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FULL PAPER

Fuzzy autocatalytic set (FACS) as fuzzy vector space

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Graphical abstract

Abstract

Dynamic model of clinical waste incineration process in Malacca has been successfully developed using Fuzzy Autocatalytic Set (FACS) of Fuzzy Graph Type 3. FACS combines the concept of fuzzy graph and autocatalytic set and has been proven to yield more precise results than using crisp graph. Next, the graph of its process has been transformed from non-coordinated FACS to coordinated FACS in Euclidean space. Some mathematical structures related to the transformation are adjacency, transition and Laplacian matrices of FACS. Furthermore, it is found that FACS to be primitive, irreducible and aperiodic. The set of adjacency matrices of FACS of Fuzzy Graph Type 3 is predicted to be seen as fuzzy vector space, and the set of Euclidean distance matrices represent coordinated FACS is also expected to be fuzzy vector space as well. As a recently developed theory, FACS has great capabilities in generating new mathematical theories and structures that would enrich the theory itself. This study is made to attempt the formulation of new algebraic structure in FACS which is fuzzy vector space.

***Keywords*:** FACS, adjacency matrix, Euclidean distance matrix, fuzzy vector space

INTRODUCTION

Fuzzy Autocatalytic Set (FACS) is a concept defined by Tahir et. al. [1] as a merge between fuzzy graph [2, 3, 4] and autocatalytic set (ACS) [5]. It is successfully implemented in the dynamic model of a clinical waste incineration process in Malacca, Malaysia. The development started since Sabariah [6] used FACS of fuzzy graph Type-3 to refine the crisp model of clinical waste incineration process, which at that time using the adopted ACS model of Jain and Krishna [5]. Beforehand, Jain and Krishna [7] defined ACS as subgraph and use it to explain the dynamics procedures of their evolution model. Yet, FACS is introduced due to the fact that crisp model is inadequate to describe the real situation of the systems.

Moreover, the type of fuzzy graph used in FACS is the fuzzy graph Type 3 from the taxonomy defined by Blue et al. [2]. Fuzzy graph Type 3 proposed by Blue et al., with crisp vertices and crisp edges but unknown edge connectivity, has been chosen to model the connectedness of clinical waste incineration process. The reason is since both the vertices and edges were similar as those of crisp graph model, therefore the autocatalytic interactions between the vertices still valid. Since the fuzzy graph used for the development of FACS is the fuzzy graph of Type 3 therefore FACS is sometimes referred similarly as "*FACS of fuzzy graph Type 3*" for the purpose of accentuation of the types. The term "*FACS of incineration process*" will be use throughout this study as a specific real problem example of FACS.

In order to explain the dynamics of the real system, the mathematical structure particularly the algebraic structure of some matrices associated with graph of FACS are explored. Three different matrices have been used to capture the properties of graph and formally represent the FACS of fuzzy graph Type-3. They are adjacency matrix, transition matrix and directed Laplacian matrix of FACS. Furthermore, it is found that FACS to be primitive, irreducible and aperiodic [8].

In 2013, FACS of Incineration process has been successfully transformed from non-coordinated FACS to coordinated FACS in Euclidean space. Now, in this paper, the axioms of a vector space become the foundation to show that FACS can be constructed as a vector space. Since FACS can be represented as fuzzy matrices [8] then there are possibilities to construct FACS and coordinated FACS as fuzzy vector spaces.

FACS OF INCINERATION PROCESS IN MALACCA

Incineration is the most common method to treat the municipal waste. It is a waste treatment technology that uses combustion process to reduce the waste's volume by only disposing ash and gases including water as the final products. Instead of another waste management method like landfill, incineration becomes a popular method because of its ability to treat hazardous components of a waste, particularly from clinical or medical waste [6].

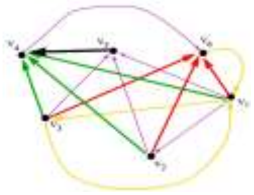

Fig. 1 Graph of FACS of Fuzzy Graph Type 3of Incineration Process

All the components involved in the combustion process are shown in Figure 1 as the set of vertices: v_1 represents waste, v_2 represents fuel, v_3 represents oxygen (O_2), v_4 represents carbon dioxide (CO_2), v_5 represents carbon monoxide (CO), and v_6 represents other gases including water (H_2O). The set of edges represent the connection between those vertices and the difference in colour and thickness represent the strength of connection by the different range of membership value [6].

The FACS model of the incineration process is a fuzzy graph of Type 3 where both vertices and edges are crisps but the edges have fuzzy connectivity [6]. This fuzzy connectivity determined the membership value for each edge and then allowed the edges to have ordered property which means the membership value represents the length of edges in FACS.

The following is the definition of FACS given by Tahir et al. [1] and the definition of the length of the edge of FACS of fuzzy graph Type 3.

Definition 2.1 [1] Fuzzy Autocatalytic Set (FACS) is a subgraph each of whose nodes has at least one incoming link with membership value $\mu(e_i) \in (0,1], \forall e_i \in E$.

Definition 2.2 [8] Let GFT3 (V, E) be a no-loop FACS of fuzzy graph of type-3 and $\theta_{ij} = \mu(e_{ij}) = \mu\left((v_i, v_j)\right) \in (0,1]$. A function $l: E \to R^+$ such that $l(\theta_{ij}) = \theta_{ij} \in R^+$ represents the length of the edge, (v_i, v_j).

In order to study more about the algebraic properties of FACS of Fuzzy Graph Type 3, Sumarni [8] considered several matrices and their relation with directed graph. These matrices are adjacency matrix, transition matrix, and Laplacian matrix. All these matrices have relation to describe the properties and behaviour of a directed graph and undirected graph. But we only consider the use of these matrices on directed graph since FACS itself is a directed fuzzy graph.

It has been shown that the adjacency matrix of FACS is a nonnegative, primitive, irreducible and aperiodic and directly it shows that FACS held the same properties [8]. However, in this paper we will define a new algebraic structure (i.e. fuzzy vector space for FACS) based from the adjacency matrix of FACS.

MATRIX REPRESENTATION OF FACS

The following matrix A shown in figure 2 is the adjacency matrix of FACS of incineration process. The entries are between zero and one since each entry represents the membership value of each edges in FACS.

$$A^{6\times 6} = \begin{bmatrix} 0 & 0 & 0.06529 & 0 & 0 & 0.13401 \\ 0.00001 & 0 & 0 & 0 & 0 & 0 \\ 0.15615 & 0 & 0 & 0 & 0 & 0 \\ 0.51632 & 0.68004 & 0.63563 & 0 & 0.99999 & 0 \\ 0.00001 & 0.00001 & 0.00002 & 0 & 0 & 0 \\ 0.32752 & 0.31995 & 0.29906 & 0.00001 & 0 & 0 \end{bmatrix}$$

Fig. 2 A 6 x 6 adjacency matrix of FACS of Incineration Process

The membership values obtained above are based on assumption of emitted gas from waste, approximation of total chemical composition during the combustion process and some calculation on the ratio of the combustion variables which have been verified by Sabariah [6].

The adjacency matrix of FACS possess the following algebraic structure: It is a square non-negative matrix, it is a non-symmetrical matrix, all entries $a_{ij} \in [0,1]$ signify the connectivity and denoted as membership values between two variables (vertices) therefore it is a fuzzy matrix, the matrix is irreducible and primitive [8].

FACS AS FUZZY VECTOR SPACE

Definition 4.1[9]**:** Fuzzy Vector Space

Let $V_{m\times n}$ denote set of all $m \times$ n fuzzy matrices over the fuzzy algebra F = [0, 1]. The operations $(+\,,\,\cdot)$ are defined on $V_{m\times n}$ as follows:

a. For any two elements $A = (a_{ij})$ and $B = (b_{ij}) \in V_{m\times n}$ define $A + B = \left(\sup\{a_{ij}, b_{ij}\}\right) = {}_{i,j}^{\vee}\,(a_{ij}, b_{ij})$ where each $a_{ij}, b_{ij} \in F$
b. For any element $A = (a_{ij}) \in V_{m\times n}$ and a scalar $k \in F$ define $kA = \left(\inf\{k, a_{ij}\}\right) = {}_{i,j}^{\wedge}\,(k, a_{ij})$

The system $V_{m\times n}$ together with these operations of component wise fuzzy addition and fuzzy multiplication is called fuzzy vector space over F and the scalars are restricted to F.

Corollary 1:

$V_{n\times n}$ is a fuzzy vector space

Proof: It is trivial for any arbitrary m and n, $V_{n\times n} \leq V_{m\times n}$ is a subspace. By definition of subspace, $V_{n\times n}$ inherit the operations in $V_{m\times n}$. ■

Next, by applying the concept of subspace and considering a special case of $n \times n$ matrices where all the diagonal entries are zero, the set of all adjacency matrices of FACS will be a subspace of $V_{n\times n}$ and it is defined as follows.

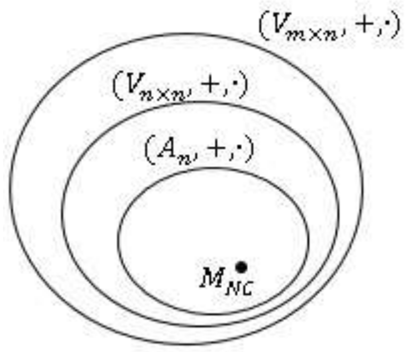

Fig. 3 A_n a subspace of $V_{n\times n}$

Definition 4.2:

$A_n = A_{n\times n}(F) = \left\{[m_{ij}]^{n\times n} : m_{ij} \in F \text{ and } m_{ii} = 0\right\}$ is the set of all $n \times n$ adjacency matrices of FACS over F = [0,1].

Theorem 1:

The set of all $n \times n$ adjacency matrices of FACS denoted by A_n is a fuzzy vector space.

Proof: Let A_n denote set of all $n \times n$ adjacency matrices of FACS over the fuzzy algebra F = [0,1], i.e. $A_n(F) = \left\{[m_{ij}]^{n\times n} : m_{ij} \in F \text{ and } m_{ii} = 0\right\}$. For any two elements $A = (a_{ij})$ and $B = (b_{ij}) \in A_n(F)$, where $A = [a_{ij}]^{n\times n} : a_{ij} \in F,\ a_{ii} = 0$ and $B = [b_{ij}]^{n\times n} : b_{ij} \in F,\ b_{ii} = 0$, then $A + B = [a_{ij}]^{n\times n} + [b_{ij}]^{n\times n}$. By definition 4.1, the entries are supremum of a_{ij} and b_{ij}, whereby the supremum of a_{ii} and b_{ii}, $\sup\{a_{ii}, b_{ii}\} = \sup\{0,0\} = 0$. Hence $A + B \in A_n$ and A_n is closed under addition.

To show that A_n is closed under scalar multiplication, take scalar $k \in F$ and $= [a_{ij}]^{n\times n} \in A_n$. $kA = k[a_{ij}]^{n\times n} = [ka_{ij}]^{n\times n}$. By definition 4.1, the entries are infimum of k and a_{ij}, whereby the infimum of k and a_{ii}, $\inf\{k, a_{ij}\} = \inf\{k, 0\} = 0$. Since $k \in F$. Hence, $kA \in A_n$ and A_n is closed under scalar multiplication. Therefore, A_n is a fuzzy vector space. ∎

Definition 4.3[9]**:** $(+\,,\ \cdot)$ on $\boldsymbol{V_{m\times n}}$

Define two operations $+$ and $\cdot$ on $V_{m\times n}$ as follows: For any two matrices A and B define:

$A + B = (\sup\{a_{ij}, b_{ij}\}) = \bigvee_{i,j} (a_{ij}, b_{ij})$ and

$AB = (\sup\{\inf\{a_{ik}, b_{kj}\}\}) = \bigvee_{k} \{\wedge(a_{ik}, b_{kj})\}$

Corollary 3: $(+\,,\ \cdot)$ on A_n

Define two operations $+$ and $\cdot$ on $V_{m\times n}$ as follows: For any two matrices A and B in fuzzy vector space A_n define :

$A + B = (\sup\{a_{ij}, b_{ij}\}) = \bigvee_{i,j} (a_{ij}, b_{ij})$ and

$AB = (\sup\{\inf\{a_{ik}, b_{kj}\}\}) = \bigvee_{k} \{\wedge(a_{ik}, b_{kj})\}$

Definition 4.4[9]: Inner Product, Fuzzy inner product space

Inner product between two elements $A = (a_{ij})$ and $B = (b_{ij}) \in V_{m\times n}$ is defined as $\langle A, B\rangle = \bigvee_{i,j} (a_{ij} \wedge b_{ij})$, satisfying the following conditions:

$\langle A, B\rangle = \langle B, A\rangle$
$\langle kA, B\rangle = k\langle A, B\rangle$, $k \in F$
$\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$, $C \in V_{m\times n}$
$\langle A, A\rangle = 0$ iff $A = 0$

$V_{m\times n}$ together with this inner product is called fuzzy inner product space.

Corollary 4:

A_n is a fuzzy inner product space.

Proof: Let A_n be a fuzzy vector space. Using definition 4.4, inner product between $A = (a_{ij})$ and $B = (b_{ij}) \in A_n$ is defined as $\langle A, B\rangle = \bigvee_{i,j} (a_{ij} \wedge b_{ij})$. We need to show it satisfy the following:

$\langle A, B\rangle = \langle B, A\rangle$
$\langle kA, B\rangle = k\langle A, B\rangle$, $k \in F$
$\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$, $C \in A_n$
$\langle A, A\rangle = 0$ iff $A = 0$

i. $\langle A, B\rangle = \bigvee_{i,j} (a_{ij} \wedge b_{ij})$
$= \bigvee_{i,j} (b_{ij} \wedge a_{ij})$
$= \langle B, A\rangle$

Take the infimum of a_{ij} and b_{ij} it is the same as infimum of b_{ij} and a_{ij}. The infimum of a_{ii} and b_{ii}, $\inf\{a_{ii}, b_{ii}\} = \inf\{b_{ii}, a_{ii}\} = 0$ by definition of A_n. Therefore, $\langle A, B\rangle = \langle B, A\rangle$

ii. $\langle kA, B\rangle = \bigvee_{i,j} (ka_{ij} \wedge b_{ij})$
$= \bigvee_{i,j} (k \wedge a_{ij} \wedge b_{ij})$, by definition of scalar multiplication
$= k \wedge \left(\bigvee_{i,j} (a_{ij} \wedge b_{ij})\right)$, since k is scalar
$= \inf\left\{k, \bigvee_{i,j} (a_{ij} \wedge b_{ij})\right\}$
$= \inf\{k, \langle A, B\rangle\}$
$= k\langle A, B\rangle$
$\therefore \langle kA, B\rangle = k\langle A, B\rangle$

iii. Let $C \in A_n$. Then,
$\langle A + B, C\rangle = \bigvee_{i,j} \left((a_{ij} \vee b_{ij}) \wedge c_{ij}\right)$, by definition of addition in A_n
$= \bigvee_{i,j} \left((a_{ij} \wedge c_{ij}) \vee (b_{ij} \wedge c_{ij})\right)$
$= \left(\bigvee_{i,j} (a_{ij} \wedge c_{ij})\right) \vee \left(\bigvee_{i,j} (b_{ij} \wedge c_{ij})\right)$
$= \langle A, C\rangle \vee \langle B, C\rangle$
$= \sup\{\langle A, C\rangle, \langle B, C\rangle\}$
$= \langle A, C\rangle + \langle B, C\rangle$
$\therefore \langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$

iv. $(\Rightarrow)$ Let $\langle A, A\rangle = 0$. By definition of inner product then $\langle A, A\rangle = \bigvee_{i,j} (a_{ij} \wedge a_{ij}) = \bigvee_{i,j} (a_{ij}) = 0$. The supremum of $a_{ij} = 0$ show that the largest entries of A is zero. Hence, $A = 0$.
$(\Leftarrow)$ Let $A = 0$, the inner product $\langle A, A\rangle = \bigvee_{i,j} (a_{ij} \wedge a_{ij}) = \bigvee_{i,j} (a_{ij}) = \bigvee_{i,j} (0) = 0\ a_{ij} = 0$ because $A = 0$ by assumption.
$\therefore \langle A, A\rangle = 0$ iff $A = 0$

Therefore, A_n is a fuzzy inner product space since the above conditions are satisfied by A_n. ∎

Definition 4.5[9]**:** Norm, Fuzzy normed linear space

The norm for every element $A \in V_{m\times n}$ is defined as $\|A\| = \langle A, A\rangle$ satisfying the following conditions:

$1 \geq \|A\| \geq 0$, $\|A\| = 0$ iff $A = 0$
$\|kA\| = k\|A\|$, $k \in F$
$\|A + B\| \leq \|A\| + \|B\|$

$V_{m\times n}$ together with this norm is called as fuzzy normed linear space.

Corollary 4:

A_n is a fuzzy normed linear space.

Proof: Let A_n be a fuzzy vector space over F = [0,1]. By definition 4.5, The norm of every element $A = (a_{ij}) \in A_n$ is defined as $\|A\| = \langle A, A\rangle = \bigvee_{i,j} (a_{ij})$.

i. $1 \geq \|A\| = \langle A, A\rangle = \bigvee_{i,j} (a_{ij}) \geq 0$ since $a_{ij} \in [0,1]$ and $\|A\| = \langle A, A\rangle = 0$ iff $A = 0$ since A_n is fuzzy inner product space. Proof by corollary 4.

ii. $\|kA\| = \langle kA, kA\rangle = \bigvee_{i,j} (ka_{ij}) = \bigvee_{i,j} (k \wedge a_{ij}) = k \wedge \left(\bigvee_{i,j} (a_{ij})\right) = k \wedge \|A\| = k\|A\|$

iii. $\|A + B\| = \langle A + B, A + B\rangle = \bigvee_{i,j} (a_{ij} \vee b_{ij})$
by definition of inner product and addition in A_n. By triangle inequality $\bigvee_{i,j} (a_{ij} \vee b_{ij}) \leq \left(\bigvee_{i,j} (a_{ij})\right) \vee \left(\bigvee_{i,j} (b_{ij})\right) \leq \|A\| \vee \|B\| \leq \|A\| + \|B\|$

Therefore, A_n is fuzzy normed linear space. ∎

COORDINATED FACS AS FUZZY VECTOR SPACE

The following matrix E is the Euclidean distance matrix of Coordinated Graph FACS of clinical waste incineration process. The entries are taken from the Euclidean distance of the edges.

$$E^{6\times 6} = \begin{bmatrix} 0 & 2.7616\ x\ 10^{5} & 20.0 & 10.0 & 2.1043\ x\ 10^{5} & 2.8127x\ 10^{-7} \\ 2.7616\ x\ 10^{5} & 0 & 2.7614\ x\ 10^{5} & 2.7617x\ 10^{5} & 0 & 1.9506\ x\ 10^{5} \\ 20.0 & 2.7614\ x\ 10^{5} & 0 & 30.0 & 0 & 20.0 \\ 10.0 & 2.7617x\ 10^{5} & 30.0 & 0 & 0.68004 & 10.0 \\ 2.1043\ x\ 10^{5} & 0.00001 & 0.00002 & 0 & 0 & 0 \\ 2.8127x10^{-7} & 1.9506\ x\ 10^{5} & 20.0 & 10.0 & 0.31995 & 0 \end{bmatrix}$$

The entries need to be fuzzified since we want to construct a fuzzy matrix. By the concept of ratio, we divide each entry by the difference between its maximum and minimum entries. The new matrix is as follows:

$$E' = \begin{bmatrix} 0 & 0.99996379 & 0.0000724 & 0.0000362096 & 0.761958214 & 1.01847\ x\ 10^{-12} \\ 0.99996379 & 0 & 0.999891371 & 1 & 0.238005576 & 0.706304088 \\ 0.0000724 & 0.999891371 & 0 & 0.000108629 & 0.76188795 & 0.0000724192 \\ 0.0000362096 & 1 & 0.000108629 & 0 & 0.761994424 & 0.0000362096 \\ 0.761958214 & 0.238005576 & 0.76188795 & 0.761994424 & 0 & 0.761958214 \\ 1.01847\ x\ 10^{-12} & 0.706304088 & 0.0000724192 & 0.0000362096 & 0.761958214 & 0 \end{bmatrix}$$

Next, we define the following definitions

Definition 5.1:
$E_n = \left\{(e_{ij})^{n\times n}, e_{ij} \in R^+\right\}$ is the set of all $n \times n$ Euclidean distance matrices of FACS.

Definition 5.2:
The set of all fuzzified $n \times n$ Euclidean distance matrices of FACS is denoted by
$E_n{}' = \left\{(e_{ij})^{n\times n}, e_{ij} \in [0,1]\right\}$

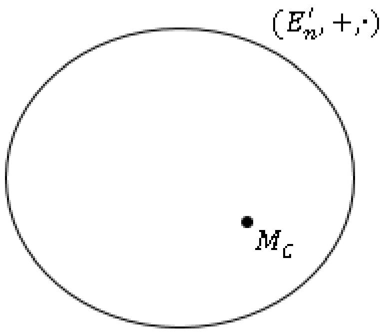

Fig. 5 E'_n is a fuzzy vector space

Theorem 1:
The set of all $n \times n$ Euclidean distance matrices of FACS denoted by $E_n{}'$ is a fuzzy vector space.

Proof: Let $E_n{}'$ denote set of all $n \times n$ adjacency matrices of FACS over the fuzzy algebra F = [0,1], i.e. $E_n{}' = \left\{(e_{ij})^{n\times n}, e_{ij} \in [0,1]\right\}$. For any two elements $A = (a_{ij})$ and $B = (b_{ij}) \in E_n{}'$ By definition 4.1, the entries are supremum of a_{ij} and b_{ij}. Hence $A + B \in E_n{}'$ and $E_n{}'$ is closed under addition.

To show that A_n is closed under scalar multiplication, take scalar $k \in F$ and $A = [a_{ij}]^{n\times n} \in E_n{}'$. $kA = k[a_{ij}]^{n\times n} = [ka_{ij}]^{n\times n}$. By definition 4.1, the entries are infimum of k and a_{ij}. Since $k \in F$. Hence, $kA \in E_n{}'$ and $E_n{}'$ is closed under scalar multiplication. Therefore, A_n is a fuzzy vector space. ■

CONCLUSION

A new algebraic structure is defined which is Fuzzy Autocatalytic Set (FACS) as fuzzy vector space. This work will be carried on to define the first isomorphism theorem for FACS

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