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Change-Detected ARIMA (1,1,1) Time Series Using the Approximated and Exact ARL of the MEWMA Scheme

Piyatida Phanthuna[a], Yupaporn Areepong[b]*

[a]Department of Mathematics and Statistics, Faculty of Science and Technology, Rajamangala University of Technology Phra Nakhon, Bangkok, 10800 Thailand; [b]Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800 Thailand

Abstract This study proposes an explicit analytical formulation for calculating the Average Run Length (ARL) of a modified Exponentially Weighted Moving Average (MEWMA) control chart applied to an Autoregressive Integrated Moving Average of order (1,1,1) time series model with exponential white noise. The developed explicit formula provides a closed-form solution that enables efficient computation and theoretical insight into the control chart's performance. The derived ARL values are validated against those obtained using the Numerical Integral Equation (NIE) method, ensuring accuracy and reliability. The comparative analysis shows that the explicit formula yields results that closely match those from the NIE method, with negligible absolute percentage differences while significantly reducing computational time. Simulation studies across various parameter settings and real-world applications, including climate and commodity price datasets, further confirm the consistency and practical utility of the proposed approach.

Keywords: Modified Exponentially Weighted Moving Average, Autoregressive Integrated Moving Average, Average Run Length, Explicit formula, Numerical Integral Equation.

Introduction

Statistical Process Control (SPC) originated in the field of industrial quality control. It is a methodological framework used to monitor, control, and improve processes through the application of statistical techniques. By understanding and managing these variations, organizations can maintain process stability, reduce waste, and improve product or service quality. Among the tools employed in SPC, control charts are perhaps the most prominent and widely used, including industrial [1], healthcare [2], financial [3], and environmental [4] monitoring. The primary objective of a control chart is to detect any deviation from the expected process behavior that may indicate the presence of special cause variation. This allows practitioners to take timely corrective actions before defects occur or escalate. Introduced by Shewhart [5] in the early 1920s at Bell Telephone Laboratories, control charts provide a visual and statistical means for monitoring process behavior over time. While the traditional Shewhart chart is effective for identifying large process shifts, more advanced methods such as the Cumulative Sum (CUSUM) chart [6] and the Exponentially Weighted Moving Average (EWMA) chart [7] have been developed to improve sensitivity in detecting small change. These charts continue to be refined and extended to address complex data structures and emerging challenges in modern industries. One notable advancement in control chart methodology is the modified Exponentially Weighted Moving Average (MEWMA) control chart. Khan *et al*. [8] proposed an extension of the traditional EWMA chart by integrating features of both the Shewhart and EWMA charts. This hybrid design enhances the chart's ability to detect small- to medium-sized shifts in the process mean and demonstrates improved performance when applied to autocorrelated data. In recent years, the integration of control charts with time series models has garnered increasing attention, particularly in contexts where process data exhibit temporal correlation. Control charts applied to time series have shown to improve detection performance and reduce false alarms in autocorrelated processes. Many researchers have explored the use of time series models [9] such as Autoregressive

***For correspondence:**
yupaporn.a@sci.kmutnb.ac.th

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(AR), Moving Average (MA), Autoregressive Moving Average (ARMA), and Autoregressive Integrated Moving Average (ARIMA) as a preprocessing step or as a foundation for constructing model-based control charts. Among the various time series models, the Autoregressive Integrated Moving Average (ARIMA) model is particularly versatile and widely used for representing non-stationary processes. ARIMA models can accommodate both trend and autocorrelation structures in time-dependent data, making them suitable for a broad range of application.

To quantitatively evaluate the performance of such control charts, one of the most widely used metrics is the Average Run Length (ARL). The ARL is defined as the expected number of observations taken before the control chart signals a process shift. There are two primary forms as: ARL_0 is the expected run length when the process is in-control. A higher ARL_0 indicates fewer false alarms. For out-of-control ARL, ARL_1 is the expected run length when the process has shifted. A lower ARL_1 reflects quicker detection of process changes. Recent studies have increasingly focused on developing explicit analytical expressions and numerical integral equation (NIE) methods for calculating the Average Run Length (ARL) in this control chart, particularly those applied to monitoring time series models. For instance, Supharakonsakun *et al*. [10] developed explicit formulas for calculating the ARL of a MEWMA control chart applied to a MA(1) process. These formulas were validated against results obtained from the NIE method, demonstrating strong agreement. Similarly, Phanthuna *et al*. [11] derived explicit ARL expressions for a MEWMA chart designed to monitor an AR(1) process with exponential white noise, with accuracy and computational efficiency confirmed through cross-validation with NIE results. In addition, Supharakonsakun *et al*. [12] presented explicit ARL formulas for a MEWMA control chart based on an ARMA(1,1) process with exponential white noise, yielding exact solutions verified by numerical evaluation. Recently, Polyeam and Phanyeam [13] proposed an explicit formula, along with a NIE approach, for determining the ARL of the EWMA control chart applied to a seasonal AR(1) model with a quadratic trend.

This study introduces an explicit formula for the computation of the Average Run Length (ARL) associated with a MEWMA control chart operating on an ARIMA(1,1,1) model. The performance of the proposed formulation is evaluated by comparison with results obtained through the Numerical Integral Equation (NIE) method, thereby assessing both its accuracy and computational efficiency. This constitutes a novel contribution to the existing literature, as no prior studies have addressed this specific formulation. The paper is structured as follows: Section 2 introduces the control chart framework and the time series model employed in this study. Sections 3 and 4 detail the methodological approach to deriving and solving the ARL problem. Section 5 reports the results obtained from the proposed method, while Section 6 demonstrates its application to real-world data. Finally, Section 7 concludes the paper and outlines potential directions for future research.

MEWMA Control Chart and ARIMA(1,1,1) Model

The MEWMA control chart, proposed by Khan *et al*. [8], has been recognized for its enhanced capability in detecting small to medium shifts in the process mean compared to the MEWMA control chart at time t, denoted as M_t, is expressed as follows

$$M_t = (1-\lambda)M_{t-1} + \lambda I_t + k\left(I_t - I_{t-1}\right);\ t = 1,2,\ldots, \tag{1}$$

where M_{t-1} denotes the preceding value of the MEWMA statistic (with M_0 initialized to υ), I_t and I_{t-1} represent the current and previous observations, respectively, from the ARIMA(1,1,1) process. The parameter k a specified constant (In the special case where k =0, the equation reduces to the conventional EWMA statistic), while λ is the exponential smoothing parameter of the MEWMA control chart, constrained within $0<\lambda<1$.

Let the mean and standard deviation of the sequence I_t be denoted by μ_I and σ_I, respectively. The center line (CL), together with the lower and upper control limits (LCL and UCL) of the MEWMA control chart, are defined as follows

$$\text{CL} = \mu_I \quad \text{and} \quad (\text{LCL, UCL}) = \mu_I \pm L\sigma_I\sqrt{(\lambda+2k\lambda+2k^2)/(2-\lambda)}, \tag{2}$$

where L is the width coefficient of the control limits.

One of the most widely employed variants of the MEWMA control chart is constructed based on time series data. In this study, particular attention is given to the ARIMA(1,1,1) model, a statistical framework for time series forecasting that incorporates one autoregressive term (p =1), first-order differencing to achieve stationarity (d =1), and one moving average term (q =1). This model integrates past observations (AR), adjusts for trends through differencing (I), and incorporates past forecast errors (MA) to generate future predictions. By combining these components, the ARIMA(1,1,1) model provides a more robust representation of time series dynamics compared to simpler models, as it captures both recent historical information and residual errors from prior forecasts. The ARIMA(1,1,1) model can be expressed using the backward shift operator B as follows

$$(1-\phi B)(1-B)I_t = \mu + (1-\theta B)\varepsilon_t, \tag{3}$$

where ϕ denotes the autoregressive coefficient parameter at $\phi \in [-1, 1]$, θ represents the moving-average coefficient parameter at $\theta \in [-1, 1]$, μ is the process mean and ε_t is a white noise, assumed to follow an exponential distribution. Based on Equation 3, the ARIMA(1,1,1) model can be algebraically reformulated as follows

$$I_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1} + (1+\phi)I_{t-1} - \phi I_{t-2}. \tag{4}$$

Consequently, by incorporating the sequence variables of the ARIMA(1,1,1) model, the MEWMA statistic can be expressed by substituting I_t from Equation (4) into Equation (1). Accordingly, the statistic M_t can be reformulated as follows

$$M_t = (1-\lambda)M_{t-1} - kI_{t-1} + (k+\lambda)(1+\phi)I_{t-1} - \phi(k+\lambda)I_{t-2} + (k+\lambda)(\mu - \theta\varepsilon_{t-1}) + (k+\lambda)\varepsilon_t. \tag{5}$$

Explicit ARL Formulation

Given the central role of the Average Run Length (ARL) in evaluating the performance of control charts, a wide range of analytical and numerical methods have been developed to compute its value under both in-control and out-of-control conditions. Conceptually, the ARL denotes the expected number of samples or time points required before a control chart signals a potential shift in the monitored process. Accordingly, a higher probability of detecting a process shift corresponds to a shorter ARL. In the context of time series models, the ARIMA(1,1,1) process, coupled with the MEWMA control chart, explicit formulas can be derived to obtain ARL values directly.

For the in-control process, the Average Run Length (ARL) is defined as the expected stopping time (τ) at a specified change-point, ω . The stopping time for detecting an out-of-control condition in a two-sided MEWMA control chart M_t can be formulated as follows

$$\tau_{l,u} = \inf\{t > 0; M_t < l \text{ or } M_t > u\},$$

where u and l are the upper and lower control limits, respectively. The ARL function of a MEWMA control chart under an ARIMA(1,1,1) model with an initial value is defined with respect to the density function at the change-point time as $A(\upsilon) = \mathrm{E}_\omega(\tau_{l,u}) < \infty$. Considering the MEWMA statistic M_t under the assumption that the process is in-control, it can be expressed as

$$l < \begin{bmatrix} (1-\lambda)M_{t-1} - kI_{t-1} + (k+\lambda)(1+\phi)I_{t-1} \\ -\phi(k+\lambda)I_{t-2} + (k+\lambda)(\mu - \theta\varepsilon_{t-1}) + (k+\lambda)\varepsilon_t \end{bmatrix} < u. \tag{6}$$

The upper and lower control limits in Equation (6) can be reformulated in terms of the variable ε_t as

$$\begin{bmatrix} \dfrac{l - (1-\lambda)M_{t-1} + kI_{t-1}}{(k+\lambda)} \\ -(1+\phi)I_{t-1} + \phi I_{t-2} - \mu + \theta\varepsilon_{t-1} \end{bmatrix} < \varepsilon_t < \begin{bmatrix} \dfrac{u - (1-\lambda)M_{t-1} + kI_{t-1}}{(k+\lambda)} \\ -(1+\phi)I_{t-1} + \phi I_{t-2} - \mu + \theta\varepsilon_{t-1} \end{bmatrix}. \tag{7}$$

Equation (7) is further represented as $LCL < \varepsilon_t < UCL$. The $A(\upsilon)$ can then be derived in the form of a Fredholm integral equation of the second kind, when the initial value is specified as $M_0 = \upsilon$ and $x = \varepsilon_t$. Accordingly, the $A(\upsilon)$ can be expressed as

$$A(\upsilon) = 1 + \int_{LCL}^{UCL} A\begin{pmatrix} (1-\lambda)\upsilon - kI_{t-1} + (k+\lambda)(1+\phi)I_{t-1} \\ -\phi(k+\lambda)I_{t-2} + (k+\lambda)(\mu - \theta\varepsilon_{t-1}) + (k+\lambda)x \end{pmatrix} f(x)\,dx, \tag{8}$$

Let the auxiliary variable y be defined as

$y=(1-\lambda)\upsilon-kI_{t-1}+(k+\lambda)(1+\phi)I_{t-1}-\phi(k+\lambda)I_{t-2}+(k+\lambda)(\mu-\theta\varepsilon_{t-1})+(k+\lambda)x.$ By applying a change of variables in the integration, Equation (8) can be transformed into the following equivalent expression

$$A(\upsilon)=1+\frac{1}{k+\lambda}\int_l^u A(y)f\left(\begin{array}{c}\dfrac{y-(1-\lambda)\upsilon+kI_{t-1}}{(k+\lambda)}\\ -(1+\phi)I_{t-1}+\phi I_{t-2}-\mu+\theta\varepsilon_{t-1}\end{array}\right)dy. \tag{9}$$

Since the ε_t is assumed to follow an exponential distribution as $\varepsilon_t\sim Exp(\beta)$, the $A(\upsilon)$ is expressed as

$$A(\upsilon)=1+\frac{1}{(k+\lambda)\beta}\left(e^{\frac{(1-\lambda)\upsilon}{(k+\lambda)\beta}}\cdot e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}\right)\int_l^u e^{\frac{-y}{(k+\lambda)\beta}}\cdot A(y)dy. \tag{10}$$

For the integral equation given in Equation 10, the Contraction Mapping Theorem [14] is employed to establish the existence and uniqueness of its solution. In the first step, Equation 10 can be expressed in the general form $A(\upsilon)=1+\int K(\upsilon,y)A(y)dy$, where the kernel function $K(\upsilon,y)$ is defined as $K(\upsilon,y)=\frac{1}{(k+\lambda)\beta}\left(e^{\frac{(1-\lambda)\upsilon}{(k+\lambda)\beta}}\cdot e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}\right)e^{\frac{-\upsilon}{(k+\lambda)\beta}}$. Let $C[l,u]$ denote the Banach space of continuous real-valued functions on the interval $[l,u]$ equipped with the supremum norm $\|A\|=\sup_{\upsilon\in[l,u]}|A(\upsilon)|$. Define the operator $T:C[l,u]\to C[l,u]$ by $TA(\upsilon)=1+\int K(\upsilon,y)A(y)dy$. Since the kernel function $K(\upsilon,y)$ is continuous in both arguments on the bounded domain $[l,u]\times[l,u]$, the integral $\int K(\upsilon,y)A(y)dy$ is well-defined for all $A\in C[l,u]$. Furthermore, as the integral of a continuous function is continuous, it follows that $TA(\upsilon)$ is continuous on $[l,u]$. Thus, T maps $C[l,u]$ into itself. Next, for any two functions $A_1,A_2\in C[l,u]$, we obtain $|TA_1(\upsilon)-TA_2(\upsilon)|=|\int K(\upsilon,y)(A_1(y)-A_2(y))dy|$. Taking the supremum over $\upsilon\in[l,u]$, it follows that

$$\|TA_1-TA_2\|\le\int|K(\upsilon,y)|\cdot|A_1(y)-A_2(y)|dy\le\sup_{\upsilon\in[l,u]}\int|K(\upsilon,y)|dy\|A_1-A_2\|\le\rho\|A_1-A_2\|.$$

where $\rho<1$. Hence, T is a contraction mapping on the Banach space $C[l,u]$. By the Banach Fixed-Point Theorem, there exists a unique function $A(\upsilon)\in C[l,u]$ such that $TA(\upsilon)=A(\upsilon)$, thereby guaranteeing the existence and uniqueness of the solution to this integral equation.

Subsequently, the integral equation is solved to obtain an explicit formula for the ARL of the MEWMA control chart constructed under the ARIMA(1,1,1) model. To facilitate the solution, Equation (10) is first transformed by introducing new variables,

$C=\int_l^u e^{\frac{-y}{(k+\lambda)\beta}}\cdot A(y)dy$ and $D(\upsilon)=e^{\frac{(1-\lambda)\upsilon}{(k+\lambda)\beta}}\cdot e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}$, such that the $A(\upsilon)$ of Equation (10) can be rearranged as follows

$$A(\upsilon)=1+\frac{D(\upsilon)}{\beta(\lambda+k)}\cdot C. \tag{11}$$

In the next step, the variable C is derived as

$$C=\int_l^u e^{\frac{-y}{(k+\lambda)\beta}}\cdot A(y)dy,$$

$$C=\int_l^u e^{\frac{-y}{(k+\lambda)\beta}}\left(1+\frac{D(y)}{(k+\lambda)\beta}\cdot C\right)dy,$$

$$C=\int_l^u e^{\frac{-y}{(k+\lambda)\beta}}dy+\int_l^u e^{\frac{-y}{(k+\lambda)\beta}}\left(\frac{D(y)}{(k+\lambda)\beta}\cdot C\right)dy,$$

$$C=-(k+\lambda)\beta\left[e^{\frac{-u}{(k+\lambda)\beta}}-e^{\frac{-l}{(k+\lambda)\beta}}\right]-\frac{e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}\cdot C}{\lambda}\left[e^{\frac{-u\lambda}{(k+\lambda)\beta}}-e^{\frac{-l\lambda}{(k+\lambda)\beta}}\right],$$

$$C=\frac{-(k+\lambda)\beta\left[e^{\frac{-u}{(k+\lambda)\beta}}-e^{\frac{-l}{(k+\lambda)\beta}}\right]}{1+\frac{e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}}{\lambda}\left[e^{\frac{-u\lambda}{(k+\lambda)\beta}}-e^{\frac{-l\lambda}{(k+\lambda)\beta}}\right]}. \tag{12}$$

The resulting expression for C from Equation (12), along with the defined variable $D(\upsilon)$, is then substituted into Equation (11), yielding the following relation

$$A(\upsilon)=1-\frac{e^{\frac{(1-\lambda)\upsilon}{(k+\lambda)\beta}}\cdot e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}\left[e^{\frac{-u}{(k+\lambda)\beta}}-e^{\frac{-l}{(k+\lambda)\beta}}\right]}{1+\frac{e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}}{\lambda}\left[e^{\frac{-u\lambda}{(k+\lambda)\beta}}-e^{\frac{-l\lambda}{(k+\lambda)\beta}}\right]},$$

$$A(\upsilon)=1-\frac{\lambda\cdot e^{\frac{(1-\lambda)\upsilon}{(k+\lambda)\beta}}\cdot e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}\left[e^{\frac{-u}{(k+\lambda)\beta}}-e^{\frac{-l}{(k+\lambda)\beta}}\right]}{\lambda+e^{\frac{-kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}\left[e^{\frac{-u\lambda}{(k+\lambda)\beta}}-e^{\frac{-l\lambda}{(k+\lambda)\beta}}\right]},$$

$$A(\upsilon)=1-\frac{\lambda\cdot e^{\frac{(1-\lambda)\upsilon}{(k+\lambda)\beta}}\left[e^{\frac{-u}{(k+\lambda)\beta}}-e^{\frac{-l}{(k+\lambda)\beta}}\right]}{\lambda\cdot e^{\frac{kI_{t-1}}{(k+\lambda)\beta}}\cdot e^{-\frac{\mu-\theta\varepsilon_{t-1}}{\beta}}\cdot e^{-\frac{(1+\phi)I_{t-1}-\phi I_{t-2}}{\beta}}+e^{\frac{-u\lambda}{(k+\lambda)\beta}}-e^{\frac{-l\lambda}{(k+\lambda)\beta}}} \tag{13}$$

Consequently, the $A(\upsilon)$ in Equation (13) yields the explicit formulas for the ARL of the MEWMA control chart corresponding to the ARIMA(1,1,1) model, evaluated at the initial process mean υ. In this formulation, λ denotes the exponential smoothing parameter, β represents the parameter of the exponential white noise, and I_{t-1} and I_{t-2} are the initial values of ARIMA(1,1,1) process. Furthermore, ϕ and θ refer to the autoregressive and moving-average coefficients, respectively, k is a specified constant in the modified EWMA statistic, and u and l represent the upper and lower control limits, respectively.

NIE Approach to ARL Evaluation

Complementarily, several approximation techniques most notably the Numerical Integral Equation (NIE) method have been widely employed to estimate ARL with a high degree of accuracy and computational accuracy and reliability. The NIE method involves solving an integral equation that characterizes the expected run length of a chart under a specific statistical distribution of the monitored variable. Because such integral equations are generally intractable to solve analytically, they are typically discretized and solved numerically using methods such as the midpoint rule, trapezoidal rule, or Simpson’s rule. Among these methods, the midpoint rule is often preferred because, although all methods produce nearly identical results, the midpoint rule requires the least computational time and features a simple and efficient implementation, as reported in several studies [15–17].

The NIE method is applied to compute the ARL of the MEWMA control chart constructed under the ARIMA(1,1,1) model, assuming an exponential distribution for the white noise process. The midpoint rule approximates the integral by dividing the area under the curve into n equal subintervals over a specified range $[l,u]$. The integral approximation can be expressed as

$$\int_{l}^{u} f(y)dy \approx \sum_{j=1}^{n} w_j f(y_j),$$

By applying this approach, the integral equation given in Equation (9) is transformed into a numerical integral equation form. The resulting numerical expression for the ARL of the MEWMA control chart under the ARIMA(1,1,1) model with an exponential white noise process can be formulated as

$$A_N(\upsilon) \approx 1 + \frac{1}{k+\lambda}\sum_{j=1}^{n} w_j \cdot A_N(y_j) \cdot f\left(\begin{array}{c}\dfrac{y_j - (1-\lambda)\upsilon + kI_{t-1}}{k+\lambda} \\ -(1+\phi)I_{t-1} + \phi I_{t-2} - \mu + \theta\varepsilon_{t-1}\end{array}\right). \qquad (14)$$

where $y_j = l + (j-0.5)w_j; j = 1,2,...,n$ is the midpoint of each subinterval and $w_j = \dfrac{u-l}{n}$ denotes the corresponding midpoint weight.

The obtained NIE-based solution is subsequently compared with the explicit ARL formula to validate its accuracy. The absolute percentage difference of expected ARL (AD%) [18] is employed to quantify the discrepancy between the expected ARL derived from the explicit formula, denoted as $EA(\upsilon)$, and that obtained from the NIE approach, denoted as $EA_N(\upsilon)$. The expected ARL is calculated as

$$\text{Expected ARL} = \frac{1}{\delta_{\max} - \delta_{\min}} \sum_{i=1}^{m} ARL(\delta_i), \quad i = 1,2,3,...,m \qquad (15)$$

where $ARL(\delta_i)$ represents the ARL corresponding to a given shift size $\delta_i > 0$, with $\delta_{\min}$ and $\delta_{\max}$ being the lower and upper bounds of the shift size, respectively. Accordingly, the absolute percentage difference (AD%) is computed as

$$\text{AD\%} = \left|\frac{EA(\upsilon) - EA_N(\upsilon)}{EA(\upsilon)}\right| \times 100\%, \qquad (16)$$

This measure provides a normalized percentage assessment of the deviation between the explicit analytical solution and the numerical approximation, thereby serving as a validation indicator for the accuracy and efficiency of the proposed explicit ARL formulation. The results obtained from this comparative analysis are presented and discussed in the following section, highlighting the performance consistency between the two approaches.

Experimental Results

For the implementation of the ARIMA(1,1,1) model with exponential white noise using a MEWMA control chart, the initial in-control Average Run Length (ARL_0) is set to 370, with control limits $[l,u]$ and an initial process mean of β_0 = 1 under an exponential distribution. The upper control limit u is determined by setting the lower control limit l to its minimum value of 0. The NIE method is computed with n = 500 discretization points. Subsequently, the process mean is adjusted for the out-of-control state using $\beta_1 = (1+\delta)\beta_0$ where δ represents the magnitude of the mean shift. The shift sizes considered are δ = 0.02, 0.04, 0.06, 0.08, 0.10, 0.12, 0.14, 0.16, 0.18, and 2.00. Different values of k, ϕ and θ are specified according to the parameter settings provided in the corresponding results tables. The initial value of the modified EWMA statistic is set as $M_0 = \upsilon = 1$. The exponential smoothing parameter is specified at λ = 0.01, 0.05, 0.10 and 0.20, respectively. $A(\upsilon)$ is computed using the explicit formula derived in Equation (13), while $A_N(\upsilon)$ is calculated using the NIE method presented in Equation (14). The ARL solutions obtained from both methods are compared using the absolute percentage difference (AD%) defined in Equation (16), and the computational processing time is recorded in seconds. Finally, the proposed methodology is applied to a real dataset to demonstrate its practical utility. The overall procedure for the study is summarized in Figure 1, which outlines the steps required to obtain all research results.

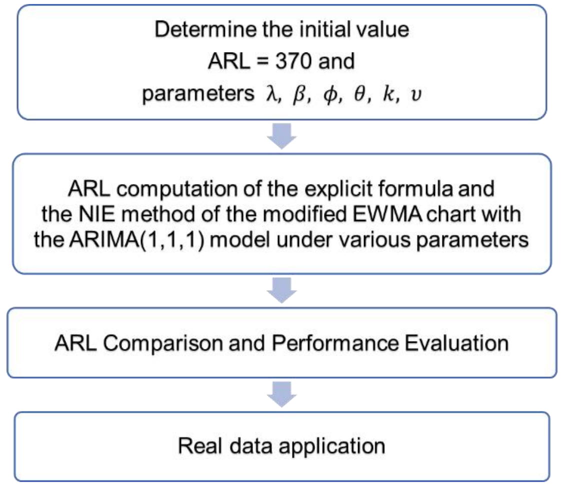

Figure 1. Research framework for ARL analysis of this study

Table 1 presents the ARL values obtained using both the explicit formula and the NIE method for the MEWMA control chart under the ARIMA(1,1,1) model. The parameter k is set to 0, 0.5, 1, and 3, while the other parameters are fixed at ϕ = 0.10, θ = 0.30, λ = 0.05. The ARL results derived from the explicit formula closely match those obtained from the NIE method. When considering the AD% values across different k levels, calculated from the ARL results of both approaches, the differences are minimal and approach zero. However, the computational processing time of the explicit formula is less than 0.001 seconds, whereas the NIE method requires more than 2.4 seconds. Furthermore, the results for k = 0, 0.5, 1 and 3 indicate that the ARL_1 values obtained from both methods decrease rapidly as the shift size increases. In contrast, when k = 0, which corresponds to the traditional EWMA control chart, the ARL_1 values decrease more slowly compared to other k values. These findings suggest that increasing k enhances the chart's ability to detect shifts more efficiently in Figure 2.

Table 2 and Figure 3 display the ARL results for the ARIMA(1,1,1) model, with ϕ = 0.15 fixed and θ varied at 0.15, 0.20, - 0.15 and - 0.20. The ARL_1 results reveal that lower θ values enable faster detection of process shifts. Similarly, Table 3 and Figure 4 show the ARL results for the model with θ = 0.25 fixed and ϕ varied at 0.20, 0.25, - 0.20 and - 0.25. The results indicate that larger ϕ values facilitate faster detection of shifts compared to smaller or negative values. Table 4 and Figure 5 illustrate the effect of varying the exponential smoothing parameter λ at 0.01, 0.05, 0.10 and 0.20. The findings demonstrate that increasing λ enhances the sensitivity of the control chart, allowing it to detect small shifts in the mean more quickly. Across all scenarios (Tables 2 - 4), the ARL and AD% values confirm the close agreement between the explicit formula and NIE method.

This consistency highlights the robustness and reliability of the explicit formulation, positioning it as a highly efficient alternative to the NIE method for ARL calculations. In particular, the explicit formulation offers a significant advantage in terms of computational speed while maintaining comparable accuracy, making it a practical and effective approach for both analysis and real-time process monitoring.

Table 1. ARL comparison of between the explicit formula and the NIE method with ϕ = 0.10, θ = 0.30, λ = 0.05 and ARL_0 = 370

Parameters	δ	ARL		Process Time (Seconds)	
		$A(\upsilon)$	$A_N(\upsilon)$	$A(\upsilon)$	$A_N(\upsilon)$
k = 0 (EWMA),	0.00	370.000944676	370.000944708	< 0.001	2.547
u = 0.0000000188843	0.02	242.076454256	242.076454242	< 0.001	2.610
	0.04	161.040106079	161.040106083	< 0.001	2.672
	0.06	108.859763555	108.859763540	< 0.001	2.532
	0.08	74.7369819882	74.7369819789	< 0.001	2.547
	0.10	52.0941701098	52.0941701024	< 0.001	2.532
	0.12	36.8596557215	36.8596557178	< 0.001	2.609
	0.14	26.4741727675	26.4741727673	< 0.001	2.609
	0.16	19.3055840708	19.3055840686	< 0.001	2.500
	0.18	14.2985759798	14.2985759800	< 0.001	2.500
	0.20	10.7618009712	10.7618009722	< 0.001	2.609
Expected ARL		37325.36327	37325.36327	**AD%**	6.2×10^{-9}
k = 0.5,	0.00	370.001070617	370.001055454	< 0.001	2.547
u = 0.24913371	0.02	92.4443955728	92.4443924124	< 0.001	2.610
	0.04	51.7841702602	51.7841686073	< 0.001	2.672
	0.06	35.5166509881	35.5166499160	< 0.001	2.532
	0.08	26.7924526929	26.7924519248	< 0.001	2.547
	0.10	21.3733585170	21.3733579338	< 0.001	2.532
	0.12	17.6935828542	17.6935823942	< 0.001	2.609
	0.14	15.0403849787	15.0403846059	< 0.001	2.609
	0.16	13.0428951450	13.0428948365	< 0.001	2.500
	0.18	11.4891642662	11.4891640067	< 0.001	2.500
	0.20	10.2493313344	10.2493311134	< 0.001	2.609
Expected ARL		14771.31933	14771.31889	**AD%**	3.0×10^{-6}
k = 1,	0.00	370.002480689	370.002460242	< 0.001	2.531
u = 0.5004165	0.02	49.5721573118	49.5721554351	< 0.001	2.828
	0.04	26.6120684714	26.6120675453	< 0.001	2.562
	0.06	18.2236501880	18.2236495947	< 0.001	2.500
	0.08	13.8838379528	13.8838375275	< 0.001	2.547
	0.10	11.2356878556	11.2356875310	< 0.001	2.531
	0.12	9.4541444104	9.4541441523	< 0.001	2.547
	0.14	8.1754649828	8.1754647717	< 0.001	2.515
	0.16	7.2143598292	7.2143596529	< 0.001	2.578
	0.18	6.4665225046	6.4665223549	< 0.001	2.704
	0.20	5.8687523126	5.8687521837	< 0.001	2.547
Expected ARL		7835.332291	7835.332037	**AD%**	3.2×10^{-6}
k = 3,	0.00	370.007657773	370.007630178	< 0.001	2.594
u = 1.5039536	0.02	30.2490955938	30.2490943780	< 0.001	2.499
	0.04	16.1612081954	16.1612076075	< 0.001	3.328
	0.06	11.2038224410	11.2038220643	< 0.001	2.984
	0.08	8.6739864629	8.6739861916	< 0.001	2.531
	0.10	7.1398873418	7.1398871333	< 0.001	2.516
	0.12	6.1106868617	6.1106866946	< 0.001	2.844
	0.14	5.3726038703	5.3726037326	< 0.001	2.453
	0.16	4.8176063874	4.8176062714	< 0.001	2.547
	0.18	4.3852213684	4.3852212691	< 0.001	2.969
	0.20	4.0389590903	4.0389590042	< 0.001	2.547
Expected ARL		4907.653881	4907.653717	**AD%**	3.3×10^{-6}

Table 2. ARL comparison of between the explicit formula and the NIE method with k = 3, λ = 0.05 and ARL$_0$ = 370

Parameters	δ	ARL		Process Time (Seconds)	
		$A(\upsilon)$	$A_N(\upsilon)$	$A(\upsilon)$	$A_N(\upsilon)$
ϕ = 0.15, θ = 0.15,	0.00	370.005514298	370.005494208	< 0.001	2.515
u = 1.2921102	0.02	28.7599029453	28.7599020979	< 0.001	2.547
	0.04	15.3380505456	15.3380501364	< 0.001	2.859
	0.06	10.6286816384	10.6286813765	< 0.001	2.516
	0.08	8.2284472408	8.2284470525	< 0.001	2.735
	0.10	6.7741184977	6.7741183531	< 0.001	2.843
	0.12	5.7990511730	5.7990510573	< 0.001	2.719
	0.14	5.1001746633	5.1001745681	< 0.001	2.719
	0.16	4.5749277105	4.5749276304	< 0.001	2.484
	0.18	4.1659232328	4.1659231642	< 0.001	2.531
	0.20	3.8385445438	3.8385444844	< 0.001	2.485
Expected ARL		4660.39111	4660.390996	**AD%**	2.4×10^{-6}
ϕ = 0.15, θ = 0.20,	0.00	370.005451626	370.005429301	< 0.001	2.531
u = 1.3591416	0.02	29.2393388320	29.2393378766	< 0.001	2.531
	0.04	15.6027056573	15.6027051958	< 0.001	2.516
	0.06	10.8135104250	10.8135101296	< 0.001	2.547
	0.08	8.3715919552	8.3715917426	< 0.001	2.547
	0.10	6.8916152490	6.8916150857	< 0.001	2.531
	0.12	5.8991460775	5.8991459468	< 0.001	2.671
	0.14	5.1876673261	5.1876672184	< 0.001	2.547
	0.16	4.6528580217	4.6528579311	< 0.001	2.547
	0.18	4.2363387233	4.2363386458	< 0.001	2.516
	0.20	3.9028905495	3.9028904823	< 0.001	2.531
Expected ARL		4739.883141	4739.883013	**AD%**	2.7×10^{-6}
ϕ = 0.15, θ = - 0.15,	0.00	370.002506743	370.002496018	< 0.001	2.516
u = 0.9544348	0.02	26.1758356227	26.1758352067	< 0.001	2.547
	0.04	13.9179134757	13.9179132754	< 0.001	2.531
	0.06	9.6386135558	9.6386134280	< 0.001	2.500
	0.08	7.4624711538	7.4624710621	< 0.001	2.563
	0.10	6.1458774461	6.1458773759	< 0.001	2.609
	0.12	5.2642063403	5.2642062843	< 0.001	2.547
	0.14	4.6329434072	4.6329433613	< 0.001	2.515
	0.16	4.1589905597	4.1589905212	< 0.001	2.516
	0.18	3.7902928692	3.7902928362	< 0.001	2.547
	0.20	3.4954673944	3.4954673660	< 0.001	2.563
Expected ARL		4234.130591	4234.130536	**AD%**	1.3×10^{-6}
ϕ = 0.15, θ = - 0.20,	0.00	370.005700654	370.005690988	< 0.001	2.468
u = 0.9075182	0.02	25.7868247264	25.7868243565	< 0.001	2.781
	0.04	13.7050917482	13.7050915702	< 0.001	2.797
	0.06	9.4905280952	9.4905279817	< 0.001	2.531
	0.08	7.3480467748	7.3480466934	< 0.001	2.516
	0.10	6.0521205841	6.0521205218	< 0.001	2.547
	0.12	5.1844554146	5.1844553650	< 0.001	2.578
	0.14	4.5633283111	4.5633282703	< 0.001	2.547
	0.16	4.0970631495	4.0970631153	< 0.001	2.547
	0.18	3.7344055871	3.7344055579	< 0.001	2.547
	0.20	3.4444579227	3.4444578975	< 0.001	2.594
Expected ARL		4170.316116	4170.316066	**AD%**	1.2×10^{-6}

Table 3. ARL comparison of between the explicit formula and the NIE method with k = 3, λ = 0.05 and ARL_0 = 370

Parameters	δ	ARL $A(\upsilon)$	ARL $A_N(\upsilon)$	Process Time (Seconds) $A(\upsilon)$	Process Time (Seconds) $A_N(\upsilon)$
ϕ = 0.20, θ = 0.25,	0.00	370.001372999	370.001357269	< 0.001	2.500
u = 1.1487197	0.02	27.7018549856	27.7018543435	< 0.001	2.625
	0.04	14.7552597781	14.7552594683	< 0.001	2.578
	0.06	10.2220069186	10.2220067206	< 0.001	2.594
	0.08	7.9136379756	7.9136378333	< 0.001	2.594
	0.10	6.5158032889	6.5158031799	< 0.001	2.547
	0.12	5.5790553151	5.5790552279	< 0.001	2.625
	0.14	4.9079247458	4.9079246741	< 0.001	2.579
	0.16	4.4037290589	4.4037289986	< 0.001	2.609
	0.18	4.0112675968	4.0112675454	< 0.001	2.609
	0.20	3.6972500555	3.6972500110	< 0.001	2.610
Expected ARL		4485.389486	4485.3894	**AD%**	1.9×10^{-6}
ϕ = 0.25, θ = 0.25,	0.00	370.007135578	370.007121410	< 0.001	2.625
u = 1.0921631	0.02	27.2701028426	27.2701022723	< 0.001	2.578
	0.04	14.5179484960	14.5179482210	< 0.001	2.516
	0.06	10.0565501908	10.0565500152	< 0.001	2.531
	0.08	7.7856235949	7.7856234687	< 0.001	2.563
	0.10	6.4108029341	6.4108028374	< 0.001	2.547
	0.12	5.4896607334	5.4896606561	< 0.001	2.578
	0.14	4.8298280190	4.8298279555	< 0.001	2.532
	0.16	4.3342033957	4.3342033424	< 0.001	2.578
	0.18	3.9484771282	3.9484770826	< 0.001	2.515
	0.20	3.6398992283	3.6398991889	< 0.001	2.532
Expected ARL		4414.154828	4414.154752	**AD%**	1.7×10^{-6}
ϕ = - 0.20, θ = 0.25,	0.00	370.002021935	370.001985234	< 0.001	2.485
u = 1.722151	0.02	31.7208241752	31.7208224921	< 0.001	2.578
	0.04	16.9777704230	16.9777696083	< 0.001	2.594
	0.06	11.7750244227	11.7750239001	< 0.001	2.609
	0.08	9.1167095326	9.1167091557	< 0.001	2.515
	0.10	7.5034522325	7.5034519425	< 0.001	2.563
	0.12	6.4205061768	6.4205059442	< 0.001	2.515
	0.14	5.6434872364	5.6434870444	< 0.001	2.516
	0.16	5.0589413451	5.0589411833	< 0.001	2.516
	0.18	4.6033344344	4.6033342957	< 0.001	2.547
	0.20	4.2383177692	4.2383176488	< 0.001	2.516
Expected ARL		5152.918387	5152.918161	**AD%**	4.4×10^{-6}
ϕ = - 0.25, θ = 0.25,	0.00	370.001574493	370.001533631	< 0.001	2.578
u = 1.8118372	0.02	32.3138519457	32.3138500427	< 0.001	2.516
	0.04	17.3075870022	17.3075860807	< 0.001	2.593
	0.06	12.0058791550	12.0058785637	< 0.001	2.516
	0.08	9.2956706512	9.2956702246	< 0.001	2.516
	0.10	7.6504192123	7.6504188838	< 0.001	2.578
	0.12	6.5457429474	6.5457426838	< 0.001	2.532
	0.14	5.7529791343	5.7529789167	< 0.001	2.547
	0.16	5.1564839684	5.1564837849	< 0.001	2.546
	0.18	4.6914864673	4.6914863100	< 0.001	2.516
	0.20	4.3188863833	4.3188862466	< 0.001	2.548
Expected ARL		5251.949343	5251.949087	**AD%**	4.9×10^{-6}

Table 4. ARL comparison of between the explicit formula and the NIE method with k = 3, ϕ = 0.30, θ = - 0.30, and ARL_0 = 370

Parameters	δ	ARL		Process Time (Seconds)	
		$A(\upsilon)$	$A_N(\upsilon)$	$A(\upsilon)$	$A_N(\upsilon)$
λ = 0.01,	0.00	370.000584597	370.000582071	< 0.001	2.531
u = 0.60223083	0.02	23.3192879905	23.3192878471	< 0.001	2.531
	0.04	12.3540410856	12.3540410155	< 0.001	2.531
	0.06	8.5477755903	8.5477755455	< 0.001	2.579
	0.08	6.6173714120	6.6173713798	< 0.001	2.563
	0.10	5.4516739949	5.4516739704	< 0.001	2.531
	0.12	4.6723252602	4.6723252406	< 0.001	2.563
	0.14	4.1151795700	4.1151795540	< 0.001	2.532
	0.16	3.6975054267	3.6975054133	< 0.001	2.563
	0.18	3.3730810273	3.3730810159	< 0.001	2.531
	0.20	3.1140575531	3.1140575433	< 0.001	2.515
Expected ARL		3763.114946	3763.114926	**AD%**	5.1×10^{-7}
λ = 0.05,	0.00	370.004939272	370.004935172	< 0.001	2.593
u = 0.5968265	0.02	22.9071943087	22.9071941684	< 0.001	2.532
	0.04	12.1382456806	12.1382456134	< 0.001	2.547
	0.06	8.4030291688	8.4030291261	< 0.001	2.531
	0.08	6.5092140762	6.5092140457	< 0.001	2.547
	0.10	5.3657830751	5.3657830519	< 0.001	2.547
	0.12	4.6013875340	4.6013875155	< 0.001	2.531
	0.14	4.0549609159	4.0549609008	< 0.001	2.563
	0.16	3.6453362500	3.6453362374	< 0.001	2.562
	0.18	3.3271708127	3.3271708020	< 0.001	2.563
	0.20	3.0731476582	3.0731476489	< 0.001	2.531
Expected ARL		3701.273474	3701.273456	**AD%**	5.0×10^{-7}
λ = 0.10,	0.00	370.001587781	370.001578852	< 0.001	2.531
u = 0.59041411	0.02	22.4146962637	22.4146961175	< 0.001	2.516
	0.04	11.8806877463	11.8806876801	< 0.001	2.578
	0.06	8.2303509686	8.2303509274	< 0.001	2.531
	0.08	6.3802170105	6.3802169814	< 0.001	2.625
	0.10	5.2633583671	5.2633583450	< 0.001	2.515
	0.12	4.5168035372	4.5168035198	< 0.001	2.547
	0.14	3.9831638368	3.9831638226	< 0.001	2.531
	0.16	3.5831403753	3.5831403635	< 0.001	2.547
	0.18	3.2724398002	3.2724397901	< 0.001	2.532
	0.20	3.0243799480	3.0243799393	< 0.001	2.547
Expected ARL		3627.461893	3627.461874	**AD%**	5.1×10^{-7}
λ = 0.20,	0.00	370.001114121	370.001087292	< 0.001	2.563
u = 0.5786504	0.02	21.4994528768	21.4994526955	< 0.001	2.734
	0.04	11.4030042804	11.4030042096	< 0.001	2.516
	0.06	7.9103152663	7.9103152249	< 0.001	2.531
	0.08	6.1412252223	6.1412251941	< 0.001	2.547
	0.10	5.0736399424	5.0736399216	< 0.001	2.562
	0.12	4.3601557050	4.3601556887	< 0.001	2.516
	0.14	3.8502129145	3.8502129014	< 0.001	2.547
	0.16	3.4679794620	3.4679794512	< 0.001	2.500
	0.18	3.1711086601	3.1711086510	< 0.001	2.531
	0.20	2.9340955756	2.9340955678	< 0.001	2.532
Expected ARL		3490.559495	3490.559475	**AD%**	5.7×10^{-7}

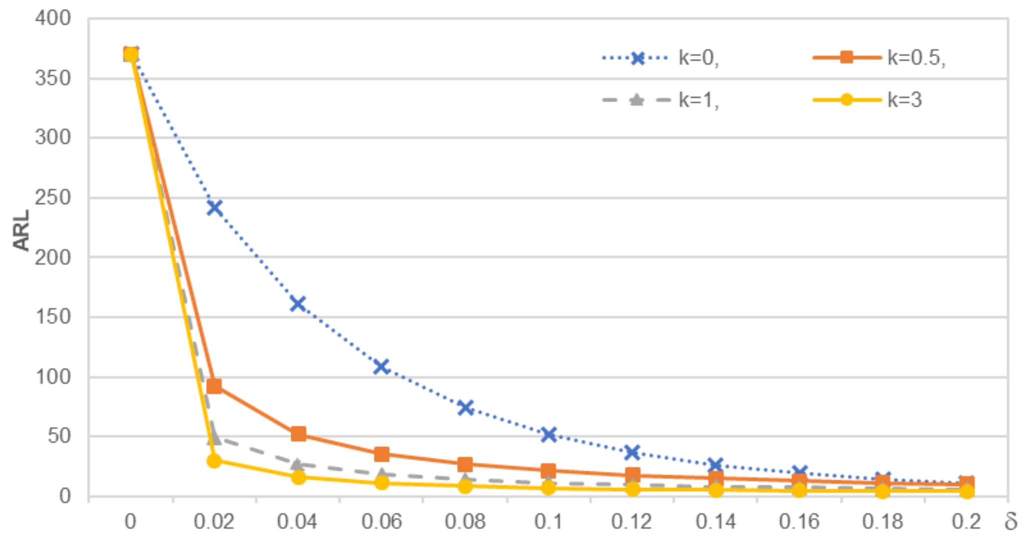

Figure 2. ARL comparison for the MEWMA control chart under the ARIMA(1,1,1) model across different k levels based on experimental results

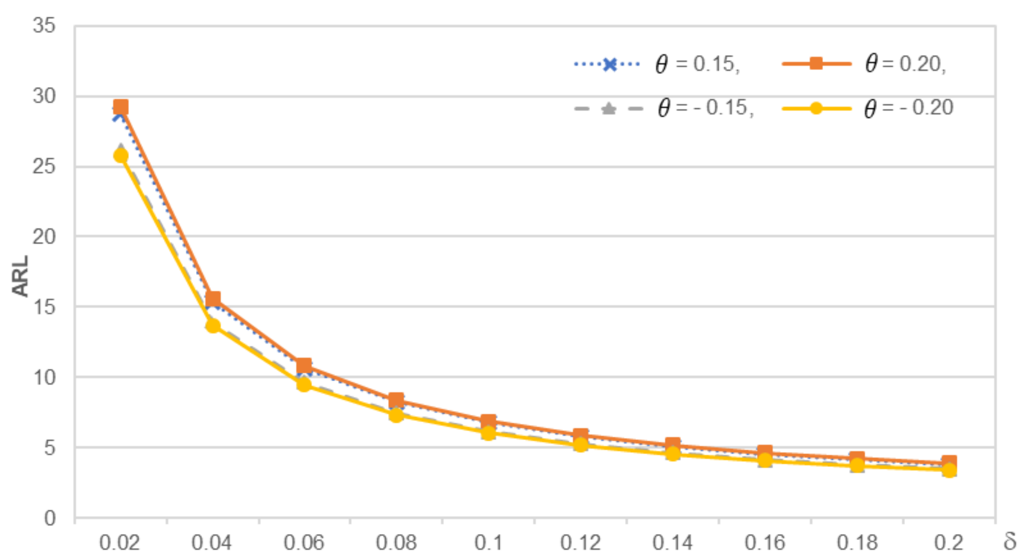

Figure 3. ARL comparison for the MEWMA control chart under the ARIMA(1,1,1) model with ϕ = 0.15 fixed and θ varied at 0.15, 0.20, - 0.15 and - 0.20 based on experimental results

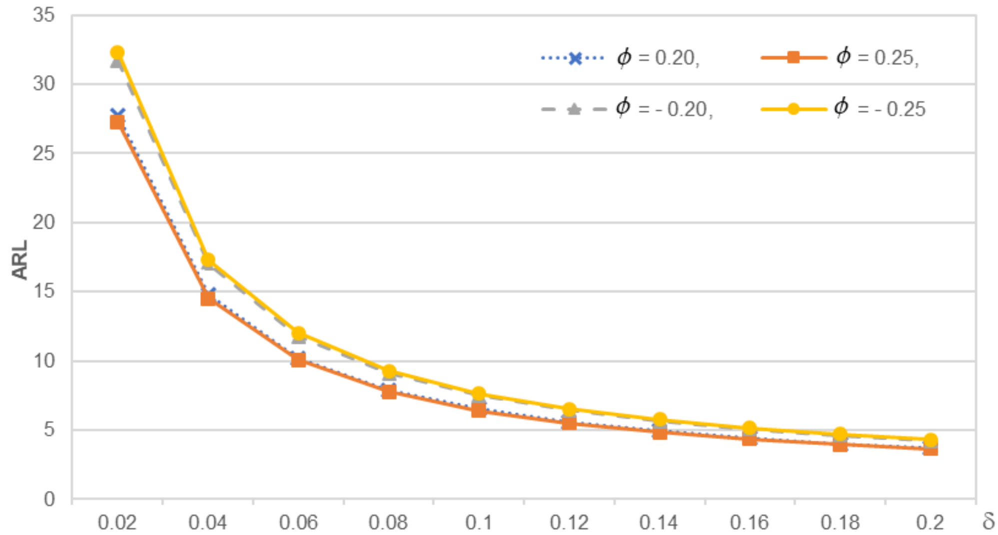

Figure 4. ARL comparison for the MEWMA control chart under the ARIMA(1,1,1) model with θ = 0.25 fixed and ϕ varied at 0.20, 0.25, - 0.20 and - 0.25 based on experimental results

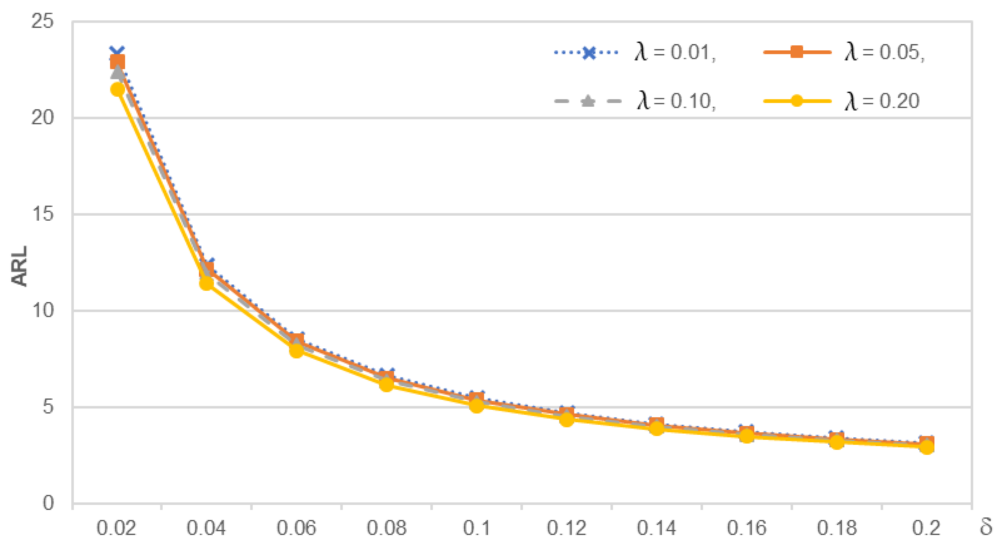

Figure 5. ARL comparison for the MEWMA control chart under the ARIMA(1,1,1) model at λ = 0.01, 0.05, 0.10 and 0.20 based on experimental results

Empirical Validation Using Real Data

To further evaluate the practical applicability and robustness of the proposed explicit formula and the NIE method, real-world time-series datasets were analyzed. These datasets were selected from distinct domains to demonstrate the flexibility of the proposed approach across different contexts as environmental monitoring and economic forecasting.

The first real dataset utilizes daily climate data from the city of Delhi, India, spanning the period from January 1, 2017, to April 24, 2017, obtained from the Indian Weather Forecasting database (https://www.kaggle.com/datasets). The dataset consists of 114 daily observations, providing sufficient data points to capture short-term variations and trends in key meteorological variables. The dataset includes four key meteorological parameters: mean temperature, humidity, wind speed, and mean pressure. Among these variables, wind speed (measured in km/h) was identified as the most suitable for analysis using the ARIMA(1,1,1) model due to its statistical properties and model fit performance. This analysis demonstrates how the proposed approach can be effectively applied to environmental data for detecting structural changes or anomalies in climatic patterns.

The second dataset consists of monthly time-series data on U.S. Rough Rice futures prices (https://www.investing.com/commodities). As a major cereal grain, rough rice represents one of the most important staple foods globally, serving as a primary source of nutrition and providing over one-fifth of the world's total caloric intake. The dataset consists of 120 monthly observations of the closing price, opening price, highest and lowest prices, as well as absolute and percentage changes in Rough Rice futures, covering the period from January 2014 to December 2023. Among these variables, the closing price of Rough Rice futures was identified as the most suitable for analysis using the ARIMA(1,1,1) model, based on its statistical characteristics and model fitting accuracy. This application highlights the method's potential use in economic and financial process monitoring, providing an efficient means to detect shifts or instabilities in commodity markets.

The wind speed variable from the daily climate dataset was fitted to the ARIMA(1,1,1) model, $\varepsilon_t \sim Exp(2.49)$, resulting in the following estimated equation

$$I_t = 0.034 + 1.426 I_{t-1} - 0.426 I_{t-2} - 0.997 \varepsilon_{t-1} + \varepsilon_t. \tag{17}$$

Similarly, the closing price of Rough Rice futures was modeled using the ARIMA(1,1,1) process, $\varepsilon_t \sim Exp(0.617)$, yielding the estimated equation

$$I_t = 1.773 I_{t-1} - 0.773 I_{t-2} - 0.849 \varepsilon_{t-1} + \varepsilon_t \tag{18}$$

Tables 5 and 6 present the ARL values of the two real datasets, obtained using both the explicit formula and the NIE method for the MEWMA control chart under the ARIMA(1,1,1) model. Both datasets were modeled using the ARIMA(1,1,1) process and analyzed within the MEWMA control chart to validate the performance and reliability of the proposed approach under real-world conditions. The ARL results across different levels of k (set to 0, 0.5, 1, and 3) exhibit trends consistent with the simulation experiments. Specifically, the results indicate that for the MEWMA control chart, the ARL_1 values obtained from both the explicit formula and the NIE method decrease sharply as the magnitude of the mean shift increases. However, for the case where k = 0, corresponding to the EWMA control chart, the ARL_1 values decrease more gradually compared to other k values. These findings reaffirm that increasing k enhances the sensitivity of the control chart, allowing it to detect smaller shifts in the process mean more effectively in Figure 6. Moreover, the close agreement between the ARL values obtained from both methods demonstrates the accuracy and computational efficiency of the proposed explicit formulation when applied to real-world time series data.

Overall, the results from both simulated and real-world datasets demonstrate the robustness and practicality of the proposed explicit ARL formulation. The strong agreement between the explicit formula and the NIE method confirms the analytical accuracy of the derived expressions, while the significantly reduced computational time highlights the method's efficiency. These findings suggest that the explicit ARL formulation provides a reliable and computationally superior alternative for monitoring processes governed by the ARIMA(1,1,1) model on the MEWMA control chart.

Table 5. ARL comparison for the wind speed of the daily climate dataset with θ = 0.997, ϕ = 0.426, λ = 0.20 and ARL_0 = 370

Parameters	δ	ARL		Process Time (Seconds)	
		$A(\upsilon)$	$A_N(\upsilon)$	$A(\upsilon)$	$A_N(\upsilon)$
k = 0 (EWMA),	0.00	370.004159509	370.004155918	< 0.001	2.593
u = 0.105791	0.02	243.783839483	243.783837361	< 0.001	2.578
	0.04	175.240289500	175.240288093	< 0.001	2.610
	0.06	132.855185881	132.855184882	< 0.001	2.563
	0.08	104.450579089	104.450578349	< 0.001	2.610
	0.10	84.340297474	84.340296907	< 0.001	2.578
	0.12	69.520804647	69.520804202	< 0.001	2.579
	0.14	58.261090080	58.261089724	< 0.001	2.562
	0.16	49.496548466	49.496548177	< 0.001	2.578
	0.18	42.538615876	42.538615638	< 0.001	2.547
	0.20	36.923689351	36.923689153	< 0.001	2.578
Expected ARL		49870.54699	49870.54662	**AD%**	7.4×10^{-7}
k = 0.5,	0.00	370.001901845	370.001319793	< 0.001	2.641
u = 1.9020345	0.02	87.993149640	87.993105435	< 0.001	2.610
	0.04	49.674226295	49.674208923	< 0.001	2.609
	0.06	34.502294780	34.502284957	< 0.001	2.547
	0.08	26.382416167	26.382409659	< 0.001	2.578
	0.10	21.332852903	21.332848195	< 0.001	2.594
	0.12	17.894086669	17.894083071	< 0.001	2.578
	0.14	15.404780129	15.404777272	< 0.001	2.594
	0.16	13.521799279	13.521796946	< 0.001	2.594
	0.18	12.049404341	12.049402396	< 0.001	2.578
	0.20	10.867798039	10.867796391	< 0.001	2.609
Expected ARL		14481.14041	14481.13566	**AD%**	3.3×10^{-5}
k = 1,	0.00	370.001861835	370.000165270	< 0.001	2.609
u = 4.2862825	0.02	74.639636815	74.639552145	< 0.001	2.609
	0.04	41.667657573	41.667626877	< 0.001	2.562
	0.06	28.986781902	28.986765208	< 0.001	2.563
	0.08	22.278210134	22.278199319	< 0.001	2.531
	0.10	18.129733727	18.129726013	< 0.001	2.562
	0.12	15.312482607	15.312476761	< 0.001	2.610
	0.14	13.275472622	13.275468005	< 0.001	2.610
	0.16	11.734835472	11.734831715	< 0.001	2.594
	0.18	10.529452586	10.529449459	< 0.001	2.594
	0.20	9.561084242	9.561081592	< 0.001	2.578
Expected ARL		12305.76738	12305.75885	**AD%**	6.9×10^{-5}
k = 3,	0.00	370.000552976	369.995977685	< 0.001	2.734
u = 14.644168	0.02	72.313557616	72.313362944	< 0.001	2.547
	0.04	40.336167056	40.336101104	< 0.001	2.547
	0.06	28.102600967	28.102566696	< 0.001	2.781
	0.08	21.645156880	21.645135399	< 0.001	2.578
	0.10	17.656889767	17.656874821	< 0.001	2.578
	0.12	14.950461291	14.950450181	< 0.001	2.562
	0.14	12.994484160	12.994475518	< 0.001	2.609
	0.16	11.515529096	11.515522148	< 0.001	2.500
	0.18	10.358546329	10.358540603	< 0.001	2.625
	0.20	9.429065218	9.429060405	< 0.001	2.578
Expected ARL		11965.12292	11965.10449	**AD%**	1.5×10^{-4}

Table 6. ARL comparison for the closing price of Rough Rice futures dataset with θ = 0.773, ϕ = 0.849, λ = 0.20 and ARL_0 = 370

Parameters	δ	ARL		Process Time (Seconds)	
		$A(\upsilon)$	$A_N(\upsilon)$	$A(\upsilon)$	$A_N(\upsilon)$
k = 0 (EWMA), u = 0.016004	0.00	370.001720951	370.001719622	< 0.001	2.609
	0.02	232.016332538	232.016331795	< 0.001	2.813
	0.04	162.065742269	162.065741792	< 0.001	2.688
	0.06	120.497600689	120.497600357	< 0.001	2.578
	0.08	93.367788069	93.36778783	< 0.001	2.562
	0.10	74.525879638	74.52587945	< 0.001	2.547
	0.12	60.846094093	60.84609395	< 0.001	2.562
	0.14	50.576829973	50.57682986	< 0.001	2.594
	0.16	42.663685363	42.66368527	< 0.001	2.656
	0.18	36.436209611	36.43620954	< 0.001	2.515
	0.20	31.449236226	31.44923616	< 0.001	2.656
Expected ARL		45222.26992	45222.2698	**AD%**	2.7×10^{-7}
k = 0.5, u = 0.27864404	0.00	370.000350323	370.000171323	< 0.001	2.625
	0.02	69.050030894	69.050021251	< 0.001	2.609
	0.04	37.866151665	37.866147834	< 0.001	2.547
	0.06	26.008363113	26.008360908	< 0.001	2.531
	0.08	19.773612696	19.773611215	< 0.001	2.594
	0.10	15.936245873	15.936244791	< 0.001	2.547
	0.12	13.341238059	13.341237226	< 0.001	2.532
	0.14	11.472518222	11.472517557	< 0.001	2.563
	0.16	10.064848460	10.064847914	< 0.001	2.562
	0.18	8.967961777	8.967961321	< 0.001	2.625
	0.20	8.090373145	8.090372758	< 0.001	2.640
Expected ARL		11028.5672	11028.56614	**AD%**	9.6×10^{-6}
k = 1, u = 0.6192117	0.00	370.000372995	369.999887000	< 0.001	2.499
	0.02	52.721103613	52.721089389	< 0.001	2.594
	0.04	28.617918895	28.617913589	< 0.001	2.594
	0.06	19.754902234	19.754899247	< 0.001	2.563
	0.08	15.153090714	15.153088724	< 0.001	2.547
	0.10	12.336944737	12.336943287	< 0.001	2.515
	0.12	10.437261544	10.437260427	< 0.001	2.578
	0.14	9.070135417	9.070134522	< 0.001	2.578
	0.16	8.039757115	8.039756379	< 0.001	2.547
	0.18	7.235790591	7.235789973	< 0.001	2.625
	0.20	6.591327256	6.591326728	< 0.001	2.531
Expected ARL		8497.911606	8497.910113	**AD%**	1.8×10^{-5}
k = 3, u = 2.0800305	0.00	370.006211689	370.004999016	< 0.001	2.531
	0.02	42.249938644	42.249917587	< 0.001	2.797
	0.04	22.879786980	22.879779531	< 0.001	2.562
	0.06	15.903617460	15.903613353	< 0.001	2.578
	0.08	12.308977739	12.308975030	< 0.001	2.594
	0.10	10.116677235	10.116675268	< 0.001	2.594
	0.12	8.639928077	8.639926563	< 0.001	2.609
	0.14	7.577474416	7.577473203	< 0.001	2.578
	0.16	6.776359934	6.776358934	< 0.001	2.562
	0.18	6.150673281	6.150672439	< 0.001	2.578
	0.20	5.648441440	5.648440718	< 0.001	2.578
Expected ARL		6912.59376	6912.591631	**AD%**	3.1×10^{-5}

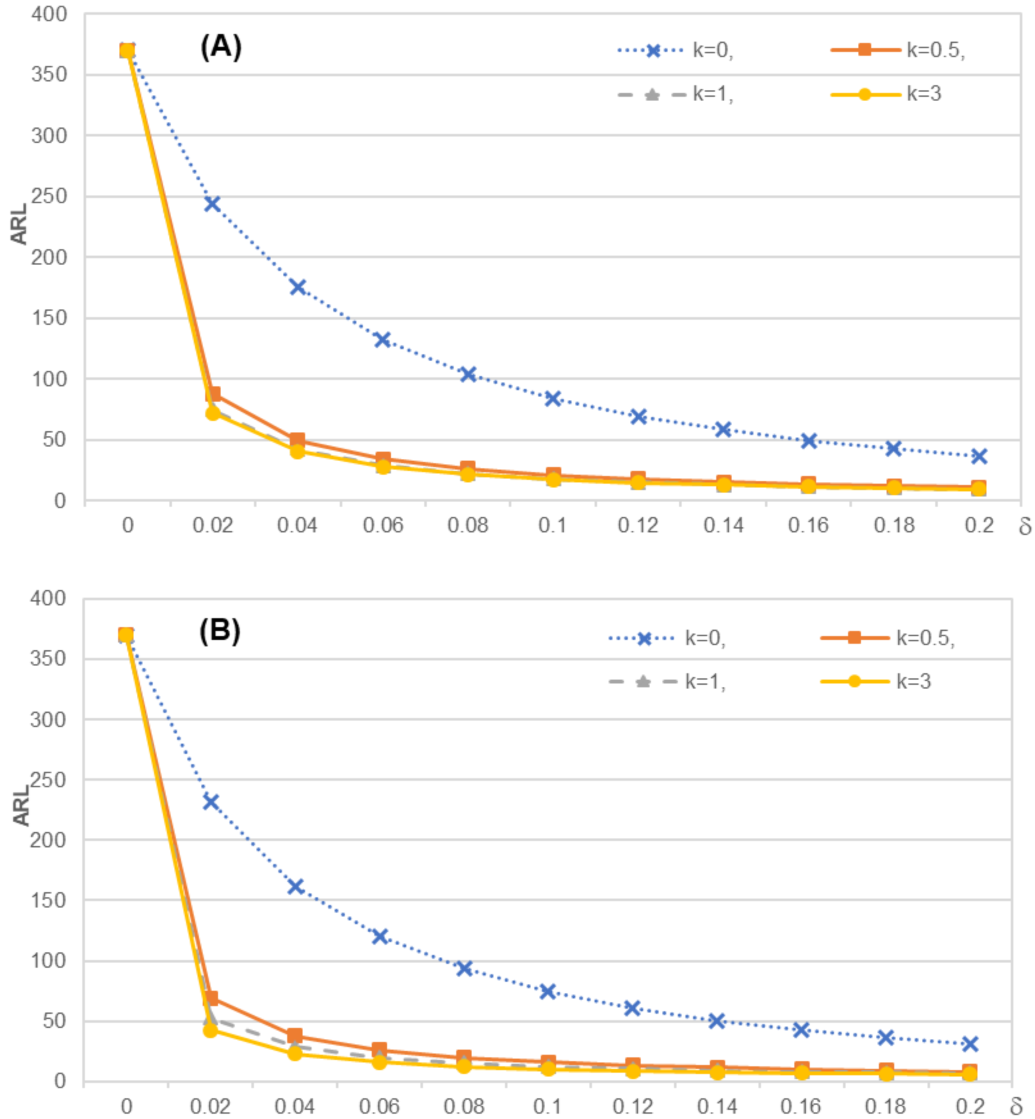

Figure 6. ARL comparison for the MEWMA control chart under the ARIMA(1,1,1) model across different k levels for **(A)** the wind speed of the daily climate dataset and **(B)** the closing price of Rough Rice futures dataset

Conclusions

This study successfully derives an explicit formula for the ARL of the MEWMA control chart based on the ARIMA(1,1,1) model with exponential white noise. The proposed formulation offers a mathematically tractable and computationally efficient method for evaluating control chart performance, overcoming the limitations of numerical estimation approaches. Comparative analyses between the explicit and NIE ARL values reveal a strong concordance, with absolute percentage differences approaching zero across various parameter configurations. Furthermore, the explicit formulation significantly reduces computation time, demonstrating its superiority in both precision and efficiency. Overall, the results demonstrate that the time series parameters ϕ and θ play a crucial role in control chart performance. Specifically, larger values of ϕ and smaller values of θ improve the shift detection capability of the proposed control chart, as evidenced by reduced ARL_1 values. Applications to real-world datasets, including daily climate data from Delhi and U.S. Rough Rice futures, further validate the model's robustness and adaptability to diverse time-dependent processes. The results indicate that increasing the sensitivity parameter k enhances detection capability for process shifts, aligning with theoretical expectations. Future research may extend this approach to more complex time series structures such as SARIMA or ARFIMA models, as well as to advanced control charts like double or triple MEWMA charts to further improve monitoring performance and adaptability.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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