximate solution of Example 2 for various values of α, β

Table 2. Comparison of the approximate solution among the Laplace-VIM and the technique [42] for problem (18), and the absolute error when $\alpha = \beta = 1$, with the exact solution

t	$\alpha = 1$ method [42]	$\alpha = \beta = 1$	Exact solution	Absolute error
0	0	0	0	0
0.1	-0.099991669294463	-0.099991666666667	-0.099988096230076	3.57×10^{-6}
0.2	-0.199866719349661	-0.199866666666667	-0.199809650709051	5.702×10^{-5}
0.3	-0.299325867756254	-0.299325	-0.299037879057834	2.871×10^{-4}
0.4	-0.397873148328419	-0.397866666666667	-0.396968548692687	8.981×10^{-4}
0.5	-0.494822498678952	-0.494791666666667	-0.492636229749632	2.155×10^{-3}
0.6	-0.589310074648311	-0.5892	-0.584844232388437	4.356×10^{-3}
0.7	-0.6803138261988690	-0.679991666666667	-0.672210969780442	7.781×10^{-3}
0.8	-0.766681408361617	-0.765866666666667	-0.753233467299964	0.013
0.9	-0.847166855032368	-0.845325	-0.826366115619003	0.019
1	-0.920475618483447	-0.916666666666667	-0.89010989010989	0.027

Conclusions and Discussion of the Results

This article aimed is to apply Laplace-VIM to solve FIDEs involving modified conformable fractional derivatives with the conformable fractional integrals, in which non-linear conformable FIDEs are solved. We exhibit the results in the form of a graphical representation to show the accuracy of the approximate obtained results in comparison with the exact results, in addition to the absolute error are presented which is exactable even so when using few iterations, such as three or four iterations. Lastly, up to our knowledge, the Laplace-VIM that is demonstrated to be a very successful and efficient method for obtaining the approximate solution of the non-linear integro-differential equations. The primary limitation of the approach discussed appears to be its challenge in addressing nonlinear problems. These issues could potentially be resolved by employing a modified approach based on utilizing Laplace-VIM for solving complicated nonlinear problems through evaluating the integral of the VIM numerically, either by using trapezoidal rule and Simpsons rule. see cited [43]

Conflicts of Interest

Regarding the publishing of this paper, the author or authors state that they have no conflicts of interest.

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References

[1] Biala, T. A., Afolabi, Y. O., & Asim, O. O. (2014). Laplace variational iteration method for integro-differential equations of fractional order. *International Journal of Pure and Applied Mathematics, 95*(3), 413–426.
[2] Abdulsahib, A. A., Fadhel, F. S., & Eidi, J. H. (2024). Approximate solution of linear fuzzy random ordinary differential equations using Laplace variational iteration method. *Iraqi Journal of Science, 65*(2), 804–817.
[3] Khalil, R., Al Horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics, 264*, 65–70. https://doi.org/10.1016/j.cam.2014.01.002.
[4] Abdeljawad, T. (2015). On conformable fractional calculus. *Journal of Computational and Applied Mathematics, 279*, 57–66.
[5] Al-Safi, M. G. S. (2024). Numerical solution for conformable fractional PDEs by using a new double conformable integral transform-decomposition method. *Iraqi Journal of Science, 65*(8), 4489–4512. https://doi.org/10.24996/ijs.2024.65.8.30.
[6] H., A., Y., B., & B., D. (2024). On the geometric and physical properties of conformable derivative. *Mathematical Sciences and Applications E-Notes, 12*(2), 60–70. https://doi.org/10.36753/mathenot.1384280.
[7] Yajima, T., & Yamasaki, K. (2012). Geometry of surfaces with Caputo fractional derivatives and applications to incompressible two-dimensional flows. *Journal of Physics A: Mathematical and Theoretical, 45*, 065201.
[8] Yajima, T., Oiwa, S., & Yamasaki, K. (2018). Geometry of curves with fractional-order tangent vector and Frenet-Serret formulas. *Fractional Calculus and Applied Analysis, 21*(6), 1493–1505.
[9] Lazopoulos, K. A., & Lazopoulos, A. K. (2016). Fractional differential geometry of curves and surfaces. *Progress in Fractional Differentiation and Applications, 2*(3), 169–186.
[10] Aydın, M. E., Mihai, A., & Yokuş, A. (2021). Applications of fractional calculus in equiaffine geometry: Plane curves with fractional order. *Mathematical Methods in the Applied Sciences, 44*(17), 13659–13669.
[11] Gözütok, U., Çoban, H. A., & Sağiroğlu, Y. (2019). Frenet frame with respect to conformable derivative. *Filomat, 33*(6), 1541–1550.
[12] Has, A., & Yılmaz, B. (2022). Special fractional curve pairs with fractional calculus. *International Electronic Journal of Geometry, 15*(1), 132–144.
[13] Has, A., Yılmaz, B., Akkurt, A., & Yıldırım, H. (2022). Conformable special curves in Euclidean 3-space. *Filomat, 36*(14), 4687–4698.
[14] Has, A., & Yılmaz, B. (2022). Effect of fractional analysis on magnetic curves. *Revista Mexicana de Física, 68*(4), 1–15.
[15] Yılmaz, B., & Has, A. (2022). Obtaining fractional electromagnetic curves in optical fiber using fractional alternative moving frame. *Optik, 260*, 169067.
[16] Yılmaz, B. (2021). A new type electromagnetic curves in optical fiber and rotation of the polarization plane using fractional calculus. *Optik, 247*, 168026.
[17] Aydın, M. E., Bektaş, M., Ögrenmiş, A. O., & Yokuş, A. (2021). Differential geometry of curves in Euclidean 3-space with fractional order. *International Electronic Journal of Geometry, 14*(1), 132–144.
[18] Aydın, M. E., & Kaya, S. (2023). Fractional equiaffine curvatures of curves in 3-dimensional affine space. *International Journal of Maps in Mathematics, 6*(1), 67–82.
[19] Ögrenmiş, M. (2022). Geometry of curves with fractional derivatives in Lorentz plane. *Journal of New Theory, 38*, 88–98.
[20] He, J. H. (1999). Variational iteration method: A kind of non-linear analytical technique; some examples. *International Journal of Non-Linear Mechanics, 34*(4), 699–708.
[21] He, J. H. (2007). Variational iteration method: Some recent results and new interpretations. *Journal of Computational and Applied Mathematics, 207*, 3–17.
[22] He, J. H., & Wu, X. H. (2007). Variational iteration method: New development and applications. *Computers & Mathematics with Applications, 54*(7–8), 881–894.
[23] Khaleel, O. I. (2014). Variational iteration method for solving multi-fractional integro differential equations. *Iraqi Journal of Science, 55*(3A), 1086–1094.
[24] Hussain, A. K., Fadhel, F. S., Yahya, Z. R., & Nursalasawati, R. (2016). Variational iteration method (VIM) for solving partial integro-differential equations. *Journal of Theoretical and Applied Information Technology, 88*(2).
[25] Wang, W.-H. (2009). An effective method for solving fractional integro-differential equations. *Acta Universitatis Apulensis, 20*, 229–235.
[26] Abbasbandy, S., & Shivanian, E. (2009). Application of the variational iteration method for system of nonlinear Volterra's integro-differential equations. *Journal of Mathematical and Computational Applications, 14*, 147–158.
[27] Abdulhussein, A. I., & Fadhel, F. S. (2025). Solution of oxygen diffusion moving boundary value problem based on variational iteration least square methods. *Al-Nahrain Journal of Science, 28*(1), 151–158.

[28] Ismael, M. S., Fadhel, F. S., & Fayadh, A. H. (2023). Solution of multi-term fractional order delay differential equations using homotopy analysis method. In *AIP Conference Proceedings, 2457*, 020018. https://doi.org/10.1063/5.0118320.

[29] Hesameddini, E., & Latifizadeh, H. (2009). Reconstruction of variational iteration algorithms using the Laplace transform. *International Journal of Nonlinear Sciences and Numerical Simulation, 10*(11–12), 1377–1382.

[30] Wu, G. C. (2012). Laplace transform overcoming principle drawbacks in application of the variational iteration method to fractional heat equations. *Thermal Science, 16*(4), 1257–1261.

[31] Martinez, H. Y., & Gomez-Aguilar, J. F. (2020). Laplace variational iteration method for modified fractional derivatives with non-singular kernel. *Journal of Applied and Computational Mechanics, 6*(3), 684–698.

[32] Elzaki, T. M. (2018). Solution of nonlinear partial differential equations by new Laplace variational iteration method. *Differential Equations: Theory and Current Research*.

[33] Singh, G., & Singh, I. (2020). New Laplace variational iterative method for solving 3D Schrödinger equations. *Journal of Mathematics and Computer Science, 10*(5), 2015–2024.

[34] Singh, G., & Singh, I. (2020). The exact solution of 3D diffusion and wave equations using new Laplace variational iterative method. *International Advanced Research in Engineering and Technology, 11*(10), 36–43.

[35] Silva, F. S., Moreira, D. M., & Moret, M. A. (2018). Conformable Laplace transform of fractional differential equations. *Axioms, 7*(3), 55. https://doi.org/10.3390/axioms7030055.

[36] Zhao, D., & Luo, M. (2017). General conformable fractional derivative and its physical interpretation. *Calcolo, 54*, 903–917.

[37] Kohlrausch, R. (1854). Theorie des elektrischen Rückstandes in der Leidener Flasche. *Annalen der Physik, 167*, 179–214.

[38] Li, M. (2018). Three classes of fractional oscillators. *Symmetry, 10*, 40.

[39] Wuttke, J. (2012). Laplace-Fourier transform of the stretched exponential function: Analytic error bounds, double exponential transform, and open-source implementation “libkww”. *Algorithms, 5*, 604–628.

[40] Metzler, R., & Klafter, J. (2002). From stretched exponential to inverse power-law: Fractional dynamics, Cole-Cole relaxation processes, and beyond. *Journal of Non-Crystalline Solids, 305*, 81–87.

[41] Younis, J., Ahmed, B., AlJazzazi, M., Al Hejaj, R., & Aydi, H. (2022). Existence and uniqueness study of the conformable Laplace transform. *Journal of Mathematics*, 2022, Article 4554065. https://doi.org/10.1155/2022/4554065.

[42] Arqub, O. A., & Maayah, B. (2019). Fitted fractional reproducing kernel algorithm for the numerical solutions of ABC fractional Volterra integro-differential equations. *Chaos, Solitons & Fractals, 126*, 394–402.

[43] AbdulSahib, A. A., Fadhel, F. S., & Abid, S. H. (2019). Modified approach for solving random ordinary differential equations. *Journal of Theoretical and Applied Information Technology, 97*(13).