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RESEARCH ARTICLE

Development of EWMA Control Chart for Detecting Changes in AR(p) with Quadratic Trend Model

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Abstract This study is intended to propose a formula for the Average Run Length (ARL) of the Exponentially Weighted Moving Average (EWMA) control chart when the observed data follow an autoregressive model of order p with quadratic trend. This research emphasizes the fundamental importance of developing precise ARL computation techniques with optimal processing efficiency, as ARL remains the predominant criterion for control chart performance evaluation. The derivation of the explicit ARL formula employs Fredholm's integral equation methodology, with solution uniqueness assured through the application of Banach's Fixed Point Theorem. Performance validation involves comparative analysis against approximate ARL values obtained via Numerical Integral Equation (NIE) approaches, specifically utilizing the Midpoint rule technique. The efficiency of the explicit formula of ARL is evaluated using two criteria: absolute percentage difference and CPU Time. The empirical results confirm that the ARL values derived from the explicit formula closely approximate those obtained via numerical integral equation methods, exhibiting an absolute percentage difference of less than 0.001%. Computationally, the proposed explicit formula achieves processing times of approximately 0.001 seconds, while the Midpoint rule method takes 2-3 seconds. In conclusion, the results demonstrate that the proposed explicit ARL formulas for EWMA charts provide accuracy comparable to the NIE method while significantly reducing computational time. This confirms the efficiency and practice applicability to the explicit formulas for monitoring real-world data, such as pneumonia cases at Siriraj Hospital.

Keywords: Explicit formulas, Numerical integral equation, Autoregressive model with quadratic trend.

Introduction

Statistical process control represents a widespread methodology for monitoring and improving the data in multiple application areas including healthcare, manufacturing, and environmental. In 1924, Shewhart [1] initiated the development of the Shewhart control chart. Subsequently, Robert [2] introduction of the Exponential Weighted Moving Average (EWMA) control chart. The EWMA control chart demonstrates enhanced performance compared to Shewhart chart, particularly in identifying slight process variations.

The term "Average Run Length (ARL)" refers to the expected number of samples taken before a control chart signals an out-of-control condition. The ARL comprises two key elements: in-control ARL (ARL0) and out-of-control ARL (ARL1). There are several principal methods for calculating ARL, such as Monte Carlo Simulations (MC), the Markov Chain Approach (MCA), Numerical Integral Equation (NIE) and the Explicit Formulas.

Calculating the ARL using integral equations is a complex analytical method, particularly valuable for control charts like the Cumulative Sum (CUSUM) and EWMA. This approach often involves formulating the ARL as the unique solution to a Fredholm integral equation of the second kind. The integral equation method provides precise, explicit ARL formulas with superior computational speed compared to simulation methods, particularly for autocorrelated time series models (autoregressive (AR) model, moving average

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(MA) model, autoregressive integrated moving average (ARIMA) model, seasonal autoregressive integrated moving average (SARIMA) model). Direct analytical solutions are not always feasible. However, NIE methods such as Midpoint rules can provide highly accurate ARL approximations. Crowder [3] developed a numerical technique for determining run length in EWMA control chart based on normal distribution, later expanding this approach to non-normal scenarios and one-sided EWMA control charts. Lucas and Saccucci (1990) investigated the EWMA control chart for monitoring process means and introduced MCA-based design methodology, including parameter values specifically for detecting small shifts [4].

Nevertheless, acknowledging the limitations of MC and MCA methods, researchers have investigated alternative techniques, such as the integral equation method. Areepong [5] developed analytical solutions for calculating the average delay and the ARL in EWMA charts with exponential distribution observations. Recently, Mititelu *et al*. [6] employed the Fredholm integral equation method to create explicit ARL formulas for specialized control charts, such as CUSUM and EWMA control charts, providing a more efficient computational alternative.

While control charts usually assume independent and identically distributed observations, specialized control charts are needed for autocorrelated processes. Several researchers have studied the method for finding the ARL using Integral Equation. In 1997, Vanbrackle and Reynolds [7] advocated for using the IE approach to determine the ARL of EWMA and CUSUM charts for an AR(1) model with additional random errors. Busaba *et al*. [8] provided analytical ARL solutions for CUSUM charts using stationary AR(1) models. Petcharat *et al*. [9] advanced this by developing explicit ARL formulas for both EWMA and CUSUM control charts applied on moving average model. Sunthornwat *et al*. [10] compared two methods for calculating the ARL of EWMA and CUSUM charts: analytical and numerical approaches. Their research showed that the analytical ARL method effectively evaluates the performance of EWMA control charts. Phanthuna *et al*. (2021) introduced a methodology for calculating the ARL of a modified EWMA control chart under a trend autoregressive (AR(1)) model. Their research compared the NIE approach with an explicit formula, concluding that the explicit formula provides superior accuracy and computational time efficiency. Additionally, their results indicated that the modified EWMA chart is more effective than the conventional EWMA scheme in detecting small to moderate shifts [11]. Phanthuna and Areepong (2021) developed an explicit formula for ARL calculation in a modified EWMA chart designed specifically for data generated by a seasonal autoregressive ($SAR(p)_L$) model with exponential residuals. They confirmed the formula's accuracy across various real-world datasets and found that the modified EWMA control chart outperforms the conventional EWMA scheme when detecting small shifts [12]. Phanyaem [13] developed explicit formulas and NIE methods to calculate the ARL for the CUSUM chart based on $SARX(P,r)_L$ model, using the Fredholm integral equation and numerical methods like the midpoint, trapezoidal, Simpson's, and Gaussian rules. Petcharat [14] later introduced an explicit ARL formula for a CUSUM chart monitoring the $SAR(P)_L$ model with a trend, applying Banach's fixed-point theorem to ensure solution existence and uniqueness. Peerajit [15] compared analytical integral equations for the ARL derived from Banach's theorem with the NIE method for a fractionally integrated moving average with exogenous variables (FIMAX) model. Karoon and Areepong [16] advanced statistical process control by proposing a Modified EWMA (MEWMA) chart and deriving an ARL formula for the autoregressive with trend and quadratic trend models with exponential white noise. The performance of the proposed method was validated against the NIE technique using both simulated and real financial data from Thailand and the United States. Areepong and Sukparungsee [17] focus on the development and comprehensive analysis of a Homogenously Weighted Moving Average (HWMA) control chart formulated to monitor autoregressive processes with a quadratic trend under zero-state conditions. The study derives an explicit expression for the ARL and evaluates its accuracy in comparison with the NIE method. Furthermore, the performance of the proposed HWMA chart is benchmarked against the Modified EWMA (MEWMA) chart for AR(1) and AR(3) models under varying levels of process mean shifts.

The major objective of this research study was to derive the explicit formula of ARL and to approximate the ARL for EWMA control chart based on an AR(p) model with quadratic trend. This study involved two techniques including the integral equation and the numerical integral equation. The structure of this paper is organized as follows: Section 1 introduces the research background and objectives. Section 2 describes the research methodology. Section 3 presents the proposed explicit formula for ARL computation, while Section 4 details the numerical integral equation (NIE) method. Section 5 discusses the results obtained from the proposed approach, and Section 6 concludes the study and offers suggestions for further work.

Methodology

In this section, the autoregressive model of order p (AR(p)) with quadratic trend is described. This model is used in the EWMA control chart to monitor changes in the process mean. The final subsection evaluates the ARL properties, which are essential for assessing the performance of the control chart.

An AR(p) Model with Quadratic Trend

An AR(p) model with quadratic trend combines two main components: an autoregressive model of order p and quadratic trend components. The AR(p) model with quadratic trend can be defined as follows (Equation 1):

$$Y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \sum_{i=1}^{p} \phi_i Y_{t-i} + \varepsilon_t \; ; t = 1,2,.. \tag{1}$$

where ϕ_i ; $i = 1,2,...,p$ are an autoregressive coefficient

$Y_{t-1}, Y_{t-2}, ..., Y_{t-p}$ are the initial values of Y_t

β_0 is a constant term

β_1 is the linear coefficient

β_2 is the quadratic coefficient

ε_t is the white noise error.

The Properties of EWMA Control Chart

In 1959, an improvement to statistical quality control was achieved by Robert [2] with the development of the EWMA control chart. It is a powerful statistical tool for monitoring processes and effectively detecting small shifts. The mathematical expression for the EWMA statistic is given as follows (Equation 2):

$$E_t = \lambda Y_t + (1-\lambda) E_{t-1}; t = 1,2,... \tag{2}$$

where E_t is the EWMA statistics

Y_t is the current observation

E_{t-1} is the previous of the EWMA statistics

λ is an exponential smoothing parameter.

The upper and lower control limits for the EWMA control chart are determined by Equation (3) and (4), respectively

$$UCL = \mu + L\sqrt{\sigma^2 \frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \tag{3}$$

$$LCL = \mu - L\sqrt{\sigma^2 \frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \tag{4}$$

where μ and σ^2 are the mean and variance of the in-control process, respectively.

The Average Run Length

The average run length (ARL) is the expected number of plotted statistics until a control chart signals an out-of-control condition. In situations where the process is in-control, the ARL should be sufficiently large to avoid too many false alarms. Conversely, for out-of-control process, the ARL should be sufficiently small to allow rapid detection of shifts. The function represents the ARL for AR(p) model with quadratic trend at an initial value $E_0 = x$, as follows (Equation 5):

$$ARL = L(x) = \mathbf{E}_\infty(\tau_b) < \infty, \tag{5}$$

where b is the constant parameter as upper control limit.

Proposed Explicit Formulas

In this section, we introduce the Fredholm integral equation of the second kind to solve for the explicit formula of ARL for the EWMA control chart. The initial ARL derived from an explicit formula method, which can be described as:

$$L(x)=1+\int_A^B L[(1-\lambda)x+\lambda\beta_0+\lambda\beta_1 t+\lambda\beta_2 t^2+\lambda\sum_{i=1}^{p}\phi_i Y_{t-i}]f(\varepsilon_1)d\varepsilon_1, \quad (6)$$

where $A=\dfrac{(1-\lambda)x-\lambda\beta_0-\lambda\beta_1 t-\lambda\beta_2 t^2-\lambda\sum_{i=1}^{p}\phi_i Y_{t-i}}{\lambda}$

and $B=\dfrac{b-(1-\lambda)x-\lambda\beta_0-\lambda\beta_1 t-\lambda\beta_2 t^2-\lambda\sum_{i=1}^{p}\phi_i Y_{t-i}}{\lambda}$.

Let $z=(1-\lambda)x+\lambda\beta_0+\lambda\beta_1 t+\lambda\beta_2 t^2+\lambda\sum_{i=1}^{p}\phi_i Y_{t-i}$.

Changing the integral variable, we obtain the integral equation as follows

$$L(x)=1+\frac{1}{\lambda}\int_0^b L(z)\times f\left(\frac{z-(1-\lambda)x}{\lambda}-\sum_{i=1}^{p}\phi_i Y_{t-i}-\beta_0-\beta_1 t-\beta_2 t^2\right)dz. \quad (7)$$

In order to obtain the following integral equation:

$$L(x)=1+\frac{1}{\lambda\alpha}\int_0^b L(z)e^{-\frac{z}{\lambda\alpha}}e^{\frac{(1-\lambda)x}{\lambda\alpha}+\frac{\sum_{i=1}^{p}\phi_i Y_{t-i}+\beta_0+\beta_1 t+\beta_2 t^2}{\alpha}}dz. \quad (8)$$

To prove the existence of a unique fixed point for the function $K:[0,b]\to[0,b]$ using Bananch's fixed point theorem, we proceed as follows:

Theorem. Let $K:[0,b]\to[0,b]$ be a contraction mapping $\forall x\in[0,b]$ on the matric space $\left([0,b],\|\cdot\|_\infty\right)$. There exists a constant κ with $0\le\kappa<1$ such that:

$$\|K(L_1)-K(L_2)\|\le\kappa\|L_1-L_2\|.$$

Then, consistent with the Banach's fixed point theorem, the existence of unique fixed point is guaranteed. consistent with the Banach's fixed point theorem, the existence of unique fixed point ARL is guaranteed.

Proof.

Firstly, define the metric space as $\left([0,b],\|\cdot\|_\infty\right)$.

Assume that $K:[0,b]\to[0,b]$ is a contraction mapping $\forall x\in[0,b]$. Define the operator K working on the function $L(x)$ as follows:

$$K(L(x))=1+\frac{1}{\lambda\alpha}\int_0^b L(z)L(x,z)dz,$$

where $L(x,z)=e^{-\frac{z}{\lambda\alpha}}e^{\frac{(1-\lambda)x}{\lambda\alpha}+\frac{\sum_{i=1}^{p}\phi_i Y_{t-i}+\beta_0+\beta_1 t+\beta_2 t^2}{\alpha}}$.

There exists a constant κ such that

$$\|K(L_1)-K(L_2)\| \le \kappa \|L_1-L_2\|;\ 0<\kappa<1.$$

For all $L_1, L_2 \in ([0,b])$.

Apply Banach' fixed point theorem, which states that if K is a contraction mapping on a metric space, then K has exactly one fixed point. Therefore, there exists a unique function $L(x)\in[0,b]$ such that

$$K(L(x)) = L(x).$$

Hence, $\|K(L_1)-K(L_2)\|_\infty \le \dfrac{1}{\lambda\alpha} \underset{x\in[0,b]}{sub} \int_0^b |L_1(z)-L_2(z)||L(x,z)|dz.$

Since $L(x,z)$ is bounded by some constant $\kappa,$ we get

$$\|K(L_1)-K(L_2)\|_\infty \le \frac{\kappa b}{\lambda\alpha}\|L_1-L_2\|_\infty .$$

where a constant $\kappa\in[0,1]$, we have $K:[0,b]\to[0,b]$ is a contraction mapping and has a fixed point. Thus, $L(x)$ is the existence and unique solution. ■

This section, we will analyse the integral equations presented in (8) to derive an explicit formula for the ARL of the EWMA control chart. This chart is applied on the AR(p) model with quadratic trend, and our approach employs the Fredholm integral equation.

The Fredholm integral equation of the second kind has the following general form:

$$L(x) = f(x) + \lambda \int_0^b L(x,z)L(z)\,dz.$$

Let $C(x) = e^{\frac{(1-\lambda)x}{\lambda\alpha} + \frac{\sum_{i=1}^{p}\phi_i Y_{t-i} + \beta_0 + \beta_1 t + \beta_2 t^2}{\alpha}}$, we have

$$L(x) = 1 + \frac{C(x)}{\lambda\alpha}\int_0^b L(z)e^{-\frac{z}{\lambda\alpha}}dz \tag{9}$$

Let $k = \int_0^b L(z)e^{-\frac{z}{\lambda\alpha}}dz$ represent an integral equation of $L(x)$. Equation (9) can be rewritten as:

$$L(x) = 1 + \frac{C(x)}{\lambda\alpha}k. \tag{10}$$

Solving for $k = \int_0^b L(z)e^{-\frac{z}{\lambda\alpha}}dz,$ we obtain:

$$\begin{aligned} k &= \int_0^b L(z)e^{-\frac{z}{\lambda\alpha}}dz \\ &= \int_0^b \left(1+\frac{C(z)}{\lambda\alpha}k\right)e^{-\frac{z}{\lambda\alpha}}dz, \\ &= \int_0^b e^{-\frac{z}{\lambda\alpha}}dz + \int_0^b \frac{C(z)}{\lambda\alpha}ke^{-\frac{z}{\lambda\alpha}}dz, \end{aligned}$$

$$= \int_0^b e^{-\frac{z}{\lambda\alpha}} dz + \frac{k}{\lambda\alpha}\int_0^b e^{\frac{(1-\lambda)x}{\lambda\alpha}+\frac{\sum_{i=1}^{p}\phi_i Y_{t-i}+\beta_0+\beta_1 t+\beta_2 t^2}{\alpha}} e^{-\frac{z}{\lambda\alpha}} dz,$$

$$= \frac{-\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)}{1+\frac{1}{\lambda}e^{\frac{\sum_{i=1}^{p}\phi_i Y_{t-i}+\beta_0+\beta_1 t+\beta_2 t^2}{\alpha}}(e^{-\frac{b}{\alpha}}-1)}. \quad (11)$$

Consequently, the integral equation of $L(x)$ can be expressed as

$$k = \frac{-\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)}{1+\frac{1}{\lambda}e^{\frac{\sum_{i=1}^{p}\phi_i Y_{t-i}+\beta_0+\beta_1 t+\beta_2 t^2}{\alpha}}(e^{-\frac{b}{\alpha}}-1)}.$$

By substituting Equation (11) into Equation (10), we derive a mathematical expression for the ARL of the EWMA control chart when the data follow an AR(p) model with quadratic trend as shown in Equation (12).

$$L(x) = 1 - \frac{\lambda e^{\frac{(1-\lambda)x}{\lambda\alpha}}(e^{-\frac{b}{\lambda\alpha}}-1)}{\lambda e^{-\frac{\sum_{i=1}^{p}\phi_i Y_{t-i}+\beta_0+\beta_1 t+\beta_2 t^2}{\alpha}}+(e^{-\frac{b}{\alpha}}-1)}. \quad (12)$$

Numerical Integral Equation Method

The following section outlines the numerical integral equation (NIE) framework for computing ARL values in EWMA control charts monitoring AR(p) model with quadratic trend, assuming exponential distribution white noise.

$$L(x) = 1 + \frac{1}{\lambda}\int_0^b L(z) \times f\left(\frac{z-(1-\lambda)x}{\lambda} - \sum_{i=1}^{p}\phi_i Y_{t-i} - \beta_0 - \beta_1 t - \beta_2 t^2\right) dz. \quad (13)$$

Therefore, we transform the integral equation into a numerical integral equation.

Given $f(a_j) = f\left(\frac{a_j-(1-\lambda)a_i}{\lambda} - d\right),$

where $d = \sum_{i=1}^{p}\phi_i Y_{t-i} + \beta_0 + \beta_1 t + \beta_2 t^2.$

A quadrature rule is applied by dividing the interval $[0,b]$ into partitions $0 \le a_1 \le a_2 \le \ldots \le a_m \le b$, where is the number of partition points and is a set of constant weights. The numerical integral equation (NIE) method for function $L(x)$ can be approximated as:

$$\tilde{L}(a_i) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_i}{\lambda} - d\right)$$

This equation can now be solved iteratively for $\tilde{L}(a_i)$ using numerical methods as follows:

$$\begin{aligned}
\tilde{L}(a_1) &\approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_1}{\lambda} - d\right)\\
\tilde{L}(a_2) &\approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_2}{\lambda} - d\right)\\
&\vdots\\
\tilde{L}(a_m) &\approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_m}{\lambda} - d\right)
\end{aligned}$$

or as a matrix $\mathbf{L}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{R}_{m\times m}\mathbf{L}_{m\times 1}$,

where $\mathbf{L}_{m\times 1} = \begin{pmatrix} \tilde{L}(a_1) \\ \tilde{L}(a_2) \\ \vdots \\ \tilde{L}(a_m) \end{pmatrix}$, $\mathbf{1}_{m\times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, $[\mathbf{R}]_{ij} \approx \frac{1}{\lambda} w_j f\left(\frac{a_j - (1-\lambda)a_i}{\lambda} - d\right)$,

and $\mathbf{I}_m = \operatorname{diag}(1,1,\ldots,1)$. If $(\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}$ there exist $\mathbf{L}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}$.

Therefore, we transform the integral equation into a numerical integral equation using Midpoint Rule. Rewriting equation (13) in a structured numerical form as follow:

$$\tilde{L}(x) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_i}{\lambda} - d\right) \quad (14)$$

where $a_j = \left(j - \frac{1}{2}\right)w_j$ and $w_j = \frac{b}{m}; j = 1,2,\ldots,m$

Comparative Study

This section presents a comprehensive comparative analysis of the proposed explicit formula against the NIE method. The optimal design of the EWMA chart depends on the appropriate selection of two essential parameters: the exponential smoothing parameter (λ) and the upper control limit (b), both exhibiting substantial influence on overall performance metrics. Consequently, various configurations of design parameters are employed to evaluate ARL performance metrics. The exponential smoothing parameters (λ) of 0.10, 0.20, and 0.30 were selected because they are commonly used in EWMA control charts and represent varying sensitivities to process shifts. Accordingly, the design configurations employ these smoothing parameters, with the control limit parameter (b) is optimized to maintain ARL_0 = 370. The ARL_1 is computed for a range of out-of-control parameter values, $\alpha_1 = \alpha_0(1+\delta)$; δ = 0.10, 0.20, 0.30, 0.40, 0.50, 1.00, 1.50, and 2.00, respectively. The numerical results regarding the proposed explicit formula are displayed in Table 1-3.

Table 1 presents the ARL values of the EWMA chart applied to AR(1) models with quadratic trend (parameter values: ϕ_1 = 0.30, β_1 = 0.10, β_2 = 0.10), at ARL_0 = 370. The configurations examined include $(\lambda$ = 0.10, b

= 0.00108), (λ = 0.20, b = 0.0444), and (λ = 0.30, b = 0.074862). The proposed formula provides ARL values that closely approximate those obtained from the NIE method across all magnitudes of shift sizes. The accuracy of the proposed method is evaluated using absolute percentage difference, which consistently maintains values below 0.001. In addition, the proposed explicit formula demonstrates superior computational time efficiency, requiring approximately 0.001 seconds, while the NIE method requires approximately 2.415 seconds to compute. Moreover, Figure 1 illustrates the performance of EWMA charts for AR(1) models with quadratic trend, demonstrating that the configuration with λ = 0.30, b = 0.074862 achieves optimal shift detection efficiency compared smoothing parameter (λ = 0.10, and 0.20).

Table 2 exhibits ARL computations for EWMA charts monitoring an AR(2) model with quadratic trend (parameter values: ϕ_1 = 0.70, ϕ_2 = 0.20, β_1 = 0.20, β_2 = 0.30), at ARL_0 = 370. Validation is conducted using three parameter sets: (λ = 0.10, b = 0.000178), (λ = 0.20, b = 0.007141), (λ = 0.30, b = 0.0119721). The explicit formula shows remarkable concordance with NIE derived ARL values throughout the entire range of shift magnitudes. In addition, the proposed explicit formula demonstrates superior computational time efficiency, requiring approximately 0.001 seconds, whereas the NIE method necessitates about 2.415 seconds for execution. Figure 2 illustrates that among the EWMA chart configurations for AR(2) models with quadratic trend, the setting of λ = 0.30, b = 0.0119721 provides the most effective shift detection when compared to the settings with λ = 0.10 and 0.20.

As shown in Table 3, the ARL values were computed for EWMA control charts designed to monitor an AR(3) model with quadratic trend, based on parameter values of ϕ_1 = 0.50, ϕ_2 = 0.30, ϕ_3 = 0.10, β_1 = 1.00, β_2 = 0.80), at ARL_0 = 370. Three distinct configurations were evaluated in this study: (λ = 0.10, b = 0.0002932), (λ = 0.20, b = 0.011814), (λ = 0.30, b = 0.019821). The ARL values estimated by the explicit formula exhibit excellent concordance with those computed by the NIE method throughout the full range of shift magnitudes. Furthermore, the proposed explicit formula exhibits significantly greater computational efficiency, with an average computational time of approximately 0.001 seconds, compared to the NIE method, which requires around 2.478 seconds to complete. Figure 3 illustrates that among the EWMA chart configurations for AR(3) models with quadratic trend, the setting of λ = 0.30, and b = 0.0119721 provides the most effective shift detection when compared to the settings with λ = 0.10 and 0.20.

In a similar manner, Tables 4 to 6 present the ARL results obtained from the proposed explicit formula in comparison with those calculated using the NIE method, with ARL_0 = 500. Table 4 presents the ARL values of the EWMA control chart applied to AR(1) models with quadratic trend (parameter values: ϕ_1 = 0.80, β_1 = 0.40, β_2 = 0.50), at ARL_0 = 500. The configurations examined include (λ = 0.10, b = 0.0000455), (λ = 0.20, b = 0.0014127), and (λ = 0.30, b = 0.0023038). The proposed formula provides the ARL values closely approximate those obtained from the NIE method across all magnitudes of shift sizes. The accuracy of the proposed method is assessed through absolute percentage difference, which consistently maintains values below 0.001. In addition, the proposed explicit formula demonstrates superior computational time efficiency, requiring approximately 0.001 seconds, whereas the NIE method necessitates approximately 2.415 seconds for execution. Furthermore, Figure 4 displays the performance assessment of EWMA charts for AR(1) models with quadratic trend, demonstrating that the configuration with λ = 0.30, b = 0.0023038 achieves optimal shift detection efficiency compared smoothing parameter (λ = 0.10, and 0.20).

Table 5 exhibits ARL computations for EWMA charts monitoring AR(2) model with quadratic trend (parameter values: ϕ_1 = 0.60, ϕ_2 = 0.10, β_1 = 0.50, β_2 = 0.80), at ARL_0 = 500. Validation is conducted using three parameter sets: (λ = 0.10, b = 0.00000873), (λ = 0.20, b = 0.0002711), (λ = 0.30, b = 0.00044201). The explicit formula remarkable concordance with NIE derived ARL values throughout the entire range of shift magnitudes. In addition, the proposed explicit formula demonstrates superior computational time efficiency, requiring approximately 0.001 seconds, whereas the NIE method necessitates approximately 2.409 seconds for execution. Figure 5 illustrates that among the EWMA control chart configurations for AR(2) models with quadratic trend, the setting of λ = 0.30 and b = 0.00044201 provides the most effective shift detection when compared to the settings with λ = 0.10 and 0.20.

Table 6 presents the ARL computations for EWMA charts monitoring an AR(3) model with quadratic trend (parameter values: ϕ_1 = 0.10, ϕ_2 = 0.30, ϕ_3 = 0.50, β_1 = 0.80, β_2 = 0.20), at ARL_0 = 500. Validation was performed using three parameter sets: (λ = 0.10, b = 0.000872), (λ = 0.20, b = 0.02748), (λ = 0.30, b =

0.045005). The explicit formula exhibits excellent agreement with the ARL values obtained via the NIE method across all shift magnitudes. In addition, the proposed explicit formula demonstrates superior computational time efficiency, requiring approximately 0.001 seconds, whereas the NIE method necessitates approximately 2.456 seconds for execution. Figure 6 illustrates that among the EWMA control chart configurations for AR(3) models with quadratic trend, the setting of λ = 0.30 and b = 0.045005 provides the most effective shift detection when compared to the settings with λ = 0.10 and 0.20.

Table 1. ARL and CPU time from the explicit formulas and the NIE methods on EWMA control chart for AR(1) model with quadratic trend at ϕ_1 = 0.30, β_1 = 1.0, β_2 = 0.10 given ARL₀ = 370

Parameters of Control Chart	Shift Size	Average Run Length (ARL)			CPU Time (Second)	
		Explicit Formulas	NIE	Absolute %Diff	Explicit Formulas	NIE
λ = 0.10	0.00	370.1323152703	370.1323152631	1.94525×10−9	0.001	2.325
b = 0.00108	0.10	130.2552374612	130.2552374591	1.61224×10−9	0.001	2.371
	0.20	54.5680741123	54.5680741116	1.28281×10−9	0.001	2.371
	0.30	26.2785420473	26.2785420471	7.61077×10−10	0.001	2.356
	0.40	14.2132446887	14.2132446886	7.03569×10−10	0.001	2.355
	0.50	8.4972299698	8.4972299697	1.17685×10−9	0.001	2.449
	1.00	1.9869614909	1.9869614909	0	0.001	2.387
	1.50	1.2782962470	1.2782962470	0	0.001	2.387
	2.00	1.1157644318	1.1157644318	0	0.001	2.434
λ = 0.20	0.00	370.0479529191	370.0479489937	1.06078×10−6	0.001	2.449
b = 0.0444	0.10	85.3136314047	85.3136307773	7.35404×10−7	0.001	2.418
	0.20	37.4792654349	37.4792652152	5.86191×10−7	0.001	2.403
	0.30	20.7117143850	20.7117133890	4.80887×10−6	0.001	2.434
	0.40	13.0550735324	13.0550734804	3.98313×10−7	0.001	2.464
	0.50	9.0024352637	9.0024352338	3.32132×10−7	0.001	2.434
	1.00	3.0069460880	3.0069460838	1.39677×10−7	0.001	2.512
	1.50	1.8710108117	1.8710108105	6.41364×10−8	0.001	2.480
	2.00	1.4887631687	1.4887631682	3.35849×10−8	0.001	2.496
λ = 0.30	0.00	370.1806506599	370.1806333883	4.66572×10−6	0.001	2.387
b = 0.074862	0.10	28.8634025198	28.8634022032	1.09689×10−6	0.001	2.449
	0.20	12.9513582458	12.9513581454	7.75208×10-7	0.001	2.450
	0.30	7.8210564651	7.8210564186	5.94549×10−7	0.001	2.434
	0.40	5.4482817307	5.4482817052	4.68037×10−7	0.001	2.340
	0.50	4.1437655781	4.1437655627	3.71643×10−7	0.001	2.387
	1.00	2.0068447891	2.0068447864	1.34540×10−7	0.001	2.387
	1.50	1.5108611824	1.5108611815	5.95687×10−8	0.001	2.465
	2.00	1.3188005290	1.3188005286	3.03306×10−8	0.001	2.371

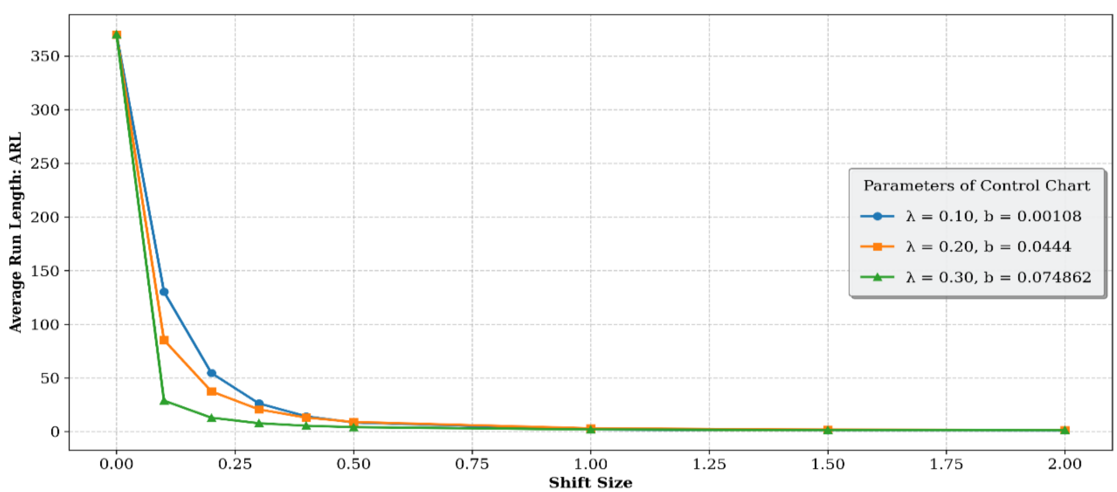

Figure 1. ARL from the explicit formulas methods on EWMA control chart for AR(1) model with quadratic trend at ϕ_1 = 0.30, β_1 = 1.0 and β_2 = 0.10 given ARL₀ = 370

Table 2. ARL and CPU time from the explicit formulas and the NIE methods on EWMA control chart for AR(2) model with quadratic trend at β_1 = 0.70, ϕ_2 = 0.20, β_2 = 2.0, β_2 = 0.30 given ARL$_0$ = 370

Parameters of Control Chart	Shift Size	Average Run Length (ARL)			CPU Time (Second)	
		Explicit Formulas	NIE	Absolute %Diff	Explicit Formulas	NIE
λ = 0.10	0.00	370.6712876175	370.6712876173	5.39648×10−9	0.001	2.324
b = 0.00178	0.10	110.2646016765	110.2646016765	0	0.001	2.387
	0.20	40.4107718408	40.4107718407	2.47463×10−9	0.001	2.403
	0.30	17.5536236852	17.5536236852	0	0.001	2.386
	0.40	8.8351145678	8.8351145678	0	0.001	2.402
	0.50	5.0804282502	5.0804282502	0	0.001	2.450
	1.00	1.3983930032	1.3983930032	0	0.001	2.496
	1.50	1.0939281994	1.0939281994	0	0.001	2.480
	2.00	1.0346776908	1.0346776908	0	0.001	2.387
λ = 0.20	0.00	370.385192649	370.3851925490	2.69989×10−8	0.001	2.387
b = 0.007141	0.10	55.6724070835	55.6724070734	1.81418×10−8	0.001	2.496
	0.20	21.6383764970	21.638376439	2.68042×10−7	0.001	2.433
	0.30	11.1471950972	11.1471950959	1.16621×10−8	0.001	2.403
	0.40	6.7491499472	6.7491499465	1.03717×10−8	0.001	2.481
	0.50	4.5749541122	4.5749541122	0	0.001	2.403
	1.00	1.7102648689	1.7102648688	5.84705×10−9	0.001	2.464
	1.50	1.2662734375	1.2662734375	0	0.001	2.418
	2.00	1.1350824658	1.1350824658	0	0.001	2.417
λ = 0.30	0.00	370.0546525967	370.0546521789	1.12902×10−7	0.001	2.340
b = 0.0119721	0.10	17.1140200510	17.1140200468	2.45413×10−8	0.001	2.403
	0.20	7.2821824609	7.2821824609	0	0.001	2.450
	0.30	4.3235385162	4.3235385157	1.15646×10−8	0.001	2.418
	0.40	3.0293639089	3.0293639086	9.90307×10−9	0.001	2.371
	0.50	2.3528317269	2.3528317267	8.50040×10−9	0.001	2.387
	1.00	1.3485721598	1.3485721598	0	0.001	2.418
	1.50	1.1536388055	1.1536388055	0	0.001	2.387
	2.00	1.0869279063	1.0869279063	0	0.001	2.418

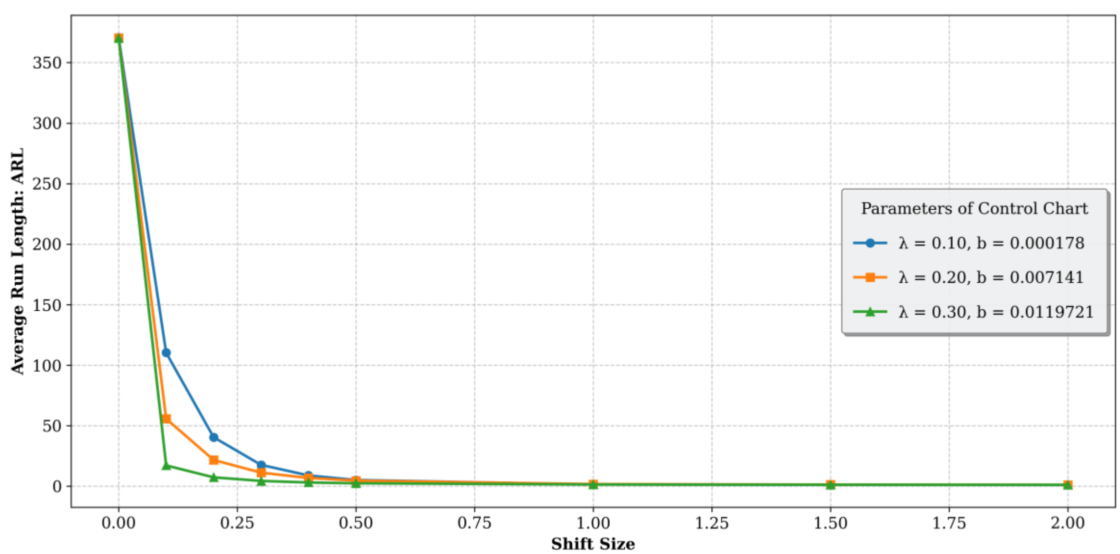

Figure 2. ARL from the explicit formulas methods on EWMA control chart for AR(2) model with quadratic trend at β_1 = 0.70, ϕ_2 = 0.20, β_2 = 2.0 and β_2 = 0.30 given ARL$_0$ = 370

Table 3. ARL and CPU time from the explicit formulas and the NIE methods on EWMA control chart for AR(3) model with quadratic trend at β_1 = 0.50, ϕ_2 = 0.30, ϕ_3 = 0.10, β_2 = 1.0, β_2 = 0.80 given ARL₀ = 370

Parameters of Control Chart	Shift Size	Average Run Length (ARL)			CPU Time (Second)	
		Explicit Formulas	NIE	Absolute %Diff	Explicit Formulas	NIE
λ = 0.10	0.00	370.0988781319	370.0988781314	1.35098×10−9	0.001	2.449
b = 0.0002932	0.10	115.3388934539	115.3388934537	1.73405×10−9	0.001	2.543
	0.20	43.8556826834	43.8556826834	0	0.001	2.481
	0.30	19.5895076215	19.5895076215	0	0.001	2.480
	0.40	10.0436643732	10.0436643732	0	0.001	2.481
	0.50	5.8229297219	5.8229297219	0	0.001	2.745
	1.00	1.5115653363	1.5115653363	0	0.001	2.512
	1.50	1.1267563834	1.1267563834	0	0.001	2.481
	2.00	1.0483753553	1.0483753553	0	0.001	2.418
λ = 0.20	0.00	370.1714772679	370.1714769937	7.40738×10−8	0.001	2.481
b = 0.011814	0.10	62.0765257738	62.0765257425	5.04216×10−8	0.001	2.542
	0.20	24.8664855362	24.8664855262	4.02148×10−8	0.001	2.465
	0.30	13.0322479936	13.0322479893	3.29951×10−8	0.001	2.496
	0.40	7.9611414598	7.9611414577	2.63781×10−8	0.001	2.465
	0.50	5.4080561339	5.4080561328	2.03400×10−8	0.001	2.496
	1.00	1.9358209619	1.9358209618	5.16577×10−9	0.001	2.449
	1.50	1.3659809023	1.3659809022	7.32075×10−9	0.001	2.450
	2.00	1.1911688170	1.1911688170	0	0.001	2.464
λ = 0.30	0.00	370.1560706331	370.1560694791	4.91823×10−9	0.001	2.449
b = 0.019821	0.10	19.4351928975	19.4351928841	4.91823×10−9	0.001	2.418
	0.20	8.3718897058	8.3718897017	4.91823×10−9	0.001	2.481
	0.30	4.9831120349	4.9831120331	4.91823×10−9	0.001	2.465
	0.40	3.4778624334	3.4778624324	4.91823×10−9	0.001	2.481
	0.50	2.6796798025	2.6796798019	4.91823×10−9	0.001	2.433
	1.00	1.4609795502	1.4609795501	4.91823×10−9	0.001	2.418
	1.50	1.2117181474	1.2117181474	4.91823×10−9	0.001	2.418
	2.00	1.1232717255	1.1232717255	4.91823×10−9	0.001	2.434

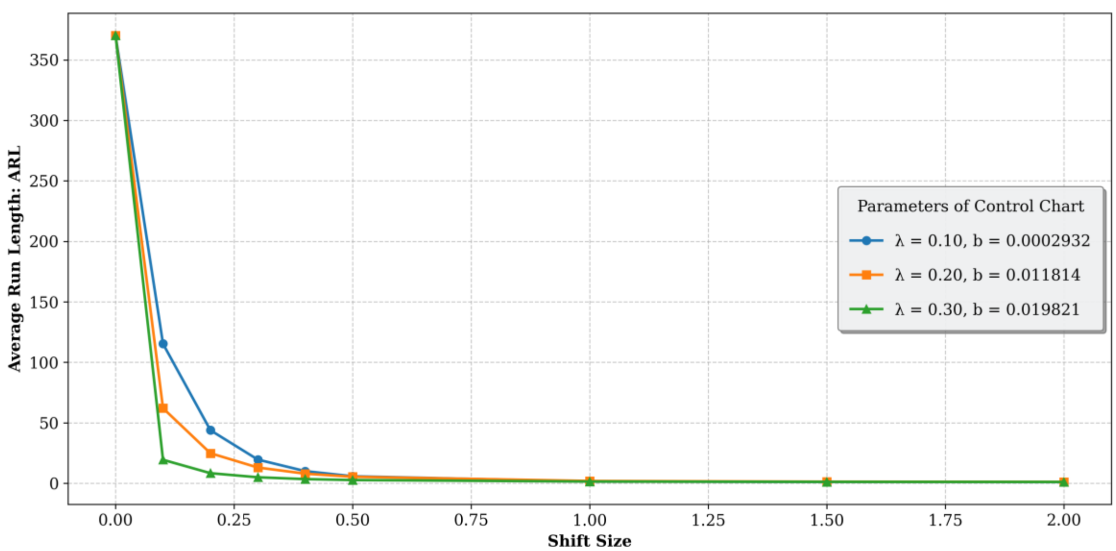

Figure 3. ARL from the explicit formulas methods on EWMA control chart for AR(3) model with quadratic trend at β_1 = 0.50, ϕ_2 = 0.30, ϕ_3 = 0.10, β_2 = 1.0 and β_2 = 0.80 given ARL₀ = 370

Table 4. ARL and CPU time from the explicit formulas and the NIE methods on EWMA control chart for AR(1) model with quadratic trend at β_1 = 0.80, β_2 = 4.0, β_2 = 0.50 given ARL₀ = 500

Parameters of Control Chart	Shift Size	Average Run Length (ARL)			CPU Time (Second)	
		Explicit Formulas	NIE	Absolute %Diff	Explicit Formulas	NIE
λ = 0.10	0.00	500.9146381017	500.9146381016	1.99600×10−11	0.001	2.402
b = 0.0000455	0.10	126.8094364020	126.8094364019	7.88599×10−11	0.001	2.450
	0.20	40.8754514186	40.8754514186	0	0.001	2.403
	0.30	16.0413040837	16.0413040837	0	0.001	2.418
	0.40	7.4977004001	7.4977004001	0	0.001	2.387
	0.50	4.1274776633	4.1274776633	0	0.001	2.511
	1.00	1.2320274656	1.2320274656	0	0.001	2.449
	1.50	1.0460595040	1.0460595040	0	0.001	2.481
	2.00	1.0153525995	1.0153525995	0	0.001	2.418
λ = 0.20	0.00	500.1725147431	500.1725147376	1.09962×10−9	0.001	2.465
b = 0.00141277	0.10	43.2566071127	43.2566071124	6.93548×10−10	0.001	2.496
	0.20	15.0788291143	15.0788291142	6.63182×10−10	0.001	2.418
	0.30	7.3385140360	7.3385140360	0	0.001	2.386
	0.40	4.3378503107	4.3378503107	0	0.001	2.450
	0.50	2.9474082066	2.9474082066	0	0.001	2.465
	1.00	1.3066627586	1.3066627586	0	0.001	2.543
	1.50	1.0991951402	1.0991951402	0	0.001	2.481
	2.00	1.0454711165	1.0454711165	0	0.001	2.433
λ = 0.30	0.00	500.6602159602	500.6602159336	5.31298×10−9	0.001	2.496
b = 0.0023038	0.10	12.1759501967	12.1759501966	8.21291×10−10	0.001	2.449
	0.20	4.9899982527	4.9899982526	2.00401×10−9	0.001	2.527
	0.30	2.9614815629	2.9614815629	0	0.001	2.496
	0.40	2.1219235352	2.1219235352	0	0.001	2.449
	0.50	1.7051257141	1.7051257141	0	0.001	2.434
	1.00	1.1453978071	1.1453978071	0	0.001	2.387
	1.50	1.0554586110	1.0554586110	0	0.001	2.465
	2.00	1.0283905354	1.0283905354	0	0.001	2.418

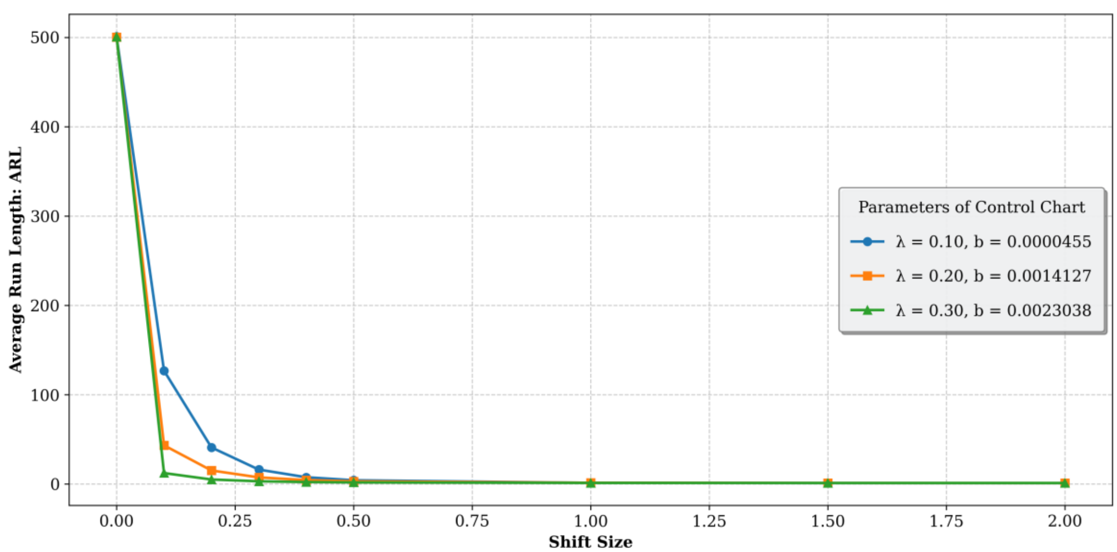

Figure 4. ARL from the explicit formulas methods on EWMA control chart for AR(1) model with quadratic trend at β_1 = 0.80, β_2 = 4.0 and β_2 = 0.05 given ARL₀ = 500

Table 5. ARL and CPU time from the explicit formulas and the NIE methods on EWMA control chart for AR(2) model with quadratic trend at β_1 = 0.60, ϕ_2 = 0.10, β_2 = 5.0, β_2 = 0.80 given ARL₀ = 500

Parameters of Control Chart	Shift Size	Average Run Length (ARL)			CPU Time (Second)	
		Explicit Formulas	NIE	Absolute %Diff	Explicit Formulas	NIE
λ = 0.10	0.00	500.5046835539	500.5046835540	1.99774×10−11	0.001	2.309
b = 0.00000873	0.10	108.6699094966	108.6699094967	9.20234×10−11	0.001	2.449
	0.20	31.1041249384	31.1041249384	0	0.001	2.434
	0.30	11.2219305443	11.2219305443	0	0.001	2.434
	0.40	5.0360290488	5.0360290488	0	0.001	2.418
	0.50	2.7970089621	2.7970089621	0	0.001	2.340
	1.00	1.1014486296	1.1014486296	0	0.001	2.450
	1.50	1.0172145896	1.0172145896	0	0.001	2.356
	2.00	1.0051033085	1.0051033085	0	0.001	2.387
λ = 0.20	0.00	500.2992241890	500.2992241881	1.79881×10−11	0.001	2.449
b = 0.0002711	0.10	32.5273630053	32.5273630053	0	0.001	2.372
	0.20	10.5536423971	10.5536423971	0	0.001	2.433
	0.30	4.9691640093	4.9691640093	0	0.001	2.465
	0.40	2.9452852799	2.9452852799	0	0.001	2.403
	0.50	2.0637862033	2.0637862033	0	0.001	2.464
	1.00	1.1314093315	1.1314093315	0	0.001	2.497
	1.50	1.0364133325	1.0364133325	0	0.001	2.415
	2.00	1.0150185173	1.0150185173	0	0.001	2.418
λ = 0.30	0.00	500.4572221692	500.4572221658	6.79385×10−10	0.001	2.371
b = 0.00044201	0.10	9.1170448373	9.1170448373	0	0.001	2.434
	0.20	3.6682804540	3.6682804540	0	0.001	2.371
	0.30	2.2163699956	2.2163699956	0	0.001	2.418
	0.40	1.6492279450	1.6492279450	0	0.001	2.418
	0.50	1.3830879951	1.3830879951	0	0.001	2.387
	1.00	1.0621705353	1.0621705353	0	0.001	2.418
	1.50	1.0203333770	1.0203333770	0	0.001	2.415
	2.00	1.0093692459	1.0093692459	0	0.001	2.325

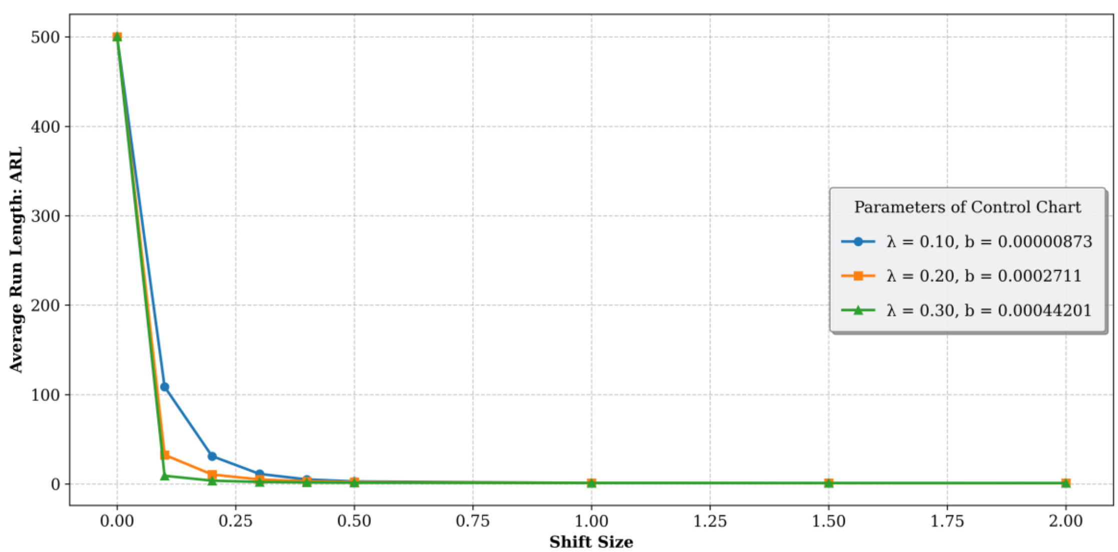

Figure 5. ARL from the explicit formulas methods on EWMA control chart for AR(2) model with quadratic trend at β_1 = 0.60, ϕ_2 = 0.10, β_2 = 5.0 and β_2 = 0.80 given ARL₀ = 500

Table 6. ARL and CPU time from the explicit formulas and the NIE methods on EWMA control chart for AR(3) model with quadratic trend at β_1 = 0.10, ϕ_2 = 0.30, ϕ_3 = 0.50, β_2 = 0.80, β_2 = 0.20 given ARL_0 = 500

Parameters of Control Chart	Shift Size	Average Run Length (ARL)			CPU Time (Second)	
		Explicit Formulas	NIE	Absolute %Diff	Explicit Formulas	NIE
λ = 0.10	0.00	500.4710958978	500.4710958915	1.25882×10−9	0.001	2.480
b = 0.000872	0.10	167.2151867663	167.2151867646	1.01666×10−9	0.001	2.449
	0.20	67.1066983749	67.1066983743	8.94114×10−9	0.001	2.418
	0.30	31.1476594253	31.1476594251	6.42103×10−9	0.001	2.512
	0.40	16.3096246430	16.3096246430	0	0.001	2.465
	0.50	9.4739900399	9.4739900398	1.05552×10−9	0.001	2.403
	1.00	2.0241600457	2.0241600457	0	0.001	2.543
	1.50	1.2745325855	1.2745325855	0	0.001	2.496
	2.00	1.1104295056	1.1104295056	0	0.001	2.480
λ = 0.20	0.00	500.2569414695	500.2569392920	4.35276×10−7	0.001	2.481
b = 0.02748	0.10	82.9128753253	82.9128750930	2.80174×10−7	0.001	2.496
	0.20	33.9315131989	33.9315131235	2.22212×10−7	0.001	2.433
	0.30	18.0910972731	18.0910972403	1.81305×10−7	0.001	2.450
	0.40	11.1595719947	11.1595719780	1.49647×10−7	0.001	2.418
	0.50	7.5952988846	7.59529887852	8.00495×10−8	0.001	2.403
	1.00	2.5402416840	2.5402416828	4.72396×10−8	0.001	2.496
	1.50	1.6413281829	1.6413281826	1.82779×10−8	0.001	2.433
	2.00	1.3500024078	1.3500024077	7.40739×10−8	0.001	2.465
λ = 0.30	0.00	500.2183056176	500.2183051544	9.25996×10−8	0.001	2.371
b = 0.045005	0.10	25.0906091741	25.0906090791	3.78628×10−9	0.001	2.527
	0.20	10.9454251321	10.9454251321	0	0.001	2.449
	0.30	6.5376185372	6.5376185238	2.04968×10−7	0.001	2.480
	0.40	4.5407097574	4.5407097502	1.58566×10−7	0.001	2.433
	0.50	3.4606646611	3.4606646568	1.24254×10−7	0.001	2.450
	1.00	1.7424250365	1.7424250357	4.59130×10−8	0.001	2.434
	1.50	1.3628327274	1.3628327272	1.46753×10−8	0.001	2.371
	2.00	1.2206355800	1.2206355799	8.19245×10−8	0.001	2.465

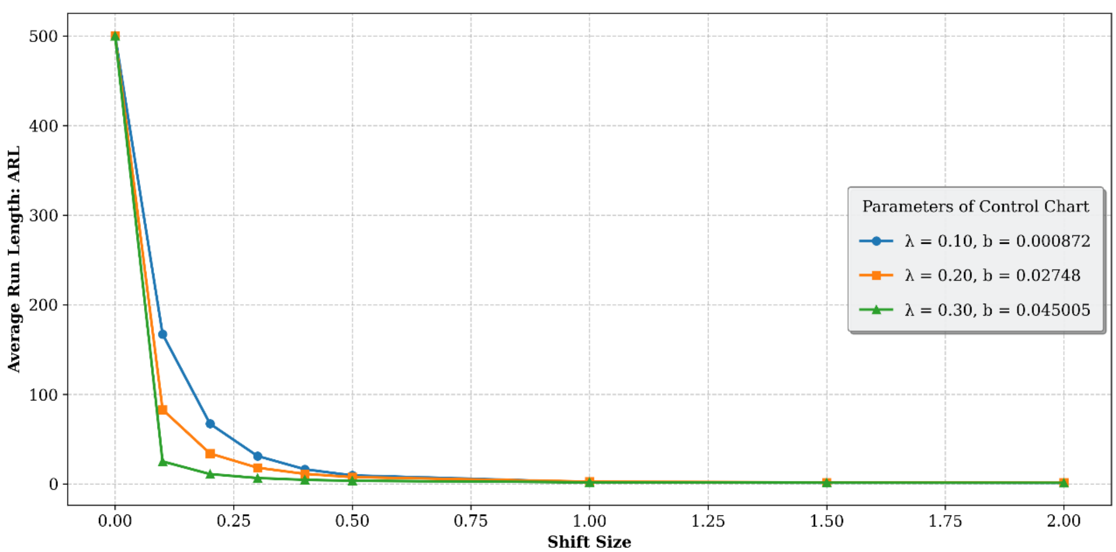

Figure 6. ARL from the explicit formulas methods on EWMA control chart for AR(3) model with quadratic trend at β_1 = 0.10, ϕ_2 = 0.30, ϕ_3 = 0.50, β_2 = 0.80 and β_2 = 0.20 given ARL_0 = 500

Empirical Application of the Proposed ARL Formulas

In this section, the proposed ARL formulas for EWMA charts are applied to real-world data to monitor pneumonia cases among patients at Siriraj Hospital in Bangkok, Thailand, using monthly observations from January 2019 - February 2025. The parameters of the AR(1) model with a quadratic trend were estimated on the EWMA chart as follows: β_1 = 0.5391, β_0 = 78.2974, β_2 = -2.4849, β_2 = 0.0426. Based on this model, the prediction equation for pneumonia cases can be formulated as:

$$Y_t = 78.2974 - 2.4849t + 0.0426t^2 + 0.5391Y_{t-1}.$$

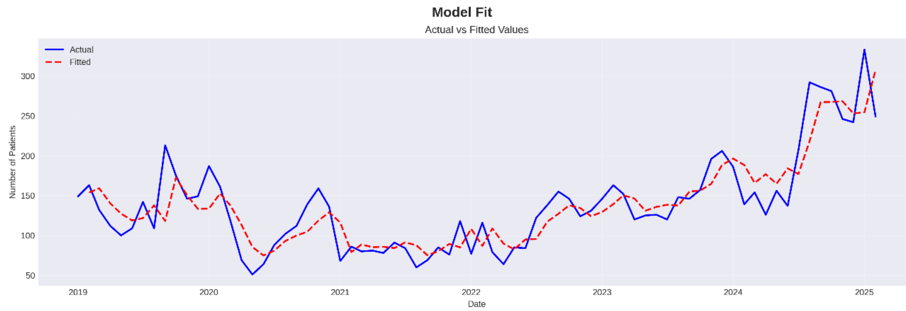

Figure 7. Actual and Fitted Number of Pneumonia Patients at Siriraj Hospital (2019–2025)

Figure 7 presents a comparison between the observed and fitted values obtained from the AR(1) model. The blue solid line represents the actual number of pneumonia cases, whereas the red dashed line indicates the fitted values estimated by the model. Overall, the fitted values closely follow the actual data trend, capturing both short-term fluctuations and the long-term increasing pattern observed from 2019 to 2025.

Figure 8. Forecasted Number of Pneumonia Patients at Siriraj Hospital for a 12-Month Period (March 2025 – February 2026) Estimated by the AR(1) with Quadratic Trend Model

Figure 8 illustrates a 12-month forecast of pneumonia cases at Siriraj Hospital obtained from the AR(1) with quadratic trend model. The solid pink line represents the actual data, while the red solid line indicates the forecasted values for the next 12 months (March 2025 – February 2026). The shaded red region corresponds to the 95% confidence interval, showing the expected range of variability around the forecasts. The results suggest a continuing upward trend in the number of pneumonia patients throughout 2025 and into early 2026.

Table 7 presents the monthly forecasts of pneumonia patients at Siriraj Hospital obtained from the AR(1) model with quadratic trend and shows the 95% confidence limits of the forecasted number of pneumonia patients from March 2025 to February 2026.

Table 7. Forecasted Number of Pneumonia Patients at Siriraj Hospital Estimated by the AR(1) Model (March 2025–February 2026)

Month/Year	Forecasted Number of Patients	Lower 95% Confidence Limit	Upper 95% Confidence Limit
1/3/2025	265.8852	207.5473	324.2232
1/4/2025	278.9379	220.5999	337.2759
1/5/2025	290.0098	231.6718	348.3478
1/6/2025	300.0992	241.7612	358.4371
1/7/2025	309.7441	251.4062	368.0821
1/8/2025	319.2348	260.8968	377.5728
1/9/2025	328.7275	270.3895	387.0654
1/10/2025	338.3065	279.9685	396.6445
1/11/2025	348.0173	289.6793	406.3553
1/12/2025	357.8844	299.5464	416.2224
1/1/2026	367.9209	309.583	426.2589
1/2/2026	378.1341	319.7961	436.4721

Table 8. ARL and CPU time from the explicit formulas and the NIE methods on EWMA control chart for AR(1) model with quadratic trend at β_1 = 0.5391, β_0 = 78.2974, β_2 = -2.4849, β_2 = 0.0426 given ARL_0 = 370

Parameters of Control Chart	Shift Size	Average Run Length (ARL)			CPU Time (Second)	
		Explicit Formulas	NIE	Absolute %Diff	Explicit Formulas	NIE
λ = 0.30	0.00	370.0430205312	370.0172454497	6.96543×10−5	0.001	2.376
b = 105.8766	0.10	354.8695170252	354.8458436000	6.67102×10−5	0.001	2.450
	0.20	340.8119661071	340.7901597506	6.39835×10−5	0.001	2.448
	0.30	327.7526252531	327.7324844323	6.14513×10−5	0.001	2.454
	0.40	315.5897358210	315.5710864772	5.90936×10−5	0.001	2.432
	0.50	304.2349001275	304.2175911851	5.68933×10−5	0.001	2.387
	1.00	257.1855612214	257.1732714608	4.77856×10−5	0.001	2.359
	1.50	221.9152160453	221.9211915336	2.69269×10−5	0.001	2.265
	2.00	194.5686809201	194.5617352680	3.56977×10−5	0.001	2.234

The evaluation of the ARL estimation methods was based on two primary criteria: ARL and computational time. For the AR(1) model with quadratic trend monitored by the EWMA control chart, ARL values were compared using the explicit formulas and the NIE approach. As presented in Table 8, both ARL and computational time are reported. Results show that the ARL_1 values obtained from the explicit and NIE methods were comparable, decreasing rapidly as the shift size became smaller. The absolute percentage difference between the two methods was below 0.0001. Nevertheless, the explicit formula required only 0.001 seconds of computation, while the NIE method required around 2-3 seconds.

Conclusions

This paper presents a method for computing the ARL of AR(p) models with quadratic trend on an EWMA control chart through explicit formulas and examined its effectiveness across different parameters and shift sizes. The ARL_1 values derived from the proposed explicit formulas exhibited strong agreement with those obtained via the NIE method, a pattern also observed in the overall ARL results. The findings of this study are consistent with previous research, confirming that the explicit formula provides results comparable to the NIE method while requiring substantially less computational time. However, the explicit formula approach required significantly less computational time compared to the NIE method. Future research could investigate the potential of this method in other interesting models.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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