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Analysis and Application of WASPAS and MABAC Techniques Based on Picture Fuzzy Soft Einstein Aggregation Operators

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Abstract: An unexpected arrival of energy in the World's climate that results in seismic waves causes an earthquake. Different technologies are used that help to resist earthquakes like a shock absorber, replaceable fuses, pendulum power, rocking core-wall and carbon-fiber wrap, etc. Fuzzy structures are valuable tools that provide some useful applications in different fields of science and technology. Picture fuzzy soft set (PFSS) has the ability to cover the abstinence grade along with the membership grade and non-membership grade, making this structure a more dominant structure than an intuitionistic fuzzy soft set. Furthermore, the Einstein t-norm (ETN) and Einstein t-conorm (ETCN) are excellent substitutes for algebraic sum and product. Because of PFSS and ETN and ETCN, we have proposed Einstein operating principles for PFS numbers. Moreover, we have developed PFS Einstein average (PFSEA) and PFS Einstein geometric (PFSEG) aggregation operators (AOs). In this article, the weighted aggregated sum product assessment (WASPAS) technique based on PFSEA and PFSEG AOs for the choice of the best technology to help buildings resist earthquakes is proposed. Additionally, the notion of the Multi-attributive border approximation area comparison (MABAC) algorithm is proposed and utilized for the classification of face recognition technologies. Moreover, the comparative analysis of the introduced conception allows us to determine the superiority of the suggested conceptions.

Keywords: WASPAS method; MABAC method; Picture fuzzy soft set; Einstein aggregation operators.

Introduction

The construction of a home or other structure that can resist earthquakes' abrupt ground shaking, limiting structural damage and human fatalities and injuries, needs more attention. To achieve the correct design target for earthquake resistance, several appropriate construction techniques are needed. Because different construction methods are used in different parts of the world, it is necessary to assess the local construction methods and resources to choose an appropriate and workable design. There is a fundamental difference between design and construction techniques. Note that advanced technologies are more effective if proper construction methods are used. Different earthquake designs incorporate the ability of a building to bend and deform without collapsing within the structure. Buildings' failure during an earthquake is due to poor construction materials and poor technologies. For a building to be earthquake-resistant, it must be properly grounded and connected to the earth through its foundation. Many modern technologies are used nowadays that help buildings resist earthquakes. The graphical representation of different technologies used that help buildings resist earthquakes is given in Figure 1.

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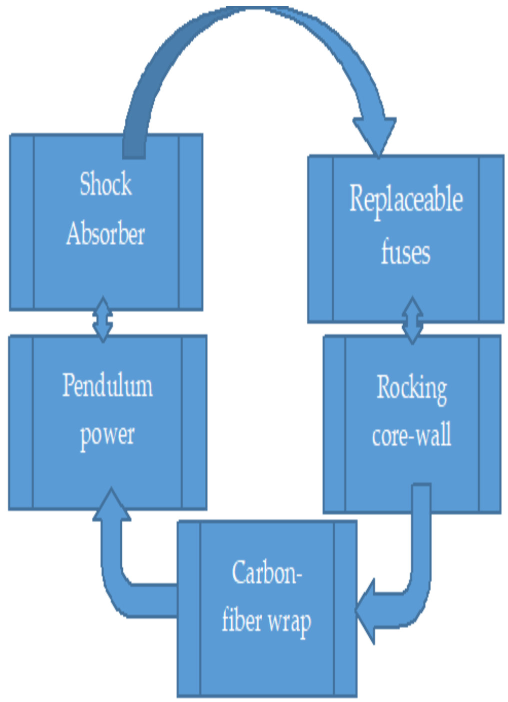

Figure 1. Different technologies that help build earthquake resistance

The MADM approach is popular in the multidisciplinary domains of social science, economics, management and decision-making. The MADM is based on a collection of theories, approaches, and techniques that are used to address the challenge of choosing the best or most practical choice from a range of available options. The fuzzy sets (FS) idea has been effectively applied to many domains ever since Zadeh [1] first introduced it. But in many decision-making complications, we have to use the non-membership grades (NMG) denoted by "$N(x)$" and these kinds of problems show the limitation of FS theory because FS uses only membership grades (MG) denoted by $\mathrm{M}(x)$. To address this difficulty, Atanassov employs the concepts of the intuitionistic fuzzy set (IFS) [2]. The IFS uses $0 \leq M(x) + N(x) \leq 1$. If $N(x) = 0$ in the aforementioned condition, then IFS reduces to FS. In FS theory, many new developments have been made for its effective use in different fields. AOs are some useful tools that convert complex given data into a single value. We may also see that fundamental algebraic products and algebraic sum are the foundations of all AOs. ETN and ETCN are good substitutes for Einstein products and Einstein sum for two real numbers. So, based on these observations, Wang et al. [3] proposed IF information AOs. Moreover, Xu et al. [4] established the notions of IF geometric AOs. Garg and Rani [5] proposed generalized geometric AOs based on t-norm rules for complex IF sets and used these notions to discuss decision-making issues. The WASPAS is one of the valuable techniques to address the ambiguous data in FS theory. The WASPAS is an MCDM method used to evaluate alternatives under multiple criteria. This approach incorporates the opinions of several scientists and has been used in a variety of sectors, including engineering, economics, and environmental management. So Schitea et al. [6] developed an IFS-based WASPAS method for the section of the Hydrogen mobility roll-up site. Keeping in view the limitations of IFS, the definition of Pythagorean FS (PyFS) has been laid out by Yager [7], and a few new improvements have been suggested to cover more advanced data and more complex situations. In PyFS we use $0 \leq M(x)^2 + N(x)^2 \leq 1$. If we replace power 2 by 1, then PyFS reduces to IFS. It means that PyFS is a more advanced structure. The WASPAS technique based on PyFS has been developed and this paradigm has been employed by Mohagheghi et al. [8] for the portfolio assessment of advanced technology programmers. Rahman et al. [9] have also created some Einstein AOs using PyFS and used them to MCDM issues to demonstrate how well they work. Moreover, the notion of q-rung orthopair FS (q-ROFS) generalizes the PyFS and uses the axiom that $0 \leq M(x)^q + N(x)^q \leq 1$ for $q \geq 1$. Based on q-ROFS, some general Einstein interactive AOs with enhanced operational laws were created by Farid et al. [10]. Furthermore, based on q-ROFs, Deveci et al. [11] initiated the notions of q-ROF Einstein-based WASPAS technique.

Soft set (SS) is the mathematical framework for dealing with uncertainty and vagueness in data. The traditional mathematical methods assume that all the data is precise and complete, which is often not the case in many real-world scenarios. To deal with vagueness and unpredictability in DM situations, Molodtsov's notion of SS [12] provides a flexible and intuitive extension of classical set theory. SS theory provides a flexible tool for dealing with uncertainty and ambiguity in data analysis, DM and knowledge representation. It can be used in various applications such as decision support systems, expert systems and image processing. Based on SS, Ali et al. [13] established some new operations in SS theory. Moreover, Maji et al. [14] used SS theory in the decision-making approach to prove its applications. The generalization of SS theory is a research direction aimed at extending the fundamental ideas and operations of SS theory to a more complex structure. One such generalization is the idea of fuzzy SS (FSS) [15], which combines the idea of FS and SS. Many such structures are developed, like intuitionistic FSS (IFSS) [16], Pythagorean FSS (PyFSS) [17] and q-rung orthopair FSS (q-ROFSS) [18]. The researchers try to find out the applications of these developed notions and Hooda et al. [19] use IFSS and provide its applications in medical diagnosis. Moreover, Muthukumar and Krishnan [20] developed the notions of similarity measures based on IF soft data and provide its applications in medical diagnosis. As AOs are mathematical functions that combine multiple input values into a single output value. They are commonly used in the decision-making process to summarize and analyze the data. Zulqarnain et al. [21] established some PyFS average and geometric AOs. Moreover, Naeem et al. [22] established the notions of TOPSIS and VIKOR techniques based on PyFSS. Moreover, interaction AOs based on PyFSS have been addressed by Zulqarnain et al. [23]. Also, some PyF Einstein AOs have been delivered by Zulqarnain et al. [24]. Based on complex PyFS information, some interaction AOs have been defined by Mahmood et al. [25]. Riaz et al. [26] delivered the TOPSIS and VIKOR techniques based on q-ROFS information and used these notions in decision-making problems. Moreover, Ali [27] used the Dombi aggregation operator under the WASPAS approach and used this work for decision-making scenarios. Ali and Popa [28] used Dombi aggregation operators and used these notions for smart city initiative prioritization. Ali et al. [29] used the TODIM method and used it for optimal gate security system selection. Note that all the above existing notions can only discuss MG and NMG in their structure. But we have to face some situations in which humans' responses can be yes, no, abstain, and rejection. Be aware that the concepts of IFSS, PyFSS, and q-ROFSS extensively addressed two parts of human opinion on an uncertain event: the yes-or-no kinds as expressed by the MG and NMG. However, abstinence grades or refusal grades are included alongside MG and NMG. As an illustration, consider voting, where one has the option to vote in support of someone, vote in opposition, abstain from casting a vote, or refuse to cast a vote. To cover all those issues, the idea of a picture fuzzy soft set (PFSS) has been introduced by Yang et al. [30]. PFSS has the ability that it can cover the AG, making this structure a more dominant structure than IFSS. Moreover, note that ETN and ETCN are great substitutes for algebraic sum and product. So, based on PFSS and ETN and ETCN, first of all, we have proposed Einstein operating rules for PFS numbers. Also, we have developed the notion of the WASPAS method and MABAC method based on PFSEA and PFSEG AOs to discuss the decision-making problems. Moreover, a comparative analysis of the established work is given that shows the superiority of the developed work. In the end, we have some concluding remarks.

The rest of the article is arranged as follows: in section 2, we have overviewed the basic notions of ETN and ETCN, PFS, PFSS, and PF Einstein weighted average AOs. In section 3, we have developed the basic operating rules for PFS numbers based on ETN and ETCN. Moreover, we have established the notion of PFSEA and PFSEG AOs and suggested their properties. In section 5, we have delivered the WASPAS method under the idea of PFSEWA and PFSEWG AOs. Also, we have developed an illustrative example to show the utilization of these ideas. In Section 6, we have proposed an algorithm for the MABAC technique and provided an example to show the application of the delivered work. Section 7 deals with a comparative study. In section 8, we have concluding remarks.
Moreover, the graphical abstract/flowchart of the proposed work is given in Figure 2.

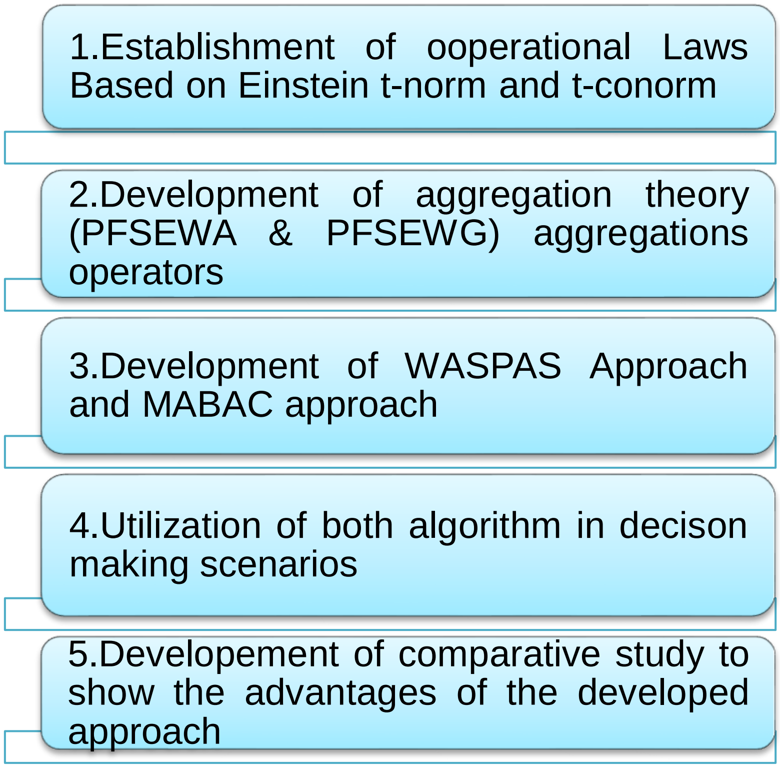

Figure 2. Flowchart of the proposed study

Preliminaries

In this portion, we will overview some fundamental notions that can be helpful in further discussion. We have revised the ideas of t-norm and t-conorm. Moreover, the definition of PFS, PFSS, and PFEWA AOs has been reviewed. The ideas of distance measures and Hamming distance for PFSNs have been revised as well.

Definition 1 [31]: A correspondence $f: [0\ 1]^2 \to [0\ 1]$ is a triangular norm if for all $a, b, c, d \in [0\ 1]$, we have

1) $f(a, b) = f(b, a)$
2) $f\big(a, f(b, c)\big) = f(f(a, b), c)$
3) $f(a, b) \geq f(c, d)$ whenever $a \geq c$ and $b \geq d$
4) $f(a, 1) = a$.

Definition 2 [31]: A function $S: [0\ 1]^2 \to [0\ 1]$ is a t-conorm if for all $a, b, c, d \in [0\ 1]$

1) $S(a, b) = S(b, a)$
2) $S\big(a, S(b, c)\big) = S(S(a, b), c)$
3) $S(a, b) \geq S(c, d)$ whenever $a \geq c$ and $b \geq d$
4) $f(a, 0) = a$.

Definition 3 [32]: For a universal set $\mho$, the structure of PFS is as follows

$$PFS = \text{Þ} = \{x, M_{\text{Þ}}(x), \mathfrak{A}_{\text{Þ}}(x), N_{\text{Þ}}(x) |\ x \in \mho\}$$

Where $M_{\text{Þ}}: \mho \to [0\ 1]$, $\mathfrak{A}_{\text{Þ}}: \mho \to [0\ 1]$ and $N_{\text{Þ}}: \mho \to [0\ 1]$ represent MG, AG and NMG respectively with $0 \leq M_{\text{Þ}}(x) + \mathfrak{A}_{\text{Þ}}(x) + N_{\text{Þ}}(x) \leq 1$.

Definition 4 [30]: A pair $(g, \mathcal{A})$ define the PFSS, where $\mho$ denotes the set of discourse and $\mathcal{S}$ represents a set of parameters and $\mathcal{A} \subseteq \mathcal{S}$, $g: \mathcal{A} \to PFS^{\mho}$ and

$$\text{Þ}_{\mathfrak{s}_{\mathcal{d}}}(a_{\hat{\imath}}) = \{\langle a_{\hat{\imath}}, M_{\mathcal{d}}(a_{\hat{\imath}}), \mathfrak{A}_{\mathcal{d}}(a_{\hat{\imath}}), N_{\mathcal{d}}(a_{\hat{\imath}})\rangle | a_{\hat{\imath}} \in \mho\}$$

Where $PFS^{\mho}$ represent the family of PFSS. Here $M_{\mathcal{d}}(a_{\hat{\imath}}), \mathfrak{A}_{\mathcal{d}}(a_{\hat{\imath}}), N_{\mathcal{d}}(a_{\hat{\imath}})$ are the MG, AG, and NMG, respectively with $0 \leq M_{\mathcal{d}}(a_{\hat{\imath}}) + \mathfrak{A}_{\mathcal{d}}(a_{\hat{\imath}}) + N_{\mathcal{d}}(a_{\hat{\imath}}) \leq 1$.

Definition 5 [33]: Assume that $\text{Þ}_{\hat{\imath}} = \left(M_{\text{Þ}_{\hat{\imath}}}, \mathfrak{A}_{\text{Þ}_{\hat{\imath}}}, N_{\text{Þ}_{\hat{\imath}}}\right)$ $(\hat{\imath} = 1, 2, \ldots, \varkappa)$ denote the family of PFNs, then PFE weighted average AOs are as follows

$$PFEWA(Þ_1, Þ_2, Þ_3, \dots, Þ_{\varkappa}) = \left(\begin{array}{c}\frac{\prod_{\hat{\imath}=1}^{\varkappa}\left(1+M_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}-\prod_{p=1}^{\varkappa}\left(1-M_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}}{\prod_{\hat{\imath}=1}^{\varkappa}\left(1+M_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}-\prod_{\hat{\imath}=1}^{\varkappa}\left(1-M_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}}, \frac{2\prod_{\hat{\imath}=1}^{\varkappa}\left(\mathfrak{A}_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}}{\prod_{\hat{\imath}=1}^{\varkappa}\left(2-\mathfrak{A}_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}+\prod_{p=1}^{\varkappa}\left(\mathfrak{A}_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}}, \\ \frac{2\prod_{\hat{\imath}=1}^{\varkappa}\left(N_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}}{\prod_{\hat{\imath}=1}^{\varkappa}\left(2-N_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}+\prod_{p=1}^{\varkappa}\left(N_{Þ_{\hat{\imath}}}\right)^{\partial_{\hat{\imath}}}}\end{array}\right)$$

Where $\partial = (\partial_1, \partial_2, \dots, \partial_{\varkappa})$ represent the weight vectors (WVs) for $Þ_{\hat{\imath}}$ with the condition that $\sum_{\hat{\imath}=1}^{\varkappa} \partial_{\hat{\imath}} = 1$ and $\partial_{\hat{\imath}} \in [0, 1]$.

Definition 6 [34]: The distance measure between two PFSNs $Þ_1$ and $Þ_2$ is the mapping $d: \mathrm{PFSS} \times \mathrm{PFSS} \to [0, 1]$ that satisfy the following properties

1) $0 \le d(Þ_1, Þ_2) \le 1$
2) $d(Þ_1, Þ_2) = 0$ iff $Þ_1 = Þ_2$
3) $d(Þ_1, Þ_2) = d(Þ_2, Þ_1)$
4) If $Þ_1 \subseteq Þ_2 \subseteq Þ_3$ then $d(Þ_1, Þ_3) \ge d(Þ_1, Þ_2)$ and $d(Þ_1, Þ_3) \ge d(Þ_2, Þ_3)$.

Definition 7 [34]: For two PFSNs, $Þ_{11} = \left(M_{Þ_{11}}, \mathfrak{A}_{Þ_{11}}, N_{Þ_{11}}\right)$ and $Þ_{12} = \left(M_{Þ_{12}}, \mathfrak{A}_{Þ_{12}}, N_{Þ_{12}}\right)$, The Hamming distance can be defined by

$$d(Þ_{11}, Þ_{12}) = \frac{1}{2\wp\varkappa}\sum_{\mathcal{\sigma}=1}^{\varkappa}\sum_{\hat{\imath}=1}^{\wp}\left[\left|M_{Þ_{11}} - M_{Þ_{12}}\right| + \left|\mathfrak{A}_{Þ_{11}} - \mathfrak{A}_{Þ_{12}}\right| + \left|N_{Þ_{11}} - N_{Þ_{12}}\right|\right]$$

Picture Fuzzy Soft Einstein Aggregation Operators

This section of the article has delivered some operating rules based on ETN and ETCN. Moreover, we will propose the PFSEA and PFSEG AOs based on the developed operational laws.

Einstein Operating Rules for PFSNs

Definition 8: For two PFSNs $Þ_{11} = \left((M_{11}, \mathfrak{A}_{11}, N_{11})\right)$ and $Þ_{12} = \left((M_{12}, \mathfrak{A}_{12}, N_{12})\right)$ and $r \ge 0$, then based on ETN and ETCN, we obtain

1) $Þ_{11} \oplus Þ_{12} = \left(\left(\frac{(1+M_{11})-(1-M_{12})}{(1+M_{11})+(1-M_{12})}, \frac{2\mathfrak{A}_{11}}{(2-\mathfrak{A}_{11})+\mathfrak{A}_{12}}, \frac{2N_{11}}{(2-N_{11})+N_{12}}\right)\right)$
2) $Þ_{11} \otimes Þ_{12} = \left(\left(\frac{2M_{11}}{(2-M_{11})+M_{12}}, \frac{2M_{11}}{(2-M_{11})+M_{12}}, \frac{(1+N_{11})-(1-N_{12})}{(1+N_{11})+(1-N_{12})}\right)\right)$
3) $rÞ_{11} = \left(\left(\frac{(1+M_{11})^r-(1-M_{11})^r}{(1+M_{11})^r+(1-M_{11})^r}, \frac{2(\mathfrak{A}_{11})^r}{(2-\mathfrak{A}_{11})^r+(\mathfrak{A}_{11})^r}, \frac{2(N_{11})^r}{(2-N_{11})^r+(N_{11})^r}\right)\right)$
4) $(Þ_{11})^r = \left(\left(\frac{2(M_{11})^r}{(2-M_{11})^r+(M_{12})^r}, \frac{2(M_{11})^r}{(2-M_{11})^r+(M_{12})^r}, \frac{(1+N_{11})^r-(1-N_{12})^r}{(1+N_{11})^r+(1-N_{12})^r}\right)\right)$

Definition 9: Let $Þ_{\hat{\imath}\mathcal{\sigma}} = \left((M_{\hat{\imath}\mathcal{\sigma}}, \mathfrak{A}_{\hat{\imath}\mathcal{\sigma}}, N_{\hat{\imath}\mathcal{\sigma}})\right)$ be the family of PFSNs, the score and accuracy function are given by

$$Sc(Þ_{\hat{\imath}\mathcal{\sigma}}) = \left(\left((M_{\hat{\imath}\mathcal{\sigma}}) - (\mathfrak{A}_{\hat{\imath}\mathcal{\sigma}}) - (N_{\hat{\imath}\mathcal{\sigma}})\right)\right)$$

And

$$Ac(Þ_{\hat{\imath}\mathcal{\sigma}}) = \left(\left((M_{\hat{\imath}\mathcal{\sigma}}) + (\mathfrak{A}_{\hat{\imath}\mathcal{\sigma}}) + (N_{\hat{\imath}\mathcal{\sigma}})\right)\right)$$

Note that for two PFSNs $Þ_{\hat{\imath}\mathcal{\sigma}}$ and $R_{\hat{\imath}\mathcal{\sigma}}$, we have

1) if $Sc(Þ_{\hat{\imath}\mathcal{\sigma}}) > Sc(R_{\hat{\imath}\mathcal{\sigma}})$ then $Þ_{\hat{\imath}\mathcal{\sigma}} > R_{\hat{\imath}\mathcal{\sigma}}$
2) if $Sc(Þ_{\hat{\imath}\mathcal{\sigma}}) < Sc(R_{\hat{\imath}\mathcal{\sigma}})$ then $Þ_{\hat{\imath}\mathcal{\sigma}} < R_{\hat{\imath}\mathcal{\sigma}}$
3) if $Sc(Þ_{\hat{\imath}\mathcal{\sigma}}) = Sc(R_{\hat{\imath}\mathcal{\sigma}})$ then

(i) if $\mathcal{A}c(Þ_{\hat{\imath}\mathcal{\sigma}}) > \mathcal{A}c(R_{\hat{\imath}\mathcal{\sigma}})$ then $Þ_{\hat{\imath}\mathcal{\sigma}} > R_{\hat{\imath}\mathcal{\sigma}}$
(ii) if $\mathcal{A}c(Þ_{\hat{\imath}\mathcal{\sigma}}) < \mathcal{A}c(R_{\hat{\imath}\mathcal{\sigma}})$ then $Þ_{\hat{\imath}\mathcal{\sigma}} < R_{\hat{\imath}\mathcal{\sigma}}$
(iii) if $\mathcal{A}c(Þ_{\hat{\imath}\mathcal{\sigma}}) = \mathcal{A}c(R_{\hat{\imath}\mathcal{\sigma}})$ then $Þ_{\hat{\imath}\mathcal{\sigma}} = R_{\hat{\imath}\mathcal{\sigma}}$.

PFSEWA Aggregation Operators

In this section of the article, we have delivered the idea of PFSEWA aggregation operators.

Definition 10: Let $Þ_{\hat{\imath}\mathcal{\sigma}} = \left((M_{\hat{\imath}\mathcal{\sigma}}, \mathfrak{A}_{\hat{\imath}\mathcal{\sigma}}, N_{\hat{\imath}\mathcal{\sigma}})\right)$ be the family of PFSNs, then the PFSEWA AOs is given as

$$\mathrm{PFSEWA}(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \oplus_{\mathcal{\sigma}=1}^{\wp} \mathscr{b}_{\mathcal{\sigma}} (\oplus_{\hat{\imath}=1}^{\varkappa} \partial_{\hat{\imath}} Þ_{\hat{\imath}\mathcal{\sigma}}) \qquad (1)$$

Where $(\hat{\imath} = 1, 2, 3, \dots, \varkappa), (\mathcal{\sigma} = 1, 2, 3, \dots, \wp)$ and $\partial_{\hat{\imath}}, \mathscr{b}_{\mathcal{\sigma}}$ denote the WVs of the experts and parameters with $\sum_{\hat{\imath}=1}^{\varkappa} \partial_{\hat{\imath}} = 1$ and $\sum_{\mathcal{\sigma}=1}^{\wp} \mathscr{b}_{\mathcal{\sigma}} = 1$.

Theorem 1: Let $Þ_{\hat{\imath}\mathcal{\sigma}} = \left((M_{\hat{\imath}\mathcal{\sigma}}, \mathfrak{A}_{\hat{\imath}\mathcal{\sigma}}, N_{\hat{\imath}\mathcal{\sigma}})\right)$ be the family of PFSNs, then the results achieved by using equation (1) is again PFSN

$$\mathrm{PFSEWA}(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \oplus_{\mathcal{J}=1}^{\wp} \mathscr{b}_{\mathcal{J}} \left(\oplus_{\hat\imath=1}^{\varkappa} \partial_{\hat\imath} Þ_{\hat\imath\mathcal{J}}\right)$$

$$= \left(\begin{array}{c} \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(1+M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(1-M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(1+M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(1-M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(2-\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(N_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(2-N_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{\varkappa}(N_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}} \end{array} \right) \quad (2)$$

Here $(\hat\imath = 1, 2, 3, \dots, \varkappa)$, $(\mathcal{J} = 1, 2, 3, \dots, \wp)$ and $\partial_{\hat\imath}, \mathscr{b}_{\mathcal{J}}$ are WVs with $\sum_{\hat\imath=1}^{\varkappa} \partial_{\hat\imath} = 1$ and $\sum_{\mathcal{J}=1}^{\wp} \mathscr{b}_{\mathcal{J}} = 1$.

Proof: We will use the mathematical induction method to prove the result

For $\varkappa = 1$, we get $\partial_{\hat\imath} = 1$

$$PFSEWA(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \oplus_{\mathcal{J}=1}^{\wp} \mathscr{b}_{\mathcal{J}} Þ_{1\mathcal{J}}$$

$$= \left(\begin{array}{c} \dfrac{\prod_{\mathcal{J}=1}^{\wp}(1+M_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}(1-M_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}(1+M_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}(1-M_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}(\mathfrak{A}_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}(2-\mathfrak{A}_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}(\mathfrak{A}_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}(N_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}(2-N_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}(2-N_{1\mathcal{J}})^{\mathscr{b}_{\mathcal{J}}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(1+M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(1-M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(1+M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(1-M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(2-\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(N_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(2-N_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat\imath=1}^{1}(N_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}} \end{array} \right).$$

Now for $\wp = 1$, we get $\mathscr{b}_{\mathcal{J}} = 1$

$$PFsEWA(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \oplus_{\hat\imath=1}^{\varkappa} \partial_{\hat\imath} Þ_{\hat\imath 1}$$

$$= \left(\begin{array}{c} \dfrac{\prod_{\hat\imath=1}^{\varkappa}(1+M_{\hat\imath 1})^{\partial_{\hat\imath}} - \prod_{\hat\imath=1}^{\varkappa}(1-M_{\hat\imath 1})^{\partial_{\hat\imath}}}{\prod_{\hat\imath=1}^{\varkappa}(1+M_{\hat\imath 1})^{\partial_{\hat\imath}} + \prod_{\hat\imath=1}^{\varkappa}(1-M_{\hat\imath 1})^{\partial_{\hat\imath}}}, \\ \dfrac{2\prod_{\hat\imath=1}^{\varkappa}(\mathfrak{A}_{\hat\imath 1})^{\partial_{\hat\imath}}}{\prod_{\hat\imath=1}^{\varkappa}(2-\mathfrak{A}_{\hat\imath 1})^{\partial_{\hat\imath}} + \prod_{\hat\imath=1}^{\varkappa}(\mathfrak{A}_{\hat\imath 1})^{\partial_{\hat\imath}}}, \\ \dfrac{2\prod_{\hat\imath=1}^{\varkappa}(N_{\hat\imath 1})^{\partial_{\hat\imath}}}{\prod_{\hat\imath=1}^{\varkappa}(2-N_{\hat\imath 1})^{\partial_{\hat\imath}} + \prod_{\hat\imath=1}^{\varkappa}(2-N_{\hat\imath 1})^{\partial_{\hat\imath}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \dfrac{\prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(1+M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(1-M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(1+M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(1-M_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(2-\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(\mathfrak{A}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(\underline{N}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(2-\underline{N}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{1}\left(\prod_{\hat\imath=1}^{\varkappa}(\underline{N}_{\hat\imath\mathcal{J}})^{\partial_{\hat\imath}}\right)^{\mathscr{b}_{\mathcal{J}}}} \end{array} \right)$$

So equation (2) is valid for $\wp = 1$ and $\varkappa = 1$.

Now suppose that the above equation holds for $\varkappa = \ell_2, \wp = \ell_1 + 1$ and for $\varkappa = \ell_2 + 1, \wp = \ell_1$, then

$$PFSEWA(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \oplus_{\mathcal{J}=1}^{\ell_1+1} \mathscr{b}_{\mathcal{J}} \left(\oplus_{\hat\imath=1}^{\ell_2} \partial_{\hat\imath} Þ_{\hat\imath\mathcal{J}}\right)$$

$$= \begin{pmatrix} \dfrac{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1+M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} - \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1-M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1+M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1-M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}, \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(2-\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}, \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(2-\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}} \end{pmatrix}$$

$$\oplus_{\mathscr{J}=1}^{\ell_1} \mathscr{b}_{\mathscr{J}}\left(\oplus_{\hat{\imath}=1}^{\ell_2+1} \partial_{\hat{\imath}}\text{Þ}_{\hat{\imath}\mathscr{J}}\right)$$

$$= \begin{pmatrix} \dfrac{\prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1+M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} - \prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1-M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1+M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1-M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}, \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(2-\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}, \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(2-\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}} \end{pmatrix}$$

Now suppose that the above equation hold for $\mathscr{n} = \ell_2 + 1, \wp = \ell_1 + 1$ then

$$PFSEWA(\text{Þ}_{11}, \text{Þ}_{12}, \text{Þ}_{13}, \dots, \text{Þ}_{\mathscr{n}\wp}) = \oplus_{\mathscr{J}=1}^{\ell_1+1} \mathscr{b}_{\mathscr{J}}\left(\oplus_{\hat{\imath}=1}^{\ell_2+1} \partial_{\hat{\imath}}\text{Þ}_{\hat{\imath}\mathscr{J}}\right)$$

$$= \oplus_{\mathscr{J}=1}^{\ell_1+1} \mathscr{b}_{\mathscr{J}}\left(\oplus_{\hat{\imath}=1}^{\ell_2} \partial_{\hat{\imath}}\text{Þ}_{\hat{\imath}\mathscr{J}} \oplus \partial_{\hat{\imath}+1}\text{Þ}_{(\ell_2+1)\mathscr{J}}\right)$$

$$= \left(\oplus_{\mathscr{J}=1}^{\ell_1+1}\oplus_{\hat{\imath}=1}^{\ell_2} \partial_{\hat{\imath}}\mathscr{b}_{\mathscr{J}}\text{Þ}_{\hat{\imath}\mathscr{J}}\right) \oplus \left(\oplus_{\mathscr{J}=1}^{\ell_1+1} \mathscr{b}_{\mathscr{J}}\partial_{\hat{\imath}+1}\text{Þ}_{(\ell_2+1)\mathscr{J}}\right)$$

$$= \begin{pmatrix} \dfrac{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1+M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} - \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1-M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1+M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1-M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}} \oplus \\ \dfrac{\prod_{\mathscr{J}=1}^{\ell_1+1}\left((1+M_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}} - \prod_{\mathscr{J}=1}^{\ell_1+1}\left((1-M_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left((1+M_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left((1-M_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}}}, \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(2-\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}} \oplus \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1+1}\left((\mathfrak{A}_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left((2-\mathfrak{A}_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left((\mathfrak{A}_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}}}, \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(2-\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}} \oplus \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1+1}\left((\underline{N}_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left((2-\underline{N}_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left((\underline{N}_{(\ell_2+1)\mathscr{J}})^{\partial(\ell_2+1)}\right)^{\mathscr{b}_{\mathscr{J}}}} \end{pmatrix} \text{ and}$$

$$= \begin{pmatrix} \dfrac{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1+M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} - \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1-M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1+M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1-M_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}, \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(2-\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\mathfrak{A}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}, \\ \dfrac{2\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}}{\prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(2-\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}} + \prod_{\mathscr{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\underline{N}_{\hat{\imath}\mathscr{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathscr{J}}}} \end{pmatrix}$$

$$= \oplus_{\mathscr{J}=1}^{\ell_1+1} \mathscr{b}_{\mathscr{J}}\left(\oplus_{\hat{\imath}=1}^{\ell_2+1} \partial_{\hat{\imath}}\text{Þ}_{\hat{\imath}\mathscr{J}}\right)$$

Hence, the result is true for $\wp = \ell_1 + 1$ and $\mathscr{n} = \ell_2 + 1$.

PFSEWG Aggregation Operators

In this section, we have proposed the basic definition for PFSEWG aggregation operators.

Definition 11: Let $Þ_{\hat{\imath}\eth} = \left((M_{\hat{\imath}\eth}, \mathfrak{A}_{\hat{\imath}\eth}, N_{\hat{\imath}\eth})\right)$ be the family of PFSNs, then the PFSEWG operator is defined as

$$\mathrm{PFSEWG}(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \otimes_{\eth=1}^{\wp} b_{\eth}(\otimes_{\hat{\imath}=1}^{\varkappa} \partial_{\hat{\imath}} Þ_{\hat{\imath}\eth}) \tag{3}$$

Where $(\hat{\imath} = 1, 2, 3, \dots, \varkappa), (\eth = 1, 2, 3, \dots, \wp)$ and $\partial_{\hat{\imath}}, b_{\eth}$ denote the WVs with $\sum_{\hat{\imath}=1}^{\varkappa} \partial_{\hat{\imath}} = 1$ and $\sum_{\eth=1}^{\wp} b_{\eth} = 1$.

Theorem 2: Let $Þ_{\hat{\imath}\eth} = \left((M_{\hat{\imath}\eth}, \mathfrak{A}_{\hat{\imath}\eth}, N_{\hat{\imath}\eth})\right)$ be the family of PFSNs, then the results achieved by using equation (3) is again PFSN

$$\mathrm{PFSEWG}(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \otimes_{\eth=1}^{\wp} b_{\eth}(\otimes_{\hat{\imath}=1}^{\varkappa} \partial_{\hat{\imath}} Þ_{\hat{\imath}\eth})$$

$$= \left(\begin{array}{c} \frac{2\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(2-M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}, \\ \frac{2\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(2-M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}, \\ \frac{\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}-\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}} \end{array}\right) \tag{4}$$

Here $(\hat{\imath} = 1, 2, 3, \dots, \varkappa), (\eth = 1, 2, 3, \dots, \wp)$ and $\partial_{\hat{\imath}}, b_{\eth}$ are the WVs with $\sum_{\hat{\imath}=1}^{\varkappa} \partial_{\hat{\imath}} = 1$ and $\sum_{\eth=1}^{\wp} b_{\eth} = 1$.

Proof: Here, we shall employ the mathematical induction method.

For $\varkappa = 1$, we get $\partial_{\hat{\imath}} = 1$

$$PFSEWG(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \otimes_{\eth=1}^{\wp} b_{\eth} Þ_{1\eth}$$

$$= \left(\begin{array}{c} \frac{2\prod_{\eth=1}^{\wp}(M_{1\eth})^{b_{\eth}}}{\prod_{\eth=1}^{\wp}(2-M_{1\eth})^{b_{\eth}}+\prod_{\eth=1}^{\wp}(M_{1\eth})^{b_{\eth}}}, \\ \frac{2\prod_{\eth=1}^{\wp}(\mathfrak{A}_{1\eth})^{b_{\eth}}}{\prod_{\eth=1}^{\wp}(2-\mathfrak{A}_{1\eth})^{b_{\eth}}+\prod_{\eth=1}^{\wp}(\mathfrak{A}_{1\eth})^{b_{\eth}}}, \\ \frac{\prod_{\eth=1}^{\wp}(1+N_{1\eth})^{b_{\eth}}-\prod_{\eth=1}^{\wp}(1-N_{1\eth})^{b_{\eth}}}{\prod_{\eth=1}^{\wp}(1+N_{1\eth})^{b_{\eth}}+\prod_{\eth=1}^{\wp}(1-N_{1\eth})^{b_{\eth}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \frac{2\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(2-M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}, \\ \frac{2\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(\mathfrak{A}_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(2-\mathfrak{A}_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(\mathfrak{A}_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}, \\ \frac{\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(1+N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}-\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(1-N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(1+N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{1}(1-N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}} \end{array}\right).$$

Now for $\wp = 1$, we get $b_{\eth} = 1$

$$PFSEWG(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \otimes_{\hat{\imath}=1}^{\varkappa} \partial_{\hat{\imath}} Þ_{\hat{\imath}1}$$

$$= \left(\begin{array}{c} \frac{2\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}1})^{\partial_{\hat{\imath}}}}{\prod_{\hat{\imath}=1}^{\varkappa}(2-M_{\hat{\imath}1})^{\partial_{\hat{\imath}}}+\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}1})^{\partial_{\hat{\imath}}}}, \\ \frac{2\prod_{\hat{\imath}=1}^{\varkappa}(\mathfrak{A}_{\hat{\imath}1})^{\partial_{\hat{\imath}}}}{\prod_{\hat{\imath}=1}^{\varkappa}(2-\mathfrak{A}_{\hat{\imath}1})^{\partial_{\hat{\imath}}}+\prod_{\hat{\imath}=1}^{\varkappa}(\mathfrak{A}_{\hat{\imath}1})^{\partial_{\hat{\imath}}}}, \\ \frac{\prod_{\hat{\imath}=1}^{\varkappa}(1+N_{\hat{\imath}1})^{\partial_{\hat{\imath}}}-\prod_{\hat{\imath}=1}^{\varkappa}(1-N_{\hat{\imath}1})^{\partial_{\hat{\imath}}}}{\prod_{\hat{\imath}=1}^{\varkappa}(1+N_{\hat{\imath}1})^{\partial_{\hat{\imath}}}+\prod_{\hat{\imath}=1}^{\varkappa}(1-N_{\hat{\imath}1})^{\partial_{\hat{\imath}}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \frac{2\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(2-M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}, \\ \frac{2\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(\mathfrak{A}_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(2-\mathfrak{A}_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(\mathfrak{A}_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}, \\ \frac{\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}-\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}}{\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}+\prod_{\eth=1}^{1}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-N_{\hat{\imath}\eth})^{\partial_{\hat{\imath}}}\right)^{b_{\eth}}} \end{array}\right)$$

So equation (4) is valid for $\wp = 1$ and $\varkappa = 1$.

Now suppose that the above equation holds for $\varkappa = \ell_2, \wp = \ell_1 + 1$ and for $\varkappa = \ell_2 + 1, \wp = \ell_1$, then

$$PFSEWG(Þ_{11}, Þ_{12}, Þ_{13}, \dots, Þ_{\varkappa\wp}) = \otimes_{\eth=1}^{\ell_1+1} b_{\eth}\left(\otimes_{\hat{\imath}=1}^{\ell_2} \partial_{\hat{\imath}} Þ_{\hat{\imath}\eth}\right)$$

$$= \left(\begin{array}{c} \frac{2\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(2-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \frac{2\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(2-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \frac{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}-\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}} \end{array} \right),$$

$$\otimes_{\mathcal{J}=1}^{\ell_1}\mathscr{b}_{\mathcal{J}}\left(\otimes_{\hat{\imath}=1}^{\ell_2+1}\partial_{\hat{\imath}}\text{Þ}_{\hat{\imath}\mathcal{J}}\right)$$

$$= \left(\begin{array}{c} \frac{2\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(2-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \frac{2\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(2-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \frac{\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}-\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}} \end{array} \right)$$

Now suppose that the above equation holds for $\nu = \ell_2 + 1, \wp = \ell_1 + 1$ then

$$PFSEWG(\text{Þ}_{11}, \text{Þ}_{12}, \text{Þ}_{13}, \ldots, \text{Þ}_{\nu\wp}) = \otimes_{\mathcal{J}=1}^{\ell_1+1}\mathscr{b}_{\mathcal{J}}\left(\otimes_{\hat{\imath}=1}^{\ell_2+1}\partial_{\hat{\imath}}\text{Þ}_{\hat{\imath}\mathcal{J}}\right)$$

$$= \otimes_{\mathcal{J}=1}^{\ell_1+1}\mathscr{b}_{\mathcal{J}}\left(\otimes_{\hat{\imath}=1}^{\ell_2}\partial_{\hat{\imath}}\text{Þ}_{\hat{\imath}\mathcal{J}}\otimes\partial_{\hat{\imath}+1}\text{Þ}_{(\ell_2+1)\mathcal{J}}\right)$$

$$= \left(\otimes_{\mathcal{J}=1}^{\ell_1+1}\otimes_{\hat{\imath}=1}^{\ell_2}\partial_{\hat{\imath}}\mathscr{b}_{\mathcal{J}}\text{Þ}_{\hat{\imath}\mathcal{J}}\right)\otimes\left(\otimes_{\mathcal{J}=1}^{\ell_1+1}\mathscr{b}_{\mathcal{J}}\partial_{\hat{\imath}+1}\text{Þ}_{(\ell_2+1)\mathcal{J}}\right)$$

$$= \left(\begin{array}{c} \frac{2\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(2-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}\otimes \\ \frac{2\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(M_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(2-M_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(M_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \frac{2\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(2-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}\otimes \\ \frac{2\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(\mathfrak{A}_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(2-\mathfrak{A}_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(\mathfrak{A}_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \frac{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}-\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}\otimes \\ \frac{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(1+N_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}-\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(1-N_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(1+N_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}+\prod_{\mathcal{J}=1}^{\ell_1+1}\left(\left(1-N_{(\ell_2+1)\mathcal{J}}\right)^{\partial_{(\ell_2+1)}}\right)^{\mathscr{b}_{\mathcal{J}}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \frac{2\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(M_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}}{\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(2-M_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}+\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(M_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}}, \\ \frac{2\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\mathfrak{A}_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}}{\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(2-\mathfrak{A}_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}+\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(\mathfrak{A}_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}}, \\ \frac{\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1+N_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}-\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1-N_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}}{\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1+N_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}+\prod_{\sigma=1}^{\ell_1+1}\left(\prod_{\hat{\imath}=1}^{\ell_2+1}(1-N_{\hat{\imath}\sigma})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_\sigma}} \end{array} \right)$$

$$= \otimes_{\sigma=1}^{\ell_1+1}\mathscr{b}_\sigma\left(\otimes_{\hat{\imath}=1}^{\ell_2+1}\partial_{\hat{\imath}}Þ_{\hat{\imath}\sigma}\right)$$

Hence, the result is true for $\wp = \ell_1 + 1$ and $\nu = \ell_2 + 1$.

The WASPAS Technique based on PFSEWA or PFSEWG Aos

In this part, we have to define an algorithm of the WASPAS technique based on PFSEWA and PFSEWG AOs. The overall discussion is deliberated as follows.

An Algorithm based on WASPAS Technique for a developed Approach

Suppose $\{al_1, al_2, al_3, \dots, al_d\}$ be set of $'d'$ alternatives and $\{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \dots, \mathfrak{s}_\wp\}$ denote the set of $'\wp'$ parameters. Also, suppose $\{q_1, q_2, q_3, \dots, q_\nu\}$ be the set of ν experts for each alternative $altr._f$ $(f = 1,2,3,\dots,d)$ against parameters $\mathfrak{s}_\sigma(\sigma = 1,2,3,\dots,\wp)$. Let $\partial = (\partial_1, \partial_2, \partial_3, \dots, \partial_\nu)^{\mathrm{T}}$ be the WVs for experts $q_{\hat{\imath}}$ and $\mathscr{b} = (\mathscr{b}_1, \mathscr{b}_2, \mathscr{b}_3, \dots, \mathscr{b}_\wp)^{\mathrm{T}}$ be the WVs for parameters $\mathfrak{s}_\sigma(\sigma = 1,2,3,\dots,\wp)$ such that $\sum_{\hat{\imath}=1}^{\nu}\partial_\sigma = 1$ and $\sum_{\sigma=1}^{\wp}\mathscr{b}_\sigma = 1$. The algorithm for the WASPAS technique is developed as

Step 1: Assume that experts evaluate each alternative for each parameter using the PFS matrix provided by

$$M = \begin{bmatrix} ((M_{11}, \mathfrak{A}_{11}, N_{11})) & ((M_{12}, \mathfrak{A}_{12}, N_{12})) & \dots & ((M_{1\wp}, \mathfrak{A}_{1\wp}, N_{1\wp})) \\ ((M_{21}, \mathfrak{A}_{21}, N_{21})) & ((M_{22}, \mathfrak{A}_{22}, N_{22})) & \dots & ((M_{2\wp}, \mathfrak{A}_{2\wp}, N_{2\wp})) \\ \vdots & \vdots & \ddots & \vdots \\ ((M_{\nu 1}, \mathfrak{A}_{\nu 1}, N_{\nu 1})) & ((M_{\nu 2}, \mathfrak{A}_{\nu 2}, N_{\nu 2})) & \dots & ((M_{\nu\wp}, \mathfrak{A}_{\nu\wp}, N_{\nu\wp})) \end{bmatrix}$$

Step 2: The normalizing procedure is given by

Case 1: For benefit type criteria

$$O_{\hat{\imath}\sigma} = \frac{O_{\hat{\imath}\sigma}}{\max_{\hat{\imath}} O_{\hat{\imath}\sigma}}$$

Where $\max_{\hat{\imath}} O_{\hat{\imath}\sigma} = \left\{\left(\max_{\hat{\imath}} M_{\hat{\imath}\sigma}, \min_{\hat{\imath}} \mathfrak{A}_{\hat{\imath}\sigma}, \min_{\varsigma} N_{\hat{\imath}\sigma}\right)\right\}$ and

$$\frac{O_{\hat{\imath}\sigma}}{\max_{\hat{\imath}} O_{\hat{\imath}\sigma}} = \left\{\left(\max\left(M_{\hat{\imath}\sigma}, \max_{\hat{\imath}} M_{\hat{\imath}\sigma}\right), \min\left(\mathfrak{A}_{\hat{\imath}\sigma}, \min_{\hat{\imath}} \mathfrak{A}_{\hat{\imath}\sigma}\right), \min\left(N_{\hat{\imath}\sigma}, \min_{\hat{\imath}} N_{\hat{\imath}\sigma}\right)\right)\right\}.$$

Case II: For cost-type criteria

Also, $\frac{\min_{\hat{\imath}} O_{\hat{\imath}\sigma}}{O_{\hat{\imath}\sigma}}$

Where $\min_{\hat{\imath}} O_{\hat{\imath}\sigma} = \left\{\left(\min_{\hat{\imath}} M_{\hat{\imath}\sigma}, \min_{\hat{\imath}} \mathfrak{A}_{\hat{\imath}\sigma}, \max_{\hat{\imath}} N_{\hat{\imath}\sigma}\right)\right\}$ and

$$\frac{\min_{\hat{\imath}} O_{\hat{\imath}\sigma}}{O_{\hat{\imath}\sigma}} = \left\{\left(\min\left(\min_{\hat{\imath}} M_{\hat{\imath}\sigma}, M_{\hat{\imath}\sigma}\right), \min\left(\min_{\hat{\imath}} \mathfrak{A}_{\hat{\imath}\sigma}, \mathfrak{A}_{\hat{\imath}\sigma}\right), \max\left(\max_{\hat{\imath}} N_{\hat{\imath}\sigma}, N_{\hat{\imath}\sigma}\right)\right)\right\}.$$

So the normalized matrix is

$$\mathrm{E} = \begin{bmatrix} O_{11} & O_{12} & \dots & O_{1\nu} \\ O_{21} & O_{22} & \dots & O_{2\nu} \\ \vdots & \vdots & \ddots & \vdots \\ O_{\wp 1} & O_{\wp 2} & \dots & O_{\wp\nu} \end{bmatrix}$$

Step 3: Find out the relative importance of the alternative as

$$\mathrm{L}_v{}^{WS} = \left(\begin{array}{c} \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1+M_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1-M_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1+M_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1-M_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(\mathfrak{A}_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(2-\mathfrak{A}_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(\mathfrak{A}_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(N_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(2-N_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(N_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}} \end{array}\right).$$

Step 4: Similarly, we have to compute

$$\mathrm{L}_v{}^{WP} = \left(\begin{array}{c} \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(M_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(2-M_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(M_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}, \\ \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1+\mathfrak{A}_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1-\mathfrak{A}_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1+\mathfrak{A}_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1-\mathfrak{A}_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}, \\ \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1+N_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1-N_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1+N_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{i=1}^{n}(1-N_{i\mathcal{J}})^{\partial_i}\right)^{b_{\mathcal{J}}}} \end{array}\right).$$

Step 5: Finally, the WASPAS technique is defined by

$$\mathrm{L}_v = \frac{\mathrm{L}_v{}^{WS} + \mathrm{L}_v{}^{WP}}{2}.$$

Step 6: Utilize the definition (9) and then order the result to find the best result.
The flowchart of the proposed WASPAS approach is given in Figure 3.

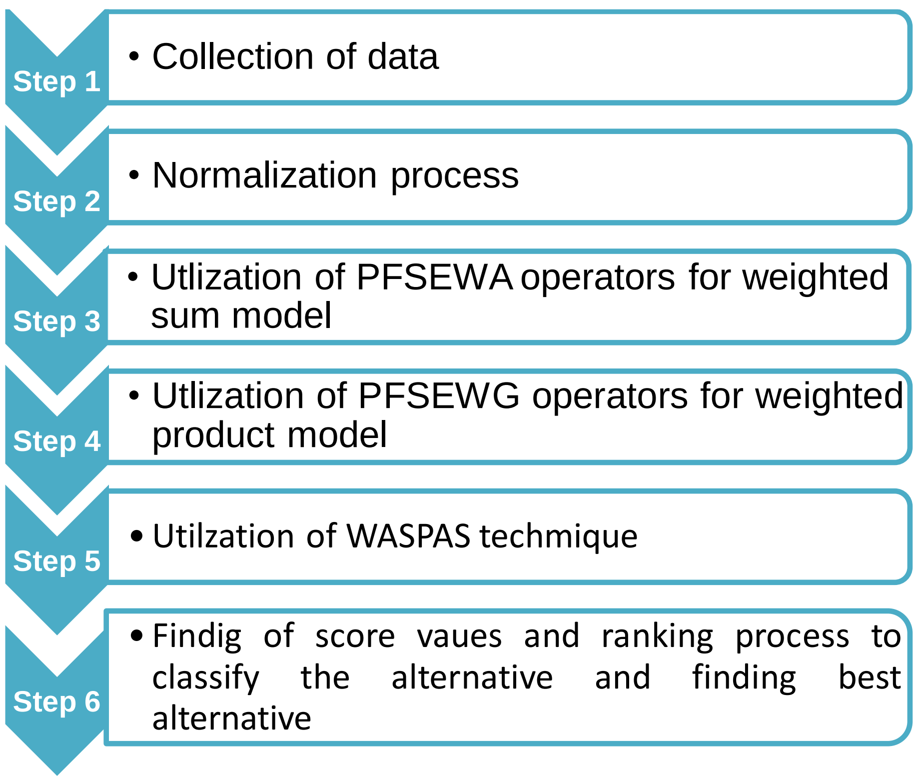

Figure 3. Flowchart of the proposed WASPAS Algorithm

Now we use the step-wise algorithm as follows

Numerical Example

Example 1: Engineers and seismologists are trying to innovate the technologies that help buildings to resist during the earthquake situation. To avoid any kind of financial and life risk damage during an earthquake, human beings want to apply the best technology in their buildings that can help them or save their buildings during an earthquake. Let us assume that four main technologies help buildings to resist earthquakes.

1. **Carbon-Fiber Wrap:** Building beams, columns, and walls can be strengthened using carbon-fiber wrap, a lightweight, high-strength material. It lessens structural damage and cracking and increases a structure's ability to support loads during earthquakes. In order to improve seismic performance, older structures are frequently retrofitted using this technology. Buildings are safer and more resilient to seismic impacts thanks to their adaptability and endurance.
2. **Shock Absorbers:** In order to absorb and disperse earthquake vibrations, buildings are equipped with specific equipment called shock absorbers. They minimize the passage of seismic energy to the structure and lessen excessive movement. During severe earthquakes, this technique enhances the comfort and stability of buildings, particularly tall ones. Consequently, there is much less chance of collapse and structural damage.
3. **Rocking Core Wall:** An innovative structural technology called a rocking core wall prevents a building from splitting or collapsing during an earthquake by allowing it to rock gently. By enabling regulated movement and assisting the building in returning to its initial position following shaking, it lessens the concentration of stress. This method reduces persistent deformation and boosts structural robustness. It works especially well for modern, high-rise structures that can withstand earthquakes.
4. **Replicable Fuses:** Replicable fuses are interchangeable structural elements intended to absorb and regulate seismic energy. These fuses deform first during seismic activity, preventing serious damage to the main structural framework. It is simple to replace the damaged fuses following the earthquake without having to rebuild the entire structure. This method increases the long-term sustainability and safety of structures while lowering repair costs.

A team of four experts is invited to assess these technologies with their WVs $\{0.17, 0.25, 0.31, 0.27\}$. Assume that the parameters of these technologies are

a) To reduce and slow down the vibratory motion
b) To contract the resonance and minimize the dynamic response
c) Reduce floor acceleration and shear forces
d) Easier to implement

Suppose that WVs for parameters are $\{0.20, 0.30, 0.21, 0.29\}$. Now we apply a step-wise algorithm:
Step 1: The evaluation of the experts is given as PFSNs given in Table 1-4.

Table 1. PFS data for $altr._1$

	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$
q_1	$((0.12, 0.12, 0.15))$	$((0.27, 0.17, 0.15))$	$((0.21, 0.23, 0.27))$	$((0.13, 0.12, 0.17))$
q_2	$((0.10, 0.26, 0.20))$	$((0.26, 0.16, 0.19))$	$((0.11, 0.11, 0.13))$	$((0.16, 0.14, 0.23))$
q_3	$((0.23, 0.27, 0.21))$	$((0.11, 0.14, 0.15))$	$((0.16, 0.11, 0.31))$	$((0.18, 0.11, 0.14))$
q_4	$(0.12, 0.15, 0.23)$	$(0.25, 0.17, 0.15)$	$(0.12, 0.25, 0.18)$	$(0.21, 0.21, 0.20)$

Table 2. PFS data for $altr._2$

	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$
q	$((0.21, 0.10, 0.30))$	$((0.17, 0.17, 0.25))$	$((0.23, 0.16, 0.19))$	$((0.13, 0.13, 0.19))$
q_2	$((0.17, 0.15, 0.13))$	$((0.26, 0.36, 0.19))$	$((0.20, 0.26, 0.27))$	$((0.15, 0.14, 0.21))$
q_3	$((0.11, 0.17, 0.37))$	$((0.21, 0.29, 0.13))$	$((0.16, 0.17, 0.11))$	$((0.18, 0.11, 0.24))$
q_4	$(0.21, 0.35, 0.11)$	$((0.19, 0.21, 0.20))$	$(0.11, 0.26, 0.37)$	$(0.11, 0.13, 0.13)$

Table 3. PFS data for $altr._3$

	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$
q_1	$((0.31, 0.20, 0.30))$	$((0.11, 0.16, 0.37))$	$((0.13, 0.16, 0.18))$	$((0.13, 0.32, 0.17))$
q_2	$((0.10, 0.26, 0.20))$	$((0.26, 0.16, 0.19))$	$((0.11, 0.21, 0.23))$	$((0.11, 0.15, 0.13))$
q_3	$((0.21, 0.17, 0.27))$	$((0.21, 0.19, 0.13))$	$((0.17, 0.21, 0.21))$	$((0.28, 0.21, 0.24))$
q_4	$((0.11, 0.15, 0.15))$	$((0.17, 0.27, 0.25))$	$((0.22, 0.25, 0.17))$	$((0.11, 0.17, 0.19))$

Table 4. PFS data for $altr._4$

	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$
q_1	$((0.32, 0.10, 0.10))$	$((0.14, 0.15, 0.17))$	$((0.21, 0.20, 0.17))$	$((0.13, 0.16, 0.19))$
q_2	$((0.13, 0.16, 0.10))$	$((0.16, 0.15, 0.16))$	$((0.13, 0.21, 0.23))$	$((0.11, 0.13, 0.16))$
q_3	$((0.31, 0.14, 0.17))$	$((0.29, 0.17, 0.13))$	$((0.19, 0.27, 0.21))$	$((0.13, 0.17, 0.18))$
q_4	$((0.11, 0.13, 0.12))$	$((0.14, 0.21, 0.22))$	$((0.15, 0.11, 0.21))$	$((0.18, 0.11, 0.14))$

Step 2: Since all parameters are of benefit type, no need to normalize the above given data.
Step 3: Now we use the formula given below to find out the relative importance of the alternatives.

$$\mathrm{L}_v{}^{WS} = \left(\left(\begin{array}{c} \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(2-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(2-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}} \end{array}\right)\right)$$

$$\mathrm{L}_1{}^{WS} = \{0.1753, 0.1601, 0.1838\}, \mathrm{L}_2{}^{WS} = \{0.1746, 0.1884, 0.1844\},$$
$$\mathrm{L}_3{}^{WS} = \{0.1758, 0.1972, 0.1995\}, \mathrm{L}_4{}^{WS} = \{0.1762, 0.1573, 0.1638\}$$

Step 4: Similarly, we have to use the formula to find out the values of $\mathrm{L}_v{}^{WP}$

$$\mathrm{L}_v{}^{WP} = \left(\left(\begin{array}{c} \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(2-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}, \\ \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}} \\ \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\varkappa}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\ell_{\mathcal{J}}}} \end{array}\right)\right)$$

$$\mathrm{L}_1{}^{WP} = \{0.1665, 0.1676, 0.1891\}, \mathrm{L}_2{}^{WP} = \{0.1689, 0.2033, 0.1989\},$$
$$\mathrm{L}_3{}^{WP} = \{0.1647, 0.2020, 0.2074\}, \mathrm{L}_4{}^{WP} = \{0.1662, 0.1623, 0.1680\}$$

Step 5: Finally, the WASPAS techniques for ordering alternatives are given by

$$\mathrm{L}_v = \frac{\mathrm{L}_v{}^{WS} + \mathrm{L}_v{}^{WP}}{2}.$$

Now

$$\mathrm{L}_1 = \{0.0857, 0.4548, 0.4821\}, \mathrm{L}_2 = \{0.0862, 0.4861, 0.4818\},$$
$$\mathrm{L}_3 = \{0.0852, 0.4965, 0.4989\}\ and\ \mathrm{L}_4 = \{0.0858, 0.4517, 0.4595\}$$

Step 6: Utilize definition (9) for score values and then order the results
$Sc(\mathrm{L}_1) = -0.8513, Sc(\mathrm{L}_2) = -0.8818, Sc(\mathrm{L}_3) = -0.9102\ and\ Sc(\mathrm{L}_4) = -0.8255.$ As we can see that $Sc(L_4) > Sc(L_1) > Sc(L_2) > Sc(L_3)$. So, $\mathbf{altr._4}$ is the best technology that helps resist earthquakes.

MABAC Technique with Known weights based on PFSE Aos

MABAC is a multi-criteria decision-making technique for evaluating and ranking alternatives based on multiple criteria. MABAC is designed to handle both quantitative and qualitative data and is particularly useful when criteria are interdependent. MABAC works by creating a reference set of an alternative called a border that represents the best and worst values for each criterion. The alternatives being evaluated are then compared to this reference set, measuring the distance between the alternatives and the border. The distance is calculated for each criterion and the total distance is used to rank the alternatives. MABAC has several advantages over other multicriteria decision analysis methods. It is easy to use and interpret and it can handle a large number of criteria. It is more flexible than other methods.

Algorithm of MABAC Technique with Known weights based on PFSE AOs

Suppose $\{al_1, al_2, al_3, \dots, al_{\wp}\}$ be set of $'\wp'$ alternatives and $\{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \dots, \mathfrak{s}_{\mathscr{n}}\}$ denote the set of $'\mathscr{n}'$ parameters. Also, suppose $\{q_1, q_2, q_3, \dots, q_d\}$ be the set of d experts for each alternative $altr._{\hat{\imath}}$ $(\hat{\imath} = 1, 2, 3, \dots, \wp)$ against parameters $\mathfrak{s}_{\mathcal{J}}$ $(\mathcal{J} = 1, 2, 3, \dots, \mathscr{n})$. Let $\partial = (\partial_1, \partial_2, \partial_3, \dots, \partial_d)^{\mathrm{T}}$ be the WVs for experts q_d and $\mathscr{b} = (\mathscr{b}_1, \mathscr{b}_2, \mathscr{b}_3, \dots, \mathscr{b}_{\wp})^{\mathrm{T}}$ be the WVs for parameters $\mathfrak{s}_{\mathcal{J}}(\mathcal{J} = 1, 2, 3, \dots, \mathscr{n})$ such that $\sum_{\hat{\imath}=1}^{\mathscr{n}} \partial_{\mathcal{J}} = 1$ and $\sum_{\mathcal{J}=1}^{\wp} \mathscr{b}_{\mathcal{J}} = 1$. The algorithm for the MABAC technique is given by

Step 1: Assume that experts evaluate each alternative for each parameter using the PFS matrix provided by

$$\left(M^{d}_{\hat{\imath}\times\mathcal{J}}\right)_{\wp\times\mathscr{n}} = \begin{bmatrix} \left((M^{d}{}_{11}, \mathfrak{A}^{d}{}_{11}, N^{d}{}_{11})\right) & \left((M^{d}{}_{12}, \mathfrak{A}^{d}{}_{12}, N^{d}{}_{12})\right) & \dots & \left((M^{d}{}_{1\mathscr{n}}, \mathfrak{A}^{d}{}_{1\mathscr{n}}, N^{d}{}_{1\mathscr{n}})\right) \\ \left((M^{d}{}_{21}, \mathfrak{A}^{d}{}_{21}, N^{d}{}_{21})\right) & \left((M^{d}{}_{22}, \mathfrak{A}^{d}{}_{22}, N^{d}{}_{22})\right) & \dots & \left((M^{d}{}_{2\mathscr{n}}, \mathfrak{A}^{d}{}_{2\mathscr{n}}, N^{d}{}_{2\mathscr{n}})\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left((M^{d}{}_{\wp1}, \mathfrak{A}^{d}{}_{\wp1}, N^{d}{}_{\wp1})\right) & \left((M^{d}{}_{\wp2}, \mathfrak{A}^{d}{}_{\wp2}, N^{d}{}_{\wp2})\right) & \dots & \left((M^{d}{}_{\wp\mathscr{n}}, \mathfrak{A}^{d}{}_{\wp\mathscr{n}}, N^{d}{}_{\wp\mathscr{n}})\right) \end{bmatrix}$$

Step 2: Normalize the matrix obtained from Step 1 by using the formula

$$(\mathrm{M}_{\hat{\imath}\times\mathcal{J}}) = \begin{cases} \left(\left((M_{11}, \mathfrak{A}_{11}, N_{11})\right)\right)^{c}; \mathit{for\ cost\ type\ parameters} \\ \left((M_{11}, \mathfrak{A}_{11}, N_{11})\right); \mathit{for\ benefit\ type\ parameters} \end{cases}$$

Where $(M_{11}, \mathfrak{A}_{11}, N_{11})^{c} = \left((N_{11}, \mathfrak{A}_{11}, M_{11})\right)$ is the complement of $\left((M_{11}, \mathfrak{A}_{11}, N_{11})\right)$.

Step 3: Normalized PFS weighted matrix $\mathscr{b}(\mathrm{M}_{\hat{\imath}\times\mathcal{J}})$ is obtained by

$$\mathscr{b}(\mathrm{M}_{\hat{\imath}\times\mathcal{J}}) = \left(\left(\begin{matrix} \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1+M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1+M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(2-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(2-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}} \end{matrix}\right)\right)$$

Step 4: Border approximation areas (BAA) and for the BAA matrix $R = [r_{\mathcal{J}}]_{1\times\mathscr{n}}$ is obtained by

$$r_{\mathcal{J}} = \left(\left(\begin{matrix} \dfrac{2\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(2-M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(M_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}, \\ \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1+\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1+\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1-\mathfrak{A}_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}} \\ \dfrac{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} - \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}}{\prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1+N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}} + \prod_{\mathcal{J}=1}^{\wp}\left(\prod_{\hat{\imath}=1}^{\mathscr{n}}(1-N_{\hat{\imath}\mathcal{J}})^{\partial_{\hat{\imath}}}\right)^{\mathscr{b}_{\mathcal{J}}}} \end{matrix}\right)\right)$$

Step 5: Evaluate the distance $[di\mathfrak{s}_{\hat{\imath}\times\mathcal{J}}]_{\wp\times\mathscr{n}}$ between each alternative and BAA by the following equations

$$di\mathfrak{s}_{\hat{\imath}\times\mathcal{J}} = \begin{cases} d\hat{\imath}\mathfrak{s}\left(\mathscr{b}(\mathrm{M}_{\mathrm{i}\times\mathrm{j}}), r_{\mathcal{J}}\right) & \hat{\imath}f\ \mathscr{b}(\mathrm{M}_{\mathrm{i}\times\mathrm{j}}) > r_{\mathcal{J}} \\ 0 & \hat{\imath}f\ \mathscr{b}(\mathrm{M}_{\mathrm{i}\times\mathrm{j}}) = r_{\mathcal{J}} \\ -d\hat{\imath}\mathfrak{s}\left(\mathscr{b}(\mathrm{M}_{\mathrm{i}\times\mathrm{j}}), r_{\mathcal{J}}\right) & \hat{\imath}f\ \mathscr{b}(\mathrm{M}_{\mathrm{i}\times\mathrm{j}}) < r_{\mathcal{J}} \end{cases}$$

Where $d\hat{\imath}\mathfrak{s}(\mathscr{b}(\mathrm{M}_{\hat{\imath}\times\mathcal{J}}), r_{\mathcal{J}})$ is the Hamming mean distance by using definition 7.

Step 6: Sum the values of each $di\mathfrak{s}_{\hat{\imath}\mathcal{J}}$ by using the following equation

$$S_{\hat{\imath}} = \sum_{\mathcal{J}=1}^{\mathscr{n}} di\mathfrak{s}_{\hat{\imath}\mathcal{J}}.$$

Moreover, the flowchart of the proposed MABAC algorithm is given in Figure 4.

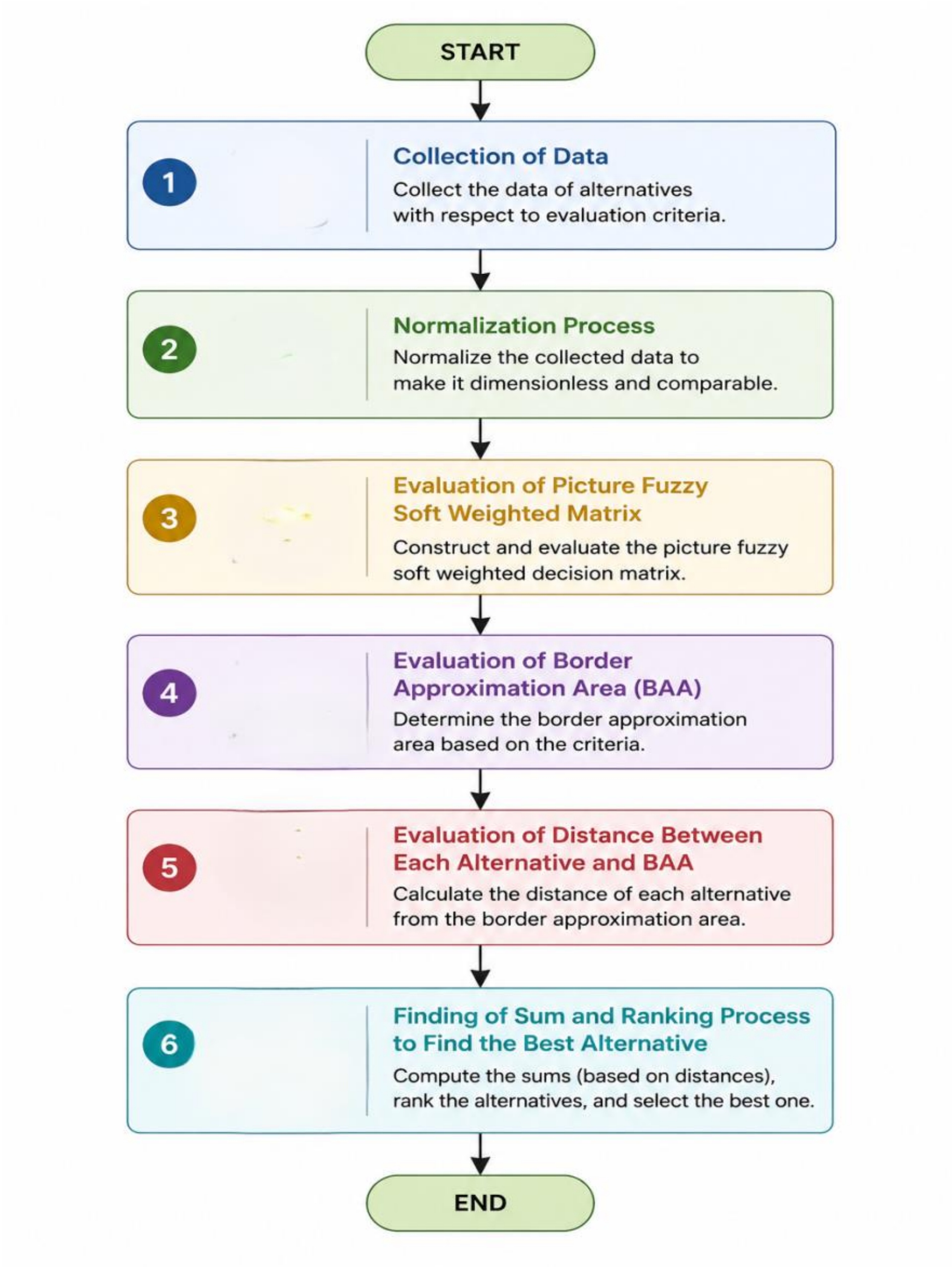

Figure 4. Flowchart of the proposed MABAC Algorithm

Now we propose an example to facilitate the above-given algorithm.

Example 2: A face recognition tool matches the faces in photographs and videos to existing identities in a database. This technology can help in detecting faces and can be considered useful for many departments. This technology matches the face, analyzes it and conforms to the identity. Facial expressions are more complex than any other biometric technologies, like fingerprint and iris scanners. Face detection technologies require highly sophisticated artificial intelligence algorithms. Assume that we have six different face recognition technologies, like $altr._1 =$ Face++, $altr._2$ =Deep Face, $altr._3$ =Amazon Rekognition, $altr._4 =$ Microsoft Azure Cognitive Services Face API, $altr._5$ =Kairos $altr._6 =$ Sense Time.

Assume that this technique has the following parameters $\mathfrak{s}_{\sigma}(\sigma = 1, 2, 3, 4)$ given by

$\mathfrak{s}_1 = \text{Precision and Accuracy}$

$\mathfrak{s}_2 = \text{Relevant facial recognition features}$

$\mathfrak{s}_3 = \text{Suitable architecture}$

$\mathfrak{s}_4 = \text{Flexible Software.}$

Assume that the WVs of these parameters are given by $(0.24, 0.36, 0.18, 0.22)$. Now we use the MABAC algorithm to select the best alternative from the set of given alternatives. The step-wise calculations are given by

Step 1: Collect the expert's evaluation for each alternative corresponding to each parameter using the PFS matrix provided in Table 5.

Table 5. PFS data

	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$
$\boldsymbol{altr._1}$	$(0.11, 0.22, 0.33)$	$(0.33, 0.24, 0.23)$	$(0.17, 0.14, 0.28)$	$(0.22, 0.17, 0.26)$
$\boldsymbol{altr._2}$	$(0.21, 0.12, 0.13)$	$(0.1, 0.32, 0.33)$	$(0.18, 0.22, 0.13)$	$(0.28, 0.32, 0.23)$
$\boldsymbol{altr._3}$	$(0.13, 0.12, 0.27)$	$(0.43, 0.13, 0.17)$	$(0.33, 0.13, 0.16)$	$(0.13, 0.17, 0.15)$
$\boldsymbol{altr._4}$	$(0.21, 0.27, 0.23)$	$(0.18, 0.17, 0.16)$	$(0.17, 0.18, 0.26)$	$(0.14, 0.15, 0.16)$
$\boldsymbol{altr._5}$	$(0.18, 0.16, 0.19)$	$(0.15, 0.12, 0.15)$	$(0.14, 0.22, 0.35)$	$(0.13, 0.12, 0.15)$
$\boldsymbol{altr._6}$	$(0.15, 0.14, 0.13)$	$(0.25, 0.24, 0.23)$	$(0.10, 0.14, 0.43)$	$(0.17, 0.14, 0.13)$

Step *2*: No need to normalize the matrix because all parameters are benefit types.

Step 3: The normalized PFS weighted matrix $\mathscr{b}(\mathrm{M}_{i\times\jmath})$ is given in Table 6.

Table 6. Normalized PFS weighted matrix

	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$
$\boldsymbol{altr._1}$	$(0.0264, 0.0528, 0.0792)$	$(0.0792, 0.0576, 0.0552)$	$(0.0648, 0.0336, 0.0672)$	$(0.0528, 0.0408, 0.0624)$
$\boldsymbol{altr._2}$	$(0.0504, 0.0288, 0.0312)$	$(0.024, 0.0768, 0.0792)$	$(0.0432, 0.0528, 0.0312)$	$(0.0672, 0.0768, 0.0552)$
$\boldsymbol{altr._3}$	$(0.0312, 0.0288, 0.0648)$	$(0.1032, 0.0312, 0.0408)$	$(0.0792, 0.0312, 0.0384)$	$(0.0312, 0.0408, 0.0360)$
$\boldsymbol{altr._4}$	$(0.0504, 0.0648, 0.0552)$	$(0.0432, 0.0408, 0.0384)$	$(0.0408, 0.0432, 0.0624)$	$(0.0336, 0.0360, 0.0384)$
$\boldsymbol{altr._5}$	$(0.0432, 0.0384, 0.0456)$	$(0.036, 0.0288, 0.036)$	$(0.0336, 0.0528, 0.084)$	$(0.0312, 0.0288, 0.0360)$
$\boldsymbol{altr._6}$	$(0.036, 0.0336, 0.0312)$	$(0.0600, 0.0576, 0.0552)$	$(0.0240, 0.0336, 0.1032)$	$(0.0408, 0.0336, 0.0312)$

Step 4: Find out border approximation areas (BAA) and for the BAA matrix $R = [r_{\jmath}]_{1\times n}$, we use the formula

$$r_{\jmath} = \left(\left(\begin{array}{c} \dfrac{2\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(M_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}}{\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(2-M_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}+\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(M_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}}, \\ \dfrac{\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(1+\mathfrak{A}_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}-\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(1-\mathfrak{A}_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}}{\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(1+\mathfrak{A}_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}+\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(1-\mathfrak{A}_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}} \\ \dfrac{\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(1+N_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}-\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(1-N_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}}{\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(1+N_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}+\prod_{\jmath=1}^{\wp}\left(\prod_{i=1}^{n}(1-N_{i\jmath})^{\partial_i}\right)^{\mathscr{b}_{\jmath}}} \end{array}\right)\right)$$

$$r_1 = (0.006943, 0.059303, 0.073682), r_2 = (0.010631, 0.070233, 0.073102)$$
$$r_3 = (0.008471, 0.059297, 0.092656), r_2 = (0.007612, 0.06161, 0.062175)$$

Step 5: Evaluate the distance $[di\mathfrak{s}_{i\times\jmath}]_{\wp\times n}$ between each alternative and BAA, measured by using Hamming distance given in definition 7 and the result is given in Table 7.

Table 7. Distance matrix

	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_4$
$\boldsymbol{altr._1}$	0.015739	0.049552	0.053741	0.033111
$\boldsymbol{altr._2}$	0.058221	0.013017	0.051341	0.040877
$\boldsymbol{altr._3}$	0.031821	0.081952	0.076541	0.035286
$\boldsymbol{altr._4}$	0.033718	0.048352	0.039341	0.037686
$\boldsymbol{altr._5}$	0.042681	0.051952	0.020141	0.041286
$\boldsymbol{altr._6}$	0.048621	0.039952	0.025885	0.046086

Step 6: Sum the values of each $dis_{\hat{\imath}\mathcal{J}}$ by using the following equation

$$S_{\hat{\imath}} = \sum_{\mathcal{J}=1}^{n} dis_{\hat{\imath}\mathcal{J}}$$

$$S_1 = 0.152144, S_2 = 0.163456, S_3 = 0.225601, S_4 = 0.159097, S_5 = 0.156001, S_6 = 0.1605044$$

We can see that $S_3 > S_2 > S_6 > S_4 > S_5 > S_1$. We can say that $altr._3 =$ Amazon Rekognition is the best technology in facial recognition.

Comparative Study

In this section of the paper, we will compare the developed work to various existing conceptions to demonstrate the application of these developed notions. Also, we will discuss how our work is more general than that of the existing notions. We compare our work with Wang and Liu's method [3], Rahman et al. method [9], Deveci et al. [11] method, Mishra et al. [35] method, Wei et al. [36] method, Zhao et al. [37] method, Jana et al. [38] method and Wang et al. [39] method. The overall result is given in Table 8. Here, we have used the results of the WASPAS and MABAC techniques given in examples 1 and 2 to compare our methods with the above existing techniques. The point-wise discussion is given below
Note that Wang and Liu [3] proposed the intuitionistic fuzzy information AOs using Einstein operations. It can be seen easily that these developed theories can rely on MG and NMG and there is no use of AG in their structure. That is the limitation of these notions, while the proposed Einstein AOs are based on PFS data that has the remarkable characteristic of considering the AG and provides more space in decision-making situations.

1) Also, Wu and Liu's methods [3] is non-parameterization structure and cannot discuss the parameterization tool that is the specialty of soft structures and their generalized forms, while we can see that the proposed AOs are based on PFS information and can discuss the parameterization tool as well. So, in both cases, we can note that the proposed approach is more reliable and more general.
2) Rahman et al. [9] method, although considering the more advanced criteria that $sum\ (MG^2, NMG^2) \in [0, 1]$, but in this situation, we have the same problems if the decision makers want to discuss the AG in its structure, then they can't discuss it under the PyF Einstein AOs proposed by Rahman et al. [9] method. Moreover, we can see that PyF Einstein AOs are also non-parameterized structures. While the proposed notion can cover all the above-mentioned problems. We can say that developed notions are more suitable.
3) Mishra et al. [35] provide the WASPAS method based on IF data, and we can see that the WASPAS technique based on the proposed AOs relies on PFS data, so the Mishra et al. [35] method cannot deal with PFS information and no result was found in this case.
4) Wei et al. [36] also established the extended WASPAS approach under PyF data, but the information given in Tables 1-4 consists of PFS data, so no result was found in this case.
5) Deveci et al. [11] proposed the WASPAS technique based on q-ROF Einstein AOs. A q-ROF structure has a weak point in that it cannot consider the AG in its structure, so the developed approach proposed by Deveci et al. [11] cannot consider the PFS data. No result was found in this case.
6) The MABAC methods given in [37], [38], and [39] consist of IF data, PyF data and q-ROF data, respectively, while the data given in Table 5 consist of PFS information, so no result is found in all of these cases. These methods are also free from parameterization tools.

Table 8. Comparative analysis results

Different Techniques	Score values	Ranking
Wang and Liu [3] technique	×	No outcome
Rahman et al. [9] Method	×	No outcome
Mishra et al. [35] technique	×	No outcome
Wei et al. [36] Method	×	No outcome
Deveci et al. [11] technique	×	No outcome
Zhao et al. [37] Method	×	No outcome
Jana et al. [38] technique	×	No outcome
Wang et al. [39] technique	×	No outcome
WASPAS method based on PFS data (Proposed)	$Sc(L_1) = -0.8513$, $Sc(L_2) = -0.8818$, $Sc(L_3) = -0.9102$ $Sc(L_4) = -0.8255$	$altr._4 > altr._1 > altr._2 > altr._3$
MABAC method based on PFS data (Proposed)	$S_1 = 0.152144$, $S_2 = 0.163456$, $S_3 = 0.225601$, $S_4 = 0.159097$, $S_5 = 0.156001$, $S_6 = 0.1605044$	$altr._3 > altr._2 > altr._6 > altr._4 > altr._5 > altr._1$

Also, the comparative study of the proposed work with some existing notions where the same kind of data can be handled effectively, but there exists a difference between the aggregation operators' theory.

A) Ahmmad and Mahmood [40] approach mainly focuses on developing prioritized average and geometric aggregation operators under the environment of PFSS. The study emphasizes handling uncertainty in medical diagnosis problems. It proposes new operational laws and demonstrates the effectiveness of the approach through a medical diagnosis algorithm. On the other hand, the established work concentrates on integrating advanced techniques such as WASPAS and MABAC with PFS Einstein aggregation operators. Unlike the first study, which mainly targets medical diagnosis, this research focuses on ranking and selecting alternatives in broader decision-making environments. The Einstein aggregation operators improve flexibility and accuracy in handling uncertain and inconsistent information during the evaluation process. A major difference between the two studies lies in their methodological framework. The Ahmmad and Mahmood approach develops prioritized aggregation operators where criteria are considered according to their priority levels, making it suitable for hierarchical medical decision systems. The proposed work integrates aggregation operators directly with WASPAS and MABAC to evaluate and rank alternatives systematically. Thus, the second approach provides stronger decision-ranking capabilities for engineering, management, and optimization problems.

B) The Dhumras and Bajaj approach [41] mainly targets agricultural robotics applications, whereas the second work has broader applicability in engineering and decision sciences. Moreover, the second approach emphasizes stronger ranking accuracy and hybrid decision-making analysis by combining two MCDM methods instead of relying only on the modified EDAS technique. Dombi aggregation operators are based on Dombi t-norm and t-conorm, which provide a highly flexible and parameterized way to model interaction among criteria in fuzzy environments. They allow smooth adjustment of aggregation behavior through a parameter that controls the degree of optimism or pessimism in decision-making. In contrast, Einstein aggregation operators are derived from Einstein's t-norm and t-conorm, offering a more structured and symmetric form of combination for fuzzy information. The proposed study is more flexible and stable as compared to the existing notion to handle the uncertain decision-making problems.

C) Razzaq and Riaz [42] proposed the theory of interactive weighted AOs by combining the soft-max mechanism with Einstein operations to handle complex decision interactions in uncertain environments. It mainly focuses on capturing interdependence among attributes using weighted and interactive structures. This property makes it suitable for dynamic decision problems. On the other hand, the proposed study integrates Einstein aggregation with WASPAS and MABAC methods for ranking and selection of alternatives in MCDM problems. The Razzaq and Riaz study emphasizes interaction modeling among criteria, while the established approach focuses more on structured ranking and comparison of alternatives using hybrid decision-making frameworks. The developed idea provides a more advanced approach as compared to the existing idea to cover the PFS data.

Moreover, the characteristic analysis of these methods is given in Table 9.

Table 9. Characteristic analysis of established work

Methods	Consider the fuzzy data	Consider the parameterization tool
Wang and Liu [3] Method	√	×
Rahman et al. [9] Method	√	×
Mishra et al. [35] Method	√	×
Wei et al. [36] Method	√	×
Deveci et al. [11] Method.	√	×
Zhao et al. [37] Method	√	×
Jana et al. [38] Method	√	×
Wang et al. [39] Method	√	×
WASPAS method based on picture fuzzy soft data (Proposed)	√	√
MABAC method based on picture fuzzy soft data (Proposed)	√	√

Conclusion

MABAC is a decision-making method used in multi-criteria decision analysis for evaluating and ranking alternatives based on multiple criteria. MABAC is designed to handle both quantitative and qualitative data and is particularly useful when criteria are interdependent. Also, WASPAS is a multi-criteria decision-making method used to evaluate and rank alternatives based on multiple criteria. This method is widely used in various fields such as engineering, economics and environmental management. So, based on the developed notion of PFSEWA and PFSEWG AOs, we have developed WASPAS and MABAC methods. Moreover, we have provided illustrative examples of the developed ideas to show the effective use of these notions in the field of seismology and facial recognition technologies based on artificial intelligence. Moreover, we have compared our work with some existing theories to show the superiority of the developed notions.

This work is limited, because if the decision maker comes up with T-spherical fuzzy data given in [43] then these developed notions cannot handle T-spherical fuzzy data. The reason is that T-spherical fuzzy data use more advanced conditions than PFS data.

We can extend these notions to bipolar fuzzy soft structure as given in [44]. Also, we can extend these notions to spherical fuzzy soft rough structures as given [45]. Some new developments can be made in this regard based on the interval-valued T-spherical fuzzy soft structure given in [46]. These notions can be extended to bipolar structures [47-50].

Author contribution statements

J.A writing original draft, T.M and K.H review and editing.

Conflicts of Interest

About the publication of this manuscript, the authors declare that they have no conflict of interest.

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