RESEARCH ARTICLE

The Dynamics of Non-homogeneous Markov Chains Associated with the *b*-bistochastic Quadratic Stochastic Operators on 1-dimensional Simplex

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Abstract This research investigates the dynamics of a quadratic stochastic operator (QSO), namely the *b*-bistochastic QSO defined on a 1-dimensional simplex as well as the dynamics of a non-homogeneous Markov chain (NHMC) associated with the said QSO. The QSO was first constructed on a 1-dimensional simplex and a fixed-point analysis was performed onto the constructed QSO. The limiting behavior of the QSO was also studied in order to find its rate of convergence. The QSO was then associated with an NHMC to determine its ergodicity by using an ergodicity coefficient called the Dobrushin ergodicity coefficient. The QSO was found to converge to its attracting fixed point at a constant rate of convergence, and the NHMC associated with the QSO was found to be weakly ergodic.

Keywords: *b*-order majorization, hyperbolicity, rate of convergence, Dobrushin ergodicity coefficient, weak ergodicity.

Introduction

It is well established that many nonlinear—often stochastic—models capture real-world phenomena [1]. Within this class, quadratic models are among the most tractable. In seminal work, Bernstein [2] introduced quadratic stochastic operators (QSOs), which arise naturally in genetic models, particularly those describing inheritance. Consequently, population evolution can be modeled using QSOs [3,4]. Building on this perspective, Lyubich [4] showed that, after several generations, selection and reproduction can substantially alter the population's dynamics. In a biological system with n species or traits, and $\mathbf{x}^{(0)} = \left(x_1^{(0)}, \dots, x_n^{(0)} \right)$ denotes the probability distribution of the species in the initial state of the population, and let $p_{ij,k}$ be the probability that individual in the ith and the jth species interbreed to produce an individual of the species k. The set of all distributions is the simplex S^{n-1}. The probability distribution of the first generation, $\mathbf{x}^{(1)} = \left(x_1^{(1)}, \dots, x_n^{(1)} \right)$ can be found by using a total probability,

$$\mathbf{x}_k^{(1)} = \sum_{i,j=1}^{n} p_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = 1, \dots, n. \tag{1}$$

This association defines an operator called the quadratic stochastic operator (QSO), and is generally denoted by V. This operator means that the population starting from the initial arbitrary state of probability distribution, $\mathbf{x}^{(0)}$ evolves to the probability distribution of the first generation, $\mathbf{x}^{(1)} = V(\mathbf{x}^{(0)})$ and continues to evolve to the second generation, $\mathbf{x}^{(2)} = V(\mathbf{x}^{(1)}) = V\left(V(\mathbf{x}^{(0)}) \right) = V^2(\mathbf{x}^{(0)})$ and so on. In other words, each QSO describes the sequence of generations in terms of probability distribution if the values of $p_{ij,k}$ and the distribution of the current generation are given. Selection and reproduction are key drivers of

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intergenerational state changes in population evolution. Models based on quadratic stochastic operators (QSOs) accommodate zygotic, bisexual organisms with gametic differentiation and remain applicable even under hermaphroditism [4]. In particular, organisms capable of functioning as both sexes and, in some cases, self-fertilizing are termed simultaneous hermaphrodites [5,6,7].

The main problem in nonlinear operator theories is studying the behavior of the nonlinear operators. Only a small number of studies on the dynamical phenomena on higher-dimensional systems are understood at present [2] despite being very important. In the case of QSO specifically, the difficulty of the problem depends on a given cubic matrix $(P_{ij,k})_{i,j,k=1}^{n}$. Even the asymptotic behavior on small dimensional systems is complicated [8]. This research aims to study the dynamics of a QSO known as the *b*-bistochastic QSO on a 1-dimensional simplex, particularly its rate of convergence and its ergodicity. We notice that such type *b*-bistochastic QSO has been first started in [9].

Any QSO is defined on an $(n-1)$-dimensional simplex,

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, \dots, x_2) \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^{0} x_i = 1, \forall i \right\}. \tag{2}$$

A general QSO $V: S^{n-1} \to S^{n-1}$ can now be defined as the following form:

$$V(\mathbf{x})_k = \sum_{i,j=1}^{n} p_{ij,k} x_i x_j, \quad k = 1, \dots, n. \tag{3}$$

The heredity coefficient $p_{ij,k}$ has the following properties:

(1) $p_{ij,k} \geq 0$
(2) $p_{ij,k} = p_{ji,k}$,
(3) $\sum_{i,j=1}^{n} p_{ij,k} = 1, \ k = 1, \dots, n.$

A general QSO is specified by Equation (3) together with the defining constraints on its probability coefficients. Imposing additional structural conditions yields distinct subclasses of QSOs. In what follows, we examine the properties of the b-bistochastic QSO..

The *b*-order Majorization

Prior to defining the *b*-bistochastic QSO, a functional has to be defined. For each $k \in \{1, \dots, n-1\}$, we define functional $\mathcal{U}_k: \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{U}_k(x_1, \dots, x_n) = \sum_{i=1}^{k} x_i. \tag{4}$$

From Equation (4), a new majorization is introduced called the *b*-order majorization [9]. For any $\mathbf{x}, \mathbf{y} \in S^{n-1}$, $\mathbf{x}$ is said to be *b*-ordered by $\mathbf{y}$ if and only if $\mathcal{U}_k(\mathbf{x}) \leq \mathcal{U}_k(\mathbf{y})$ for all $k \in \{1, \dots, n-1\}$, and is denoted as $x \leq^b y$. For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S^{n-1}$, the *b*-order majorization satisfies the following conditions:

(1) $\mathbf{x} \leq^b \mathbf{x}$,
(2) $\mathbf{x} \leq^b \mathbf{y}, \mathbf{y} \leq^b \Leftrightarrow \mathbf{x} = \mathbf{y}$,
(3) $\mathbf{x} \leq^b \mathbf{y}, \mathbf{y} \leq^b \mathbf{z} \Leftrightarrow \mathbf{x} \leq^b \mathbf{z}$.

Additionally, it also has the following properties:

(1) $\mathbf{x} \leq^b \mathbf{y} \Leftrightarrow \alpha\mathbf{x} \leq^b \alpha\mathbf{y}, \forall \alpha > 0$,
(2) $\mathbf{x} \leq^b \mathbf{y}, \alpha < \beta \Rightarrow \alpha\mathbf{x} \leq^b \beta\mathbf{y}, \forall \alpha, \beta > 0$.

The Description of the *b*-bistochastic QSO

Given the b-order majorization, a b-bistochastic QSO is precisely the object specified in Definition 1.

Definition 1 [9]: Let V be a QSO. Then, if V satisfies $V(\mathbf{x}) \leq^b \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$, then V is called a *b*-bistochastic QSO.

The following theorem describes the general properties of the *b*-bistochastic QSO.

Theorem 1 [9]: Let V be a *b*-bistochastic QSO defined on S^{n-1}, then the following statements hold:

(1) $\sum_{m=1}^{k} \sum_{i,j=1}^{n} p_{ij,m} \leq kn, \ k \in \{1, \dots, n-1\}$,
(2) $p_{ij,k} = 0$ for all $i, j \in \{k+1, \dots, n\}$ where $k \in \{1, \dots, n-1\}$,
(3) $p_{nn,n} = 1$,
(4) for every $\mathbf{x} \in S^{n-1}$, one has*:*
    (i) $V(\mathbf{x})_k = \sum_{l=1}^{k} p_{ll,k} x_l^2 + 2 \sum_{l=1}^{k} \sum_{j=l+1}^{n} p_{lj,k} x_l x_j$ where $k = \overline{1, n-1}$,

(ii) $V(\mathbf{x})_n = x_n^2 + \sum_{l=1}^{n-1} p_{ll,n} x_l^2 + 2 \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} p_{lj,n} x_l x_j$,

(5) $p_{lj,l} \leq \frac{1}{2}$ for all $j \geq l+1$, $l \in \{1, \dots, n-1\}$.

The behavior of the *b*-bistochastic QSO was studied on a 1-dimensional simplex, which is defined as follows.

For a population distribution with 2 species, the 1-dimensional simplex
$$S^1 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0, x_1 + x_2 = 1\}. \tag{5}$$

Throughout this paper, only the 1-dimensional simplex will be considered.

Materials and Methods

Fixed Point

To describe the dynamics of the *b*-bistochastic QSO, we need to establish the definition of a fixed point as follows.

Definition 3 [10]. Let $x \in \mathbb{R}^n$ be a point of a function $V: \mathbb{R}^n \to \mathbb{R}^n$. Then, x is called a fixed point of the function V if $V(x) = x$.

If an operator V has multiple fixed points, then the set of its fixed points is denoted as $\text{Fix}(V)$ [5]. A fixed point of a QSO can be further identified as hyperbolic or non-hyperbolic.

To identify the hyperbolicity of the fixed point of a QSO in S^1, we need to rewrite Equation (3) on S^1 as
$$V(\mathbf{x})_1 = p_{11,1} x_1^2 + 2 p_{12,1} x_1 x_2 + p_{22,1} x_2^2. \tag{6}$$
Now, we let $p_{11,1} = a$, $p_{12,1} = b$ and $p_{22,1} = c$ be arbitrary coefficients such that $a, b, c \in [0,1]$. By expressing $x_2 = 1 - x_1$, we can rewrite Equation (6) as follows:
$$V(\mathbf{x})_1 = (a - 2b + c)x_1^2 + 2(b - c)x_1 + c. \tag{7}$$
We can now define a hyperbolicity coefficient for the QSO V defined on S^1 as follows.

Definition 4 [3]. Let λ be defined as
$$\lambda = 2(a - 2b + c)\xi + 2(b - c). \tag{8}$$
Then, λ is called the hyperbolicity coefficient of the fixed point ξ.

The fixed point ξ is said to be a hyperbolic fixed point if $|\lambda| \neq 1$ and is non-hyperbolic otherwise. If the fixed point ξ is hyperbolic, then it can be further identified as either an attracting fixed point or a repelling fixed point. This behavior of the hyperbolic fixed point ξ of the operator V is also determined using the hyperbolicity coefficient λ.

Theorem 2 [3]: If $|\lambda| < 1$, then ξ is said to be an attracting fixed point. If $|\lambda| > 1$, then ξ is said to be a repelling fixed point.

The behavior of the hyperbolic fixed point ξ can also be determined by using the discriminant, Δ of Equation (7).

Lemma 1 [3]: If $0 < \Delta < 4$, then ξ is an attracting fixed point and if $4 < \Delta < 5$, then ξ is a repelling fixed point.

The interval $(0,5)$ contains all possible values of Δ, as proven by Mukhamedov and Ganikhodjaev [3].

The Regularity of a *b*-bistochastic QSO

It was elaborated that any QSO is said to be a regular QSO if every arbitrary initial point of the QSO converges to an arbitrary point.

Definition 5 [8]. A QSO V is said to be a regular QSO if for any arbitrary point of the QSO, $\lim_{m \to \infty} V^m(\mathbf{x})$ exists. If otherwise, then V is said to be a non-regular QSO.

For the *b*-bistochastic QSO defined on S^1, with the functional in Equation (4) and the *b*-order majorization, it is evident that
$$\mathcal{U}_k\big(V^{m+1}(\mathbf{x})\big) \leq \mathcal{U}_k\big(V^m(\mathbf{x})\big), \quad k = \overline{1, n-1}, \tag{9}$$

for all $m \in \mathbb{N}$. This implies that the sequence $\{\mathcal{U}_k(V^m(\mathbf{x}))\}$ is monotone decreasing and due to its boundedness, it can be concluded that its limit exists. Following this implementation, the following theorem is obtained.

Theorem 3 [11]: Let V be a *b*-bistochastic QSO defined on S^{n-1}. Then, for every $\mathbf{x} \in S^{n-1}$, $\lim\limits_{m \to \infty} V^m(\mathbf{x})$ exists.

Hence, the *b*-bistochastic QSO is generally considered a regular QSO. Therefore, the *b*-bistochastic QSO defined on S^1 is also considered a regular QSO.

The Fixed Points of a *b*-bistochastic QSO on 1-dimensional Simplex

Mukhamedov and Embong [9] mentioned that the fixed point of a *b*-bistochastic QSO is the point the iterations converge to. In other words, if a point ξ is the limit of a *b*-bistochastic QSO operator V, then it is also the fixed point of V. This is based on the following fact: let V be a QSO fined on S^{n-1} and let $\lim\limits_{m \to \infty} V^m(\mathbf{x}) = \xi$. Then, ξ is a fixed point of V.

The following proposition defines one of the fixed points of a *b*-bistochastic QSO.

Proposition 1 [9]: Let V be a b-bistochastic QSO. Then, $\xi = (0,0,\dots,1)$ is one of its fixed points.

From Proposition 1, it is evident that $(0,1)$ is a fixed point of the *b*-bistochastic QSO on S^1. To determine its behavior, we will apply the properties in Theorem 1 into Equation (7) to get

$$V(\mathbf{x}) = (a - 2b)x^2 + 2bx. \tag{10}$$

By applying the fixed point definition into Equation (10), we obtain the following calculation.

$$(a - 2b)x^2 + 2bx = x$$
$$x = \frac{1 - 2b}{a - 2b}. \tag{11}$$

Substituting $a = 1$, we will get $x = 1$. Therefore, $(1,0)$ is also a fixed point of the *b*-bistochastic QSO defined on S^1. Hence, $(0,1)$ and $(1,0)$ are some of the fixed points of the said operator. □

However, $(0,1)$ and $(1,0)$ are not the only fixed points of the *b*-bistochastic QSO on S^1 [2]. The following theorem by the authors describes the set of fixed points of a *b*-bistochastic QSO defined on S^1.

Theorem 4 [2]: Let V be a *b*-bistochastic QSO defined on S^1 and let $\mathbf{x} = (x_1, x_2) \in S^1$. Then, one has

$$\mathrm{Fix}(V) = \begin{cases} \{(0,1)\} & \text{if } a \neq 1, \\ \{(0,1),(1,0)\} & \text{if } a = 1, b \neq \frac{1}{2}, \\ \{(x_1, x_2) \mid x_1 + x_2 = 1\}: x_1, x_2 \in [0,1] & \text{if } a = 1, b = \frac{1}{2}. \end{cases} \tag{12}$$

Moreover,

$$\lim_{m \to \infty} V^m(\mathbf{x}) = \begin{cases} \{(0,1)\} & \text{if } a \neq 1 \text{ or } a = 1, b \neq \frac{1}{2} \text{ for } x \neq 1 \\ \{(x_1, x_2)\} & \text{if } a = 1, b = \frac{1}{2}. \end{cases} \tag{13}$$

With the fixed points of the *b*-bistochastic QSO on S^1 well defined, several cases need to be considered when determining the rate of convergence of the operator. This is due to the existence of multiple fixed points of the said operator.

The Rate of Convergence of an Operator on 1-dimensional Simplex

The rate of convergence of an operator describes the rate at which the operator converges to its attracting fixed point [8] as its iteration goes unbound. Specifically, for a QSO defined on S^1, the rate of convergence of the QSO is described in the following theorem.

Theorem 5 [3]: If $\{x_m\}_0^\infty$ is a trajectory, then the rate of convergence of the trajectory is derived as

$$x_m \approx C(x_0)\lambda^m, \tag{14}$$

where C is a constant and x_0 is the initial point of the trajectory.

Hence, we need to identify the values of λ in order to determine the rate of convergence for the *b*-bistochastic QSO on S^1.

Non-homogeneous Markov Chains

Normally, a non-homogeneous Markov chain (NHMC) is associated with linear operators. However, such association is not strictly limited to linear operators, as the NHMC may also be associated with non-linear operators [9], such as the QSO. The association of the NHMC with the QSO is crucial as it is a great tool in identifying some properties of the QSO. For example, the NHMC has previously been associated with the *F*-stochastic operator to determine its absolute continuity [12]. Its applications to quantum Markov chains has been implemented in [13,14].

Prior to defining the NHMC, a vector sequence from a collection of vectors needs to be defined as follows.

Definition 6 [12]. For a given collection of vectors $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \subset S^{n-1}$, each vector is defined for every m-th generations as

$$\mathbf{x}_k^{(m)} = \left(x_{k,1}^{(m)}, x_{k,2}^{(m)}, \ldots, x_{k,n}^{(m)} \right), \qquad k \in \{1, \ldots, n\}, \tag{15}$$

With these vectors, a sequence $\{\mathbb{P}_{m;X}\}$ of stochastic matrices can be defined as follows:

$$\mathbb{P}_{m;X} = \begin{pmatrix} x_{1,1}^{(m)} & x_{1,2}^{(m)} & \cdots & x_{1,n}^{(m)} \\ x_{2,1}^{(m)} & x_{2,2}^{(m)} & \cdots & x_{2,n}^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}^{(m)} & x_{n,2}^{(m)} & \cdots & x_{n,n}^{(m)} \end{pmatrix}. \tag{16}$$

For simplicity, one can write $\mathbb{P}_{m;X}$ as $\left(P_{ij,X}^{(m)} \right)_{i,j=1}^n$ where $P_{ij,X}^{(m)} = x_{ij}^{(m)}$. It is obvious that $\mathbb{P}_{m;X}$ is an $n \times n$ stochastic matrix. With the generating sequence $\{\mathbb{P}_{m;X}\}$, an NHMC can now be defined for $k < m$ as

$$P_X^{k,m} := \mathbb{P}_{m;X} \mathbb{P}_{m-1;X} \cdots \mathbb{P}_{k+1;X}. \tag{17}$$

Every NHMC is now identifiable with its generating sequence. We notice that there is also other kind of construction of NHMC associated with quadratic processes [3,11,15,16].

To construct the NHMC associated with the *b*-bistochastic QSO defined on S^1, a set of vectors $X = \{\mathbf{x}_1, \mathbf{x}_2\}$ should be considered such that $\mathbf{x}_1 = (x_{1,1}, x_{1,2})$ and $\mathbf{x}_2 = (x_{2,1}, x_{2,2})$. Then, the NHMC generating sequence is defined as:

$$\mathbb{P}_{m;X} = \begin{pmatrix} x_{1,1}^{(m)} & x_{1,2}^{(m)} \\ x_{2,1}^{(m)} & x_{2,2}^{(m)} \end{pmatrix}. \tag{18}$$

The Absolute Continuity of NHMCs

A different measure that relates one NHMC to another needs to be considered since NHMCs depend on their initial state of $\mathbf{x}$. Such measure that relates two NHMCs with different initial states is called the absolute continuity.

Let $(\Omega, \mathfrak{F})$ be a measurable space with two different measures μ and ν. The following definitions describe the relation of the measures within this measurable space.

Definition 7 [11]. μ is said to be *a*bsolutely continuous with respect to ν, denoted as $\mu \ll \nu$, if $\mu(A) = 0$ whenever $\nu(A) = 0$, such that $A \in \mathfrak{F}$. If $\mu \ll \nu$ and $\nu \ll \mu$, then μ and ν are equivalent, denoted as $\mu \sim \nu$.

Mukhamedov and Embong [9] investigated locally absolutely continuity. Namely, a measure μ is locally absolutely continuous with respect to the measure ν, denoted as $\mu \ll^{loc} \nu$, if $\mu_k \ll \nu_k$ for every $k \geq 1$.

From the definition of absolute continuity, two relations that may relate two NHMCs can be further derived, known as singularity.

Definition 8 [9]. Two measures μ and ν are singular (or orthogonal, $\mu \perp \nu$) if there exists a set $A \in \mathfrak{F}$ such that $\mu(A) = 1$ and $\nu(A^C) = 1$.

From these definitions, the following theorem of equivalence-singularity dichotomy were obtained.

Theorem 6 [12]: Let μ and ν be two measures such that $\mu \ll^{loc} \nu$. Then, either μ and ν are equivalent or singular. These alternatives are determined by

$$\mu \sim \nu \Leftrightarrow \sum_{s,t \in S} \left(\sqrt{P_{m,\mu}(s,t)} - \sqrt{P_{m,\nu}(s,t)} \right)^2 < \infty, \forall m \geq 1. \tag{19}$$

The dichotomy in Equation (19) is denoted as $D_n^2(\mu, \nu)$ for simplicity. For two NHMCs X and Y, the dichotomy can be further simplified as

$$D_{X,Y} \coloneqq \sum_{m=1}^{\infty} \sum_{i,j=1}^{n} \left(\sqrt{P_{ij;X}^{(m)}} - \sqrt{P_{ij;Y}^{(m)}} \right)^2, \tag{20}$$

which can be further separated as follows

$$\sum_{m=1}^{\infty} \sum_{i=1}^{n} \left(\sqrt{P_{i,1;X}^{(m)}} - \sqrt{P_{i,1;Y}^{(m)}} \right)^2 + \sum_{m=1}^{\infty} \sum_{i=1}^{n} \sum_{j=2}^{n} \left(\sqrt{P_{ij;X}^{(m)}} - \sqrt{P_{ij;Y}^{(m)}} \right)^2. \tag{21}$$

Remark

To identify the absolute continuity of NHMCs associated with the b-bistochastic QSO on S^1, we let $P_{ij;X}^{(m)}$ and $Q_{ij;Y}^{(m)}$ be two NHMCs associated with the b-bistochastic QSO defined on S^1 where $X = \{\mathbf{x}, \mathbf{y}\} \subset S^1$ and $Y = \{\mathbf{u}, \mathbf{v}\} \subset S^1$. The NHMCs are firstly denoted as

$$P_{ij;X}^{(m)} = \begin{pmatrix} x_1^{(m)} & x_2^{(m)} \\ y_1^{(m)} & y_2^{(m)} \end{pmatrix}, \tag{22}$$

and

$$Q_{ij;Y}^{(m)} = \begin{pmatrix} u_1^{(m)} & u_2^{(m)} \\ v_1^{(m)} & v_2^{(m)} \end{pmatrix}, \tag{23}$$

respectively. By applying the equivalence-singularity dichotomy in Theorem 6 on the NHMCs in Equations (22) and (23), we have

$$\left(\sqrt{P_{11;X}^{(m)}} - \sqrt{Q_{11;Y}^{(m)}} \right)^2 = \left(\sqrt{x_1^{(m)}} - \sqrt{u_1^{(m)}} \right)^2$$
$$\leq \left(\sqrt{x_1^{(m)}} + \sqrt{u_1^{(m)}} \right)^2. \tag{24}$$

The relation in (24) will be used to determine the absolute continuity of the NHMCs later on.

The Ergodicity of NHMCs

The ergodicity of a dynamical system is related to the long-term behavior of the system. The ergodicity of NHMCs in particular refers to the asymptotic behavior of the NHMCs as the number of transitions increase.

The Dobrushin ergodicity coefficient can be calculated to determine the ergodicity [12] of a stochastic matrix. For a given stochastic matrix $\mathbb{T} = (T_{ij})_{i,j=1}^{n}$, its Dobrushin ergodicity coefficient is obtained by either of these two formulas:

$$\delta(\mathbb{T}) = \frac{1}{2} \max_{i<j} \sum_{k=1}^{n} |t_{ik} - t_{jk}|, \tag{25}$$

or

$$\delta(\mathbb{T}) = 1 - \min_{i<j} \sum_{k=1}^{n} \{t_{ik}, t_{jk}\}. \tag{26}$$

From Equation (25) and Equation (26), if $\lim_{m \to \infty} \delta(P_X^{k,m}) = 0$ for every $k \in \mathbb{N}$, then the NHMC $\{P_X^{k,m}\}$ is considered to be weakly ergodic.

Results and Discussion

The Rate of Convergence of the *b*-bistochastic QSO on 1-dimensional Simplex

The rate of convergence of the b-bistochastic QSO on S^1 is as follows.

Theorem 7: Let $V = (a - 2b)x^2 + 2bx$ be a b-bistochastic QSO defined on S^1 where $a \in [0,1]$ and

$b \in \left[0, \frac{1}{2}\right)$. Then, any given initial point x_0 of the operator converges to the fixed point $\xi = (0,1)$ at a constant rate $C(x_0)(2b)^m$.

Proof

Consider several cases of the values of a in Equation (10). Let $b < \frac{1}{2}$ for all cases. When $a = 0$, the operator takes the form

$$V(\mathbf{x}) = 2bx - 2bx^2, \tag{27}$$

and whenever $0 < a < 1$, the operator remains as in Equation (10). Substituting the fixed point $\xi_1 = (0,1)$ into Equation (8), we get λ as follows:

$$\lambda = 2(a - 2b)(0) + 2b$$
$$= 2b < 1. \tag{28}$$

When $a = 1$, the operator takes the form

$$V(\mathbf{x}) = 2bx + (1 - 2b)x^2. \tag{29}$$

Substituting the fixed point $\xi_1 = (0,1)$ yields the same result as in Equation (28), while substituting the fixed point $\xi_2 = (0,1)$ will result in

$$\lambda = 2(1 - 2b)(1) + 2b$$
$$= 2(1 - b) > 1. \tag{30}$$

Clearly $\xi_1 = (0,1)$ is an attracting fixed point and $\xi_2 = (1,0)$ is a repelling fixed point. Since the rate of convergence only considers the attracting fixed point and $\lambda = 2b$, the proof is complete. □

It is crucial to note that this rate of convergence of the b-bistochastic QSO on S^1 is for when $b < \frac{1}{2}$ only.

The Ergodicity of NHMC Associated with the *b*-bistochastic QSO on 1-dimensional Simplex

The ergodicity of NHMC associated with the b-bistochastic QSO on S^1 is as follows.

Theorem 8: Let $P_X^{k,m}$ be an NHMC associated with the b-bistochastic QSO defined on S^1 with $a \in [0,1]$ and $b \in \left[0, \frac{1}{2}\right)$. Then, $P_X^{k,m}$ is weakly ergodic.

Proof

From Theorem 7, the attracting fixed point of a *b*-bistochastic QSO defined on S^1 is $\xi = (0,1)$ and any initial point of the operator converges to ξ. Hence, the set of vectors can now be defined as

$$\mathbf{x}_1^{(m)} = (0,1), \tag{31}$$
$$\mathbf{x}_2^{(m)} = (0,1), \tag{32}$$

as m goes to ∞, and the NHMC generating sequence can then be written as

$$\mathbb{P}_{m;X} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{33}$$

From Equation (33), it is obvious that $\mathbb{P}_{m;X}\mathbb{P}_{m-1;X} = \mathbb{P}_{m;X}$ and therefore, the NHMC from Equation (17) can be defined as

$$P_X^{k,m} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

Using the Dobrushin ergodicity coefficient in Equation (25) to identify the ergodicity of the NHMC results in the following calculation.

$$\delta\left(P_X^{k,m}\right) = \frac{1}{2}\max_{i<j}\sum_{k=1}^{2}|x_{ik} - x_{jk}|$$
$$= \frac{1}{2}(|0 - 0| + |1 - 1|) = 0. \tag{35}$$

Similarly, using the Dobrushin ergodicity coefficient in Equation (26) will result in the following calculation.

$$\delta\left(P_X^{k,m}\right) = 1 - \min_{i<j}\sum_{k=1}^{2}\min\{x_{ik}, x_{jk}\}$$
$$= 1 - [(0) + (1)] = 0. \tag{36}$$

Since $\delta\left(P_X^{k,m}\right) = 0$ for every $k \in \mathbb{N}$, then the proof is now complete. □

The NHMC exerts a weak ergodicity only if it is associated with the *b*-bistochastic QSO on S^1 with $b < \frac{1}{2}$. Future research should be dedicated to obtain the ergodicity of the NHMC where $b = \frac{1}{2}$.

The Absolute Continuity of NHMCs Associated with the *b*-bistochastic QSO on 1-dimensional Simplex

Recall the NHMCs $P_{ij;X}^{(m)}$ and $Q_{ij;Y}^{(m)}$ from Equations (22) and (23). From Theorem 7, it is apparent that

$$x_1^{(m)} \to 0, y_1^{(m)} \to 0, u_1^{(m)} \to 0, v_1^{(m)} \to 0, \tag{37}$$

and

$$x_2^{(m)} \to 1, y_2^{(m)} \to 1, u_2^{(m)} \to 1, v_2^{(m)} \to 1, \tag{38}$$

where, as m goes to ∞, each of the state converges to the respective values at the rate of $(2b)^m$. Applying Theorem 7 into the relation in (24), we have

$$
\begin{aligned}
\left(\sqrt{P_{11;X}^{(m)}} - \sqrt{Q_{11;Y}^{(m)}} \right)^2 &= \left(\sqrt{x_1^{(m)}} - \sqrt{u_1^{(m)}} \right)^2 \\
&\leq \left(\sqrt{x_1^{(m)}} + \sqrt{u_1^{(m)}} \right)^2 \\
&< \left(\sqrt{(2b)^m} + \sqrt{(2b)^m} \right)^2 \\
&= \left(2(2b)^{\frac{m}{2}} \right)^2 \\
&= 4(2b)^m.
\end{aligned}
\tag{39}
$$

Similar calculations are obtained for each state of the NHMC. By using the values from these calculations, the equivalence-singularity dichotomy can be expressed as

$$D_{X,Y} = 16 \sum_{m=1}^{\infty} (2b)^m, \tag{40}$$

and since $16 \sum_{m=1}^{\infty} (2b)^m < \infty$, the two measures are either absolutely continuous or singular.

By comparing $P_{ij;X}^{(m)}$ and $Q_{ij;Y}^{(m)}$ as m goes to ∞, we have

$$P_{ij;X}^{(m)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \tag{41}$$

and

$$Q_{ij;Y}^{(m)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{42}$$

From this comparison, it can be concluded that the two NHMCs are equivalent. Hence, NHMCs associated with the *b*-bistochastic QSO on S^1 are absolutely continuous and will exert similar behavior regardless of their differing initial states. Figure 1 illustrates the rate of convergence of the *b*-bistochastic QSO on S^1 as well as its absolute continuity.
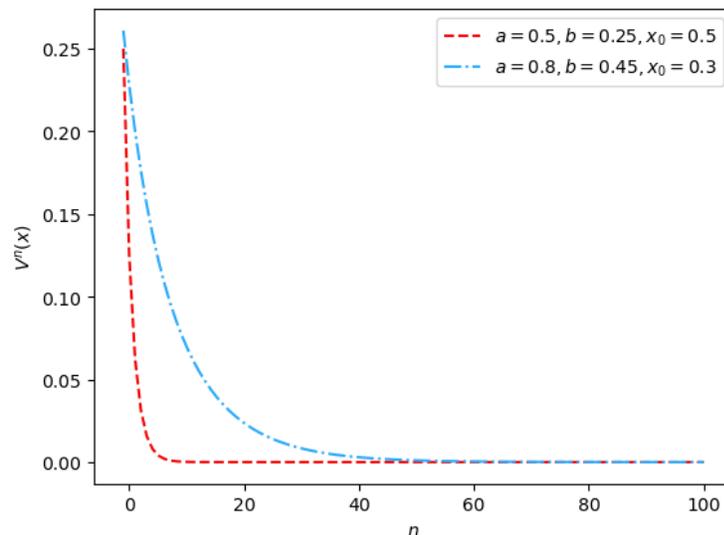
Figure 1. The iterations of two *b*-bistochastic QSOs defined on S^1 with different values of parameters a, b and initial states x_0.

Although the two trajectories in Figure 1 start from different initial states and evolve under different parameters, they both converge to the attracting fixed point 0. A heuristic analogy of this is, if a fixed

inherited biological trait, such as the human height, is predetermined by the laws of reproduction, then the average height of the human population will converge to that fixed value after many generations, independent of the initial distribution of the heights.

For other studies that explored the behavior and applications of quadratic stochastic operators (QSOs) with probabilistic interactions may be referred in Zada and Shah [17]. A paper by A.F. and Rosli [18] analyses the behaviour of two-dimensional b-bistochastic systems, while Saburov and Yusof [19] examined the uniqueness of fixed points and its stability. Apart from the biological applications, Ahlberg, de la Riva and Griffiths [20] established the convergence in Voronoi percolation model.

Conclusions

The rate of convergence of the *b*-bistochastic QSO, specifically on a 1-dimensional simplex S^1, was obtained and the QSO is said to converge to the attracting fixed point $\xi = (0,1)$ at a constant rate $C(x_0)(2b)^m$. However, this is only true for when $b < \frac{1}{2}$ as the operator would behave differently otherwise. The ergodicity of the NHMC associated with the *b*-bistochastic QSO on S^1 was identified to be weakly ergodic, regardless of the initial state of the operator as the NHMCs associated with the QSO were found to be absolutely continuous.

It is recommended to further continue the study of the *b*-bistochastic QSO on S^1 wth the specific parameter value of $b = \frac{1}{2}$. For this case, it is suspected that the hyperbolicity coefficient indicates that the fixed point of the operator is non-hyperbolic. Further investigation is required with the aid of computer programming to simulate the trajectory of the operator as well as its dichotomy. Several cases should be considered, where the parameter value of *a* varies. Interesting results are expected when investigating each case. Once its behavior is fully understood, the study of the *b*-bistochastic QSO can be expanded to include the QSO defined on a general $(n-1)$-dimensional simplex, S^{n-1} for $n = 3,4,\dots,m$ instead of only the 1-dimensional simplex.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgment

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