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RESEARCH ARTICLE

Bipolar Pythagorean Neutrosophic Generalized Weighted Hamacher Heronian Mean (BPN-GWHHM) for MCDM in Business Resilience

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Abstract This study proposes the Bipolar Pythagorean Neutrosophic-Generalized Weighted Hamacher Heronian Mean (BPN-GWHHM) operator to improve decision-making in multi-criteria decision-making (MCDM) effectively handling uncertainty, indeterminacy, and bipolar information. The method enhances the aggregation process by capturing the complex interrelationships between positive and negative elements while maintaining sensitivity to data fluctuations through generalized weighting. The proposed operator is implemented to financial performance evaluation among firms, demonstrating robustness and practical reliability. Sensitivity analysis is utilized to confirm its stability under varying parameter values.

Keywords: Bipolar Pythagorean Neutrosophic Set, Hamacher Operator, Heronian Mean, MCDM, Financial Performance Evaluation.

Introduction

Multi-Criteria Decision-Making (MCDM) is crucial for assessing complex financial performance, as it involves the simultaneous consideration of multiple factors such as profitability, liquidity, and risk. It offers a structured framework to analyze and prioritize these criteria, supporting more informed and objective decision-making. In today's dynamic and highly competitive business environment, financial stability and performance are vital for the growth and long-term sustainability of enterprises aiming to generate revenue and profit. It refers to the company's ability to manage financial obligations [1] and is often measured by analysing debt levels [2], liquidity [3], and financial risk management [4]. Stable firms can attract investors; however, it requires lower financial risk, which is measured by debt level compared to the equity. The stability has also been influenced by interest coverage ratio, liquidity ratios (current ratio and quick ratio), and cash flow management. Some important keys in financial performance should be focused on as they reflect the overall profitability [5], efficiency, and return on investment. Financial performance evaluation has been applied by various aggregation operators. Jana *et al*. [6] applied a Multiple Attribute Decision-Making (MADM) approach using intuitionistic fuzzy Dombi hybrid operators to evaluate enterprise financial performance. Besides, Mohamed & Elsayed [7] integrated Bipolar Neutrosophic Sets (BNS) with MCDM methods to rank companies on the Egyptian Exchange (EGX), demonstrating a robust and adaptable financial performance evaluation framework for investment decision-making. Fuzzy and neutrosophic environment gain researcher's attention in solving complex decision-making.

In real-world applications, decision-making often involves multiple conflicting criteria which need structured approaches to balance various factors [8]. Objectives, alternatives, and consequences

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frequently impact decision-making plans, which are made more difficult by human judgment's inaccuracy and ambiguity. These difficulties can lead to biases, misinterpretations, inconsistent results, and other problems in the decision-making process [9]. The Bipolar Pythagorean Neutrosophic Set (BPNS) theory addresses such limitations by combining bipolarity, neutrosophic logic, and Pythagorean fuzzy sets. It provides an effective framework for assessing positive and negative membership degrees, as well as indeterminacy and non-membership degrees in a variety of criteria. As it involves managing contradicting data, BPNS also successfully overcomes the limitations of both traditional fuzzy sets and intuitionistic fuzzy sets [10]. BPNS and its expansions, including interval neutrosophic set [11], interval-valued neutrosophic set [12], single-valued neutrosophic set [13], and interval-valued Pythagorean neutrosophic set (IVPNS) [14], provide sophisticated tools for handling ambiguity [14]. These are derived from neutrosophic set theory [15]. These frameworks have been applied in diverse fields, including medical data analysis [16], big data analysis [17], finance and management science [18], and supply chain management [19], effectively addressing the limitations of traditional set theories.

Aggregation operators (AOs) are crucial tools in decision-making [20], used to combine several criteria or preferences into a single result. Numerous AOs provide different approaches to deal with the intricacies and uncertainties [21], including the Maclaurin symmetric mean [22], ordered weighted average [23], Choquet integral [24], Dombi [25], Hamacher [26], Einstein [27], and Frank [28] operators. The Hamacher operator provides a versatile extension to standard algebraic operators and was developed by Wolfgang Hamacher in the 1970s. Compared to conventional algebraic and Einstein operators, these operators are better able to handle intricate interactions and uncertainty [29]. Within the conceptual framework of Archimedean t-norms and t-conorms, Hamacher operator offer a more generalized t-norm and t-conorm. The Heronian mean (HM) is useful in decision-making since it is remarkable for capturing the relationships between input values [30]. By providing numerous criteria, the Generalized Heronian Mean (GHM) expands on HM and allows decision-makers more flexibility in handling complicated relationships. The Generalized Weighted Heronian Mean (GWHM), which considers weighting factors to indicate the relative relevance of criteria, was developed as HM and GHM do not take input weights into consideration [31]. In MCDM applications, GWHM improves the Heronian mean-based model's applicability and flexibility [32][33]. These fundamental aggregation operations are combined with fuzzy and neutrosophic frameworks to improve suitability for MCDM.

By combining fuzzy and neutrosophic set frameworks with aggregation operators, some academics have improved decision-making techniques to deal with complexity and uncertainty. Jamil *et al*. [34, 35] developed Hamacher operations based on BNS, like BN Hamacher weighted geometric (BNHWG), BN Hamacher ordered weighted geometric (BNHOWG), and BN Hamacher hybrid geometric (BNHHG). However, this aggregation operator has limitations in computational complexity, as real-world decision-making requires large-scale datasets. Mei *et al*. [36] proposed an enhanced single-valued neutrosophic Hamacher weighted averaging (ISNHWA) operator and grey relational analysis, but the effectiveness of proposed operator appears restricted to certain types of issues in decision-making problems. Nithya Sri *et al*. [37] constructed Einstein aggregation operators under the Bipolar Linear Diophantine Fuzzy Hypersoft Set framework for social data. Abdul Halim *et al*. [38] introduced the rough neutrosophic Shapley weighted Einstein averaging aggregation operator, which combines Shapley fuzzy measures and Einstein operators. Conversely, both operators may struggle in managing highly uncertain data as Einstein aggregation operator tends to be highly sensitive to extreme values.

Moreover, Thilagayathy & Mohanaselvi [39] defined T-spherical fuzzy Hamacher Heronian operators, including weighted geometric, ordered geometric, and hybrid geometric forms. Nevertheless, the operator's capacity to completely capture uncertainty representations is constrained. Yaacob *et al*. [40] formulated Dombi-based Heronian mean aggregation operators within the BNS to address the limitations of the existing Dombi operator in the context of BNS. Even so, Dombi operations under BNS are prone to being sensitive towards extreme values, which can undermine the stability of uncertainty degrees. Asif *et al*. [41] and Paul *et al*. [42] both utilized the idea of Pythagorean fuzzy Hamacher aggregation operators for MCDM studies. Nevertheless, the hesitancy and indeterminacy inherent in complex decision-making circumstances are not entirely captured by Pythagorean fuzzy sets. Many current operators fail to entirely capture hesitancy, indeterminacy, and conflicting information, which are critical in evaluating financial performance. On the other hand, some operators, notably those based on Einstein and Dombi operators, tend to be overly sensitive to outlier data and lead to instability in the results. To address these limitations, this study proposes the Bipolar Pythagorean Neutrosophic Generalized Weighted Heronian Mean (BPN-GWHHM) operator, which combines the strengths of Heronian and Hamacher operators as a flexible solution for MCDM applications under the BPNS model.

In this study, two aggregation operators, the Heronian mean and the Hamacher operators, are utilized for integrating BPNS to evaluate financial performance. The following outlines the structure of this research study: The basic ideas of BPNS, Hamacher, and Heronian mean operators are introduced in

Section 2. In Section 3, the BPN-GWHHM aggregation operator is proposed. Section 4 demonstrates a numerical example and sensitivity analysis to evaluate the efficiency of the proposed model. The final section provides the overall conclusion and suggestions for further research study. Multi-criteria decision-making's strength lies in its adaptability to practical circumstances. Such trade-offs frequently occur in a variety of areas, including supplier selection in supply chain management [43], project prioritization in resource allocation [44], and environmental impact evaluation in sustainability assessments [45]. For example, supply chain management requires weighting factors such as cost, quality, and reliability in order to choose the best supplier, and resource allocation entails balancing project feasibility, potential returns, and resource constraints. The intricacy of these issues highlights the need for strong and adaptable methodologies. MCDM approaches provide structured frameworks that assist decision-makers in addressing complex problems, supporting strategic planning, and aligning solutions with organizational or societal goals.

Preliminaries

This section provides a comprehensive overview of the fundamental concepts and definitions related to BPNS, Hamacher operators, and Heronian mean operators.

Bipolar Pythagorean Neutrosophic Set

The following definitions of bipolar neutrosophic set are based on [46].

Definition 1. Let z be a nonvoid set. A BPNS with components D in Z is defined as:

$$D = \{\langle z, \alpha_D^+(z), l_D^+(z), \beta_D^+(z), \alpha_D^-(z), l_D^-(z), \beta_D^-(z) \rangle : z \in Z\} \tag{1}$$

where $\alpha_D^+(z), l_D^+(z), \beta_D^+(z): Z \to [0,1]$ and $\alpha_D^-(z), l_D^-(z), \beta_D^-(z): Z \to [-1,0]$. $\alpha_D^+(z), l_D^+(z), \beta_D^+(z)$ represents positive membership degree, while $\alpha_D^-(z), l_D^-(z), \beta_D^-(z)$ represents negative membership degree of an element $z \in Z$.

Heronian Mean

The generalized Heronian mean (GHM) aggregation operator presented as follows:

Definition 2. Generalized weighted Heronian mean (GWHM) [47].

Let $w_i = (w_1, w_2, \dots, w_n)^T$ be the weight vector of a collection of non-negative numbers $x_i (i = 1,2, \dots, n)$, and satisfying $p, q \geq 0$, $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$. Then,

$$GWHM^{p,q}(x_1, x_2, \dots, x_n) = \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(w_i x_i)^p \otimes (w_j x_j)^q\right)\right]^{\frac{1}{p+q}} \tag{2}$$

Hamacher Operations under BPNSs

Definition 3. Hamacher t-norm and t-conorm [26].

The Hamacher product, $\otimes$, is a t-norm and the Hamacher sum, $\oplus$, is a t-conorm, are two real numbers a_i and a_j are defined as follows:

$$T(a_i, a_j) = a_i \otimes a_j = \frac{a_i a_j}{\varsigma + (1-\varsigma)(a_i + a_j - a_i a_j)}, \varsigma > 0 \tag{3}$$

$$T^*(a_i, a_j) = a_i \oplus a_j = \frac{a_i + a_j - a_i a_j - (1-\varsigma) a_i a_j}{1 - (1-\varsigma) a_i a_j}, \varsigma > 0 \tag{4}$$

Definition 4. Hamacher operations of bipolar Pythagorean neutrosophic sets.

Let $A_1 = \langle t_1^+, i_1^+, f_1^+, t_1^-, i_1^-, f_1^- \rangle$ and $A_2 = \langle t_2^+, i_2^+, f_2^+, t_2^-, i_2^-, f_2^- \rangle$, be any two BPNSs and $\delta > 0$ be any real number, then we define basic Hamacher operators with $\tau > 0$.

$$A_1 \oplus_H A_2 = \left\langle \begin{array}{l} \sqrt{\frac{(t_1^+)^2+(t_2^+)^2-(t_1^+)^2(t_2^+)^2-(1-\tau)(t_1^+)^2(t_2^+)^2}{1-(1-\tau)(t_1^+)^2(t_2^+)^2}}, \frac{i_1^+ i_2^+}{\tau+(1-\tau)(i_1^+ + i_2^+ - i_1^+ i_2^+)}, \frac{f_1^+ f_2^+}{\tau+(1-\tau)(f_1^+ + f_2^+ - f_1^+ f_2^+)}, \\ \frac{-t_1^- t_2^-}{\tau+(1-\tau)(t_1^- + t_2^- - t_1^- t_2^-)}, \frac{-(-i_1^- - i_2^- - i_1^- i_2^- - (1-\tau) i_1^- i_2^-)}{1-(1-\tau) i_1^- i_2^-}, -\sqrt{\frac{(f_1^-)^2+(f_2^-)^2-(f_1^-)^2(f_2^-)^2-(1-\tau)(f_1^-)^2(f_2^-)^2}{1-(1-\tau)(f_1^-)^2(f_2^-)^2}} \end{array} \right\rangle \tag{5}$$

$$A_1 \otimes_H A_2 = \left\langle \begin{array}{l} \frac{t_1^+ t_2^+}{\tau+(1-\tau)(t_1^+ + t_2^+ - t_1^+ t_2^+)}, \frac{i_1^+ + i_2^+ - i_1^+ i_2^+ - (1-\tau) i_1^+ i_2^+}{1-(1-\tau) i_1^+ i_2^+}, \sqrt{\frac{(f_1^+)^2+(f_2^+)^2-(f_1^+)^2(f_2^+)^2-(1-\tau)(f_1^+)^2(f_2^+)^2}{1-(1-\tau)(f_1^+)^2(f_2^+)^2}}, \\ -\sqrt{\frac{(t_1^-)^2+(t_2^-)^2-(t_1^-)^2(t_2^-)^2-(1-\tau)(t_1^-)^2(t_2^-)^2}{1-(1-\tau)(t_1^-)^2(t_2^-)^2}}, \frac{i_1^- i_2^-}{\tau+(1-\tau)(i_1^- + i_2^- - i_1^- i_2^-)}, \frac{-(f_1^- f_2^-)}{\tau+(1-\tau)(f_1^- + f_2^- - f_1^- f_2^-)} \end{array} \right\rangle \tag{6}$$

$$\delta \cdot A_1 = \left\langle \begin{matrix} \sqrt{\frac{(1+(\tau-1)(t_1^+)^2)^\delta-(1-(t_1^+)^2)^\delta}{(1+(\tau-1)(t_1^+)^2)^\delta+(\tau-1)(1-(t_1^+)^2)^\delta}}, \frac{\tau(i_1^+)^\delta}{(1+(\tau-1)(1-i_1^+))^\delta+(\tau-1)(i_1^+)^\delta}, \frac{\tau(f_1^+)^\delta}{(1+(\tau-1)(1-f_1^+))^\delta+(\tau-1)(f_1^+)^\delta}, \\ \frac{-\tau|t_1^-|^\delta}{(1+(\tau-1)(1+t_1^-))^\delta+(\tau-1)|t_1^-|^\delta}, \frac{-\tau|i_1^-|^\delta}{(1+(\tau-1)(1+i_1^-))^\delta+(\tau-1)|i_1^-|^\delta}, -\sqrt{\frac{(1+(\tau-1)(f_1^-)^2)^\delta-(1+(f_1^-)^2)^\delta}{(1+(\tau-1)(f_1^-)^2)^\delta-(\tau-1)(1+(f_1^-)^2)^\delta}} \end{matrix} \right\rangle \tag{7}$$

$${A_1}^\delta = \left\langle \begin{matrix} \frac{(t_1^+)^\delta}{(1+(\tau-1)(1-t_1^+))^\delta+(\tau-1)(t_1^+)^\delta}, \frac{(1+(\tau-1)i_1^+)^\delta-(1-i_1^+)^\delta}{(1+(\tau-1)i_1^+)^\delta+(\tau-1)(1-i_1^+)^\delta}, \sqrt{\frac{(1+(\tau-1)(-f_1^+)^2)^\delta-(1-(-f_1^+)^2)^\delta}{(1+(\tau-1)(-f_1^+)^2)^\delta+(\tau-1)(1-(-f_1^+)^2)^\delta}}, \\ -\sqrt{\frac{(1+(\tau-1)(t_1^-)^2)^\delta-(1-(t_1^-)^2)^\delta}{(1+(\tau-1)(t_1^-)^2)^\delta+(\tau-1)(1-(t_1^-)^2)^\delta}}, \frac{-\tau|i_1^-|^\delta}{(1+(\tau-1)(1+i_1^-))^\delta+(\tau-1)|i_1^-|^\delta}, \frac{-\tau|f_1^-|^\delta}{(1+(\tau-1)(1+f_1^-))^\delta+(\tau-1)|f_1^-|^\delta} \end{matrix} \right\rangle \tag{8}$$

Proposed Bipolar Pythagorean Neutrosophic Generalized Weighted Hamacher Heronian Mean (BPN-GWHHM)

This section outlines the proposed BPN-GWHHM operator. Additionally, the associated properties and proofs for the aggregation approach are presented.

Definition 5. Let $p, q \geq 0$ and $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2, \dots, n)$ be a collection of BPNSs. The BPN-GWHHM operator defined as follows:

$$BPN - GWHHM^{p,q}(A_1, A_2, \dots, A_n) = \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(w_iA_i)^p \otimes (w_jA_j)^q\right)\right]^{\frac{1}{p+q}} \tag{9}$$

Theorem 1. *Let $p, q \geq 0$ and $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2, \dots, n)$ be a collection of BPNSs. Based on the Hamacher operations in Definition 4, the aggregated value of the BPN-GWHHM operator under BPNS information, as defined in Definition 5, is given as follows:*

$$BPN - GWHHM^{p,q}(A_1, A_2, \dots, A_n) = \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(w_iA_i)^p \otimes (w_jA_j)^q\right)\right]^{\frac{1}{p+q}} =$$

$$\left(\begin{matrix} \frac{(\sum_{i=1}^n\sum_{j=1}^n\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\sum_{i=1}^n\sum_{j=1}^n\left(1-(t_i^+)^2\right)^{pw_i}\left(1-\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^n\sum_{j=1}^n\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^n\sum_{j=1}^n\left(1-(t_i^+)^2\right)^{pw_i}\left(1-\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}, \\ \frac{\tau(\sum_{i=1}^n\sum_{j=1}^n(i_i^+)^{pw_i}\left(i_j^+\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^n\sum_{j=1}^n\left(1+(\tau-1)(1-i_i^+)\right)^{pw_i}\left(1+(\tau-1)\left(1-i_j^+\right)\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^n\sum_{j=1}^n(i_i^+)^{pw_i}(i_j^+)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}, \\ \sqrt{\frac{\tau(\sum_{i=1}^n\sum_{j=1}^n(f_i^+)^{pw_i}(f_j^+)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^n\sum_{j=1}^n(1+(\tau-1)(1-f_i^+))^{pw_i}(1+(\tau-1)(1-f_j^+))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^n\sum_{j=1}^n(f_i^+)^{pw_i}(f_j^+)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}}, \\ -\sqrt{\frac{\tau(\sum_{i=1}^n\sum_{j=1}^n|t_i^-|^{pw_i}|t_j^-|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^n\sum_{j=1}^n(1+(\tau-1)(1+t_i^-))^{pw_i}(1+(\tau-1)(1+t_j^-))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^n\sum_{j=1}^n|t_i^-|^{pw_i}|t_j^-|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}}, \\ -\frac{\tau(\sum_{i=1}^n\sum_{j=1}^n|i_i^-|^{pw_i}|i_j^-|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^n\sum_{j=1}^n(1+(\tau-1)(1+i_i^-))^{pw_i}(1+(\tau-1)(1+i_j^-))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^n\sum_{j=1}^n|i_i^-|^{pw_i}|i_j^-|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}, \\ \frac{(\sum_{i=1}^n\sum_{j=1}^n(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\sum_{i=1}^n\sum_{j=1}^n(1+\left(f_i^-\right)^2)^{pw_i}(1+\left(f_j^-\right)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^n\sum_{j=1}^n(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\tau-1)(\sum_{i=1}^n\sum_{j=1}^n(1+\left(f_i^-\right)^2)^{pw_i}(1+\left(f_j^-\right)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}} \end{matrix} \right) \tag{10}$$

Proof Using Eq. (7),

$$w_iA_i = \left(\begin{matrix} \sqrt{\frac{(1+(\tau-1)(t_i^+)^2)^{w_i}-(1-(t_i^+)^2)^{w_i}}{(1+(\tau-1)(t_i^+)^2)^{w_i}+(\tau-1)(1-(t_i^+)^2)^{w_i}}}, \frac{\tau(i_i^+)^{w_i}}{(1+(\tau-1)(1-i_i^+))^{w_i}+(\tau-1)(i_i^+)^{w_i}}, \frac{\tau(f_i^+)^{w_i}}{(1+(\tau-1)(1-f_i^+))^{w_i}+(\tau-1)(f_i^+)^{w_i}}, \\ \frac{-\tau|t_i^-|^{w_i}}{(1+(\tau-1)(1+t_i^-))^{w_i}+(\tau-1)|t_i^-|^{w_i}}, \frac{-\tau|i_i^-|^{w_i}}{(1+(\tau-1)(1+i_i^-))^{w_i}+(\tau-1)|i_i^-|^{w_i}}, -\sqrt{\frac{(1+(\tau-1)(f_i^-)^2)^{w_i}-(1+(f_i^-)^2)^{w_i}}{(1+(\tau-1)(f_i^-)^2)^{w_i}-(\tau-1)(1+(f_i^-)^2)^{w_i}}} \end{matrix}\right)$$

$$w_jA_j = \left(\begin{matrix} \sqrt{\frac{(1+(\tau-1)(t_j^+)^2)^{w_j}-(1-(t_j^+)^2)^{w_j}}{(1+(\tau-1)(t_j^+)^2)^{w_i}+(\tau-1)(1-(t_j^+)^2)^{w_i}}}, \frac{\tau(i_j^+)^{w_j}}{(1+(\tau-1)(1-i_j^+))^{w_j}+(\tau-1)(i_j^+)^{w_j}}, \frac{\tau(f_j^+)^{w_j}}{(1+(\tau-1)(1-f_j^+))^{w_j}+(\tau-1)(f_j^+)^{w_j}}, \\ \frac{-\tau|t_j^-|^{w_j}}{(1+(\tau-1)(1+t_j^-))^{w_j}+(\tau-1)|t_j^-|^{w_j}}, \frac{-\tau|i_j^-|^{w_j}}{(1+(\tau-1)(1+i_j^-))^{w_j}+(\tau-1)|i_j^-|^{w_j}}, -\sqrt{\frac{(1+(\tau-1)(f_j^-)^2)^{w_j}-(1+(f_j^-)^2)^{w_j}}{(1+(\tau-1)(f_j^-)^2)^{w_j}-(\tau-1)(1+(f_j^-)^2)^{w_j}}} \end{matrix}\right)$$

From Eq. (8),

$$(w_iA_i)^p =$$

$$\left(\begin{matrix} \frac{((1+(\tau-1)(t_i^+)^2)^{w_i})^p-((1-(t_i^+)^2)^{w_i})^p}{((1+(\tau-1)(t_i^+)^2)^{w_i})^p+(\tau-1)((1-(t_i^+)^2)^{w_i})^p}, \frac{\tau((i_i^+)^{w_i})^p}{((1+(\tau-1)(1-i_i^+))^{w_i})^p+(\tau-1)((i_i^+)^{w_i})^p}, \sqrt{\frac{\tau((f_i^+)^{w_i})^p}{((1+(\tau-1)(1-f_i^+))^{w_i})^p+(\tau-1)((f_i^+)^{w_i})^p}}, \\ -\sqrt{\frac{\tau(|t_i^-|^{w_i})^p}{((1+(\tau-1)(1+t_i^-))^{w_i})^p+(\tau-1)(|t_i^-|^{w_i})^p}}, \frac{-\tau(|i_i^-|^{w_i})^p}{((1+(\tau-1)(1+i_i^-))^{w_i})^p+(\tau-1)(|i_i^-|^{w_i})^p}, \frac{-((1+(\tau-1)(f_i^-)^2)^{w_i})^p-((1+(f_i^-)^2)^{w_i})^P}{((1+(\tau-1)(f_i^-)^2)^{w_i})^p-(\tau-1)((1+(f_i^-)^2)^{w_i})^p} \end{matrix}\right)$$

$$(w_jA_j)^q =$$

$$\left(\begin{matrix} \frac{((1+(\tau-1)(t_j^+)^2)^{w_j})^q-((1-(t_j^+)^2)^{w_j})^q}{((1+(\tau-1)(t_j^+)^2)^{w_j})^q+(\tau-1)((1-(t_j^+)^2)^{w_j})^q}, \frac{\tau((i_j^+)^{w_j})^q}{((1+(\tau-1)(1-i_j^+))^{w_j})^q+(\tau-1)((i_j^+)^{w_j})^q}, \sqrt{\frac{\tau((f_j^+)^{w_j})^q}{((1+(\tau-1)(1-f_j^+))^{w_j})^q+(\tau-1)((f_j^+)^{w_j})^q}}, \\ -\sqrt{\frac{\tau(|t_j^-|^{w_j})^q}{((1+(\tau-1)(1+t_j^-))^{w_j})^q+(\tau-1)(|t_j^-|^{w_j})^q}}, \frac{-\tau(|i_j^-|^{w_j})^q}{((1+(\tau-1)(1+i_j^-))^{w_j})^q+(\tau-1)(|i_j^-|^{w_j})^q}, \frac{-((1+(\tau-1)(f_j^-)^2)^{w_j})^q-((1+(f_j^-)^2)^{w_j})^q}{((1+(\tau-1)(f_j^-)^2)^{w_j})^q-(\tau-1)((1+(f_j^-)^2)^{w_j})^q} \end{matrix}\right)$$

Referring to Eq. (6),

$$(w_iA_i)^p \otimes (w_jA_j)^q = \left(\begin{matrix} \frac{(1+(\tau-1)(t_i^+)^2)^{pw_i}(1+(\tau-1)(t_j^+)^2)^{qw_j}-(1-(t_i^+)^2)^{pw_i}(1-(t_j^+)^2)^{qw_j}}{(1+(\tau-1)(t_i^+)^2)^{pw_i}(1+(\tau-1)(t_j^+)^2)^{qw_j}+(\tau-1)(1-(t_i^+)^2)^{pw_i}(1-(t_j^+)^2)^{qw_j}}, \\ \frac{\tau(i_i^+)^{pw_i}(i_j^+)^{qw_j}}{(1+(\tau-1)(1-i_i^+))^{pw_i}(1+(\tau-1)(1-i_j^+))^{qw_j}+(\tau-1)(i_i^+)^{pw_i}(i_j^+)^{qw_j}}, \\ \sqrt{\frac{\tau(f_i^+)^{pw_i}(f_j^+)^{qw_j}}{(1+(\tau-1)(1-f_i^+))^{pw_i}(1+(\tau-1)(1-f_j^+))^{qw_j}+(\tau-1)(f_i^+)^{pw_i}(f_j^+)^{qw_j}}}, \\ -\sqrt{\frac{\tau|t_i^-|^{pw_i}|t_j^-|^{qw_j}}{(1+(\tau-1)(1+t_i^-))^{pw_i}(1+(\tau-1)(1+t_j^-))^{qw_j}+(\tau-1)|t_i^-|^{pw_i}|t_j^-|^{qw_j}}}, \\ -\frac{\tau|i_i^-|^{pw_i}|i_j^-|^{qw_j}}{(1+(\tau-1)(1+i_i^-))^{pw_i}(1+(\tau-1)(1+i_j^-))^{qw_j}+(\tau-1)|i_i^-|^{pw_i}|i_j^-|^{qw_j}}, \\ -\frac{(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j}-(1+(f_i^-)^2)^{pw_i}(1+(f_j^-)^2)^{qw_j}}{(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j}-(\tau-1)(1+(f_i^-)^2)^{pw_i}(1+(f_j^-)^2)^{qw_j}} \end{matrix}\right)$$

Based on Eq. (5),

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(w_iA_i)^p \otimes (w_jA_j)^q =$$

$$
\begin{pmatrix}
\sqrt{\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^+)^2)^{pw_i}(1+(\tau-1)\left(t_j^+\right)^2)^{qw_j}-\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^+)^2)^{pw_i}(1-\left(t_j^+\right)^2)^{qw_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(t_i^+)^2)^{pw_i}(1+(\tau-1)\left(t_j^+\right)^2)^{qw_j}+(\tau-1)\sum_{i=1}^{n}\sum_{j=1}^{n}(1-(t_i^+)^2)^{pw_i}(1-\left(t_j^+\right)^2)^{qw_j}}}, \\
\frac{\tau\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^+)^{pw_i}(i_j^+)^{qw_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-i_i^+))^{pw_i}(1+(\tau-1)(1-i_j^+))^{qw_j}+(\tau-1)\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^+)^{pw_i}(i_j^+)^{qw_j}}, \\
\frac{\tau\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^+)^{pw_i}(f_j^+)^{qw_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-f_i^+))^{pw_i}(1+(\tau-1)(1-f_j^+))^{qw_j}+(\tau-1)\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^+)^{pw_i}(f_j^+)^{qw_j}}, \\
-\frac{\tau\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^-|^{pw_i}|t_j^-|^{qw_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+t_i^-))^{pw_i}(1+(\tau-1)(1+t_j^-))^{qw_j}+(\tau-1)\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^-|^{pw_i}|t_j^-|^{qw_j}}, \\
-\frac{\tau\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^{pw_i}|i_j^-|^{qw_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+i_i^-))^{pw_i}(1+(\tau-1)(1+i_j^-))^{qw_j}+(\tau-1)\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^{pw_i}|i_j^-|^{qw_j}}, \\
-\sqrt{\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j}-\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(f_i^-)^2)^{pw_i}(1+\left(f_j^-\right)^2)^{qw_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j}-(\tau-1)\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(f_i^-)^2)^{pw_i}(1+\left(f_j^-\right)^2)^{qw_j}}}
\end{pmatrix}
$$

From Eq. (7),

$$
\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(w_iA_i)^p\otimes\left(w_jA_j\right)^q\right)=
$$

$$
\begin{pmatrix}
\sqrt{\frac{(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2}{n(n+1)}}-(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^+)^2\right)^{pw_i}\left(1-\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^+)^2\right)^{pw_i}\left(1-\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2}{n(n+1)}}}}, \\
\frac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^+)^{pw_i}(i_j^+)^{qw_j})^{\frac{2}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(1-i_i^+)\right)^{pw_i}\left(1+(\tau-1)\left(1-i_j^+\right)\right)^{qw_j})^{\frac{2}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^+)^{pw_i}(i_j^+)^{qw_j})^{\frac{2}{n(n+1)}}}, \\
\frac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^+)^{pw_i}(f_j^+)^{qw_j})^{\frac{2}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-f_i^+))^{pw_i}(1+(\tau-1)(1-f_j^+))^{qw_j})^{\frac{2}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^+)^{pw_i}(f_j^+)^{qw_j})^{\frac{2}{n(n+1)}}}, \\
-\frac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^-|^{pw_i}|t_j^-|^{qw_j})^{\frac{2}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+t_i^-))^{pw_i}(1+(\tau-1)(1+t_j^-))^{qw_j})^{\frac{2}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^-|^{pw_i}|t_j^-|^{qw_j})^{\frac{2}{n(n+1)}}}, \\
-\frac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^{pw_i}|i_j^-|^{qw_j})^{\frac{2}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+i_i^-))^{pw_i}(1+(\tau-1)(1+i_j^-))^{qw_j})^{\frac{2}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^{pw_i}|i_j^-|^{qw_j})^{\frac{2}{n(n+1)}}}, \\
-\sqrt{\frac{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)\left(f_i^-\right)^2)^{pw_i}(1+(\tau-1)\left(f_j^-\right)^2)^{qw_j})^{\frac{2}{n(n+1)}}-(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+\left(f_i^-\right)^2)^{pw_i}(1+\left(f_j^-\right)^2)^{qw_j})^{\frac{2}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j})^{\frac{2}{n(n+1)}}-\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+\left(f_i^-\right)^2)^{pw_i}(1+\left(f_j^-\right)^2)^{qw_j})^{\frac{2}{n(n+1)}}}}
\end{pmatrix}
$$

According to Eq. (8),

$$
\left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(w_iA_i)^p\otimes(w_jA_j)^q\right)\right]^{\frac{1}{p+q}}=
$$

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$$\left(\begin{array}{c}
\dfrac{(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)(t_j^+)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^+)^2\right)^{pw_i}\left(1-(t_j^+)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^+)^2\right)^{pw_i}\left(1-\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}},\\
\dfrac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^+)^{pw_i}(i_j^+)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(1-i_i^+)\right)^{pw_i}\left(1+(\tau-1)\left(1-i_j^+\right)\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^+)^{pw_i}(i_j^+)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}},\\
\sqrt{\dfrac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^+)^{pw_i}(f_j^+)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-f_i^+))^{pw_i}(1+(\tau-1)(1-f_j^+))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^+)^{pw_i}(f_j^+)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}},\\
-\sqrt{\dfrac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^-|^{pw_i}|t_j^-|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+t_i^-))^{pw_i}(1+(\tau-1)(1+t_j^-))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^-|^{pw_i}|t_j^-|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}},\\
-\dfrac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^{pw_i}|i_j^-|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+i_i^-))^{pw_i}(1+(\tau-1)(1+i_j^-))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^-|^{pw_i}|i_j^-|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}},\\
\dfrac{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(f_i^-)^2)^{pw_i}(1+\left(f_j^-\right)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^-)^2)^{pw_i}(1+(\tau-1)(f_j^-)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+\left(f_i^-\right)^2)^{pw_i}(1+\left(f_j^-\right)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}
\end{array}\right)$$

Therefore, the *Eq. (10)* has been proven. A numerical example is provided to demonstrate the application of the BPN-GWHHM operator on BPNSs.

Example 1. Assume $A_1=\langle 0.6,0.4,0.3,-0.2,-0.7,-0.1\rangle$, $A_2=\langle 0.5,0.4,0.2,-0.6,-0.8,-0.4\rangle$, and $A_3=\langle 0.4,0.5,0.2,-0.3,-0.2,-0.5\rangle$ be a collection of BPNSs, and $p=1$, $q=2$, $\tau=0.4$, $w=(w_1,w_2,w_3)=(0.2,0.3,0.5)$. Then, calculate the $BPN-GWHHM^{p,q}(A_1,A_2,A_3)$ value by applying the *Eq. (10)*. The equation determines the t^+ is as follows:

$$t^+=\frac{(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)(t_j^+)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^+)^2\right)^{pw_i}\left(1-(t_j^+)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^+)^2\right)^{pw_i}\left(1-\left(t_j^+\right)^2\right)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}$$

$$=\frac{(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1+(0.4-1)(t_i^+)^2\right)^{w_i}\left(1+(0.4-1)(t_j^+)^2\right)^{2w_j})^{\frac{2(1+2)}{3(3+1)}}-(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1-(t_i^+)^2\right)^{w_i}\left(1-(t_j^+)^2\right)^{2w_j})^{\frac{2(1+2)}{3(3+1)}}}{(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1+(0.4-1)(t_i^+)^2\right)^{w_i}\left(1+(0.4-1)\left(t_j^+\right)^2\right)^{2w_j})^{\frac{2(1+2)}{3(3+1)}}+(0.4-1)(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1-(t_i^+)^2\right)^{w_i}\left(1-\left(t_j^+\right)^2\right)^{2w_j})^{\frac{2(1+2)}{3(3+1)}}}$$

$$=\frac{(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1+(-0.6)(t_i^+)^2\right)^{w_i}\left(1+(-0.6)(t_j^+)^2\right)^{2w_j})^{\frac{1}{2}}-(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1-(t_i^+)^2\right)^{w_i}\left(1-(t_j^+)^2\right)^{2w_j})^{\frac{1}{2}}}{(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1+(-0.6)(t_i^+)^2\right)^{w_i}\left(1+(-0.6)\left(t_j^+\right)^2\right)^{2w_j})^{\frac{1}{2}}+(-0.6)(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1-(t_i^+)^2\right)^{w_i}\left(1-\left(t_j^+\right)^2\right)^{2w_j})^{\frac{1}{2}}}$$

$$=\frac{\begin{pmatrix}(1+(-0.6)(t_1^+)^2)^{w_1}(1+(-0.6)(t_1^+)^2)^{2w_1}+\\(1+(-0.6)(t_1^+)^2)^{w_1}(1+(-0.6)(t_2^+)^2)^{2w_2}+\\(1+(-0.6)(t_1^+)^2)^{w_1}(1+(-0.6)(t_3^+)^2)^{2w_3}+\\(1+(-0.6)(t_2^+)^2)^{w_2}(1+(-0.6)(t_2^+)^2)^{2w_2}+\\(1+(-0.6)(t_2^+)^2)^{w_2}(1+(-0.6)(t_3^+)^2)^{2w_3}+\\(1+(-0.6)(t_3^+)^2)^{w_3}(1+(-0.6)(t_3^+)^2)^{2w_3}\end{pmatrix}^{\frac{1}{2}}-\begin{pmatrix}(1-(t_1^+)^2)^{w_1}(1-(t_1^+)^2)^{2w_1}+\\(1-(t_1^+)^2)^{w_1}(1-(t_2^+)^2)^{2w_2}+\\(1-(t_1^+)^2)^{w_1}(1-(t_3^+)^2)^{2w_3}+\\(1-(t_2^+)^2)^{w_2}(1-(t_2^+)^2)^{2w_2}+\\(1-(t_2^+)^2)^{w_2}(1-(t_3^+)^2)^{2w_3}+\\(1-(t_3^+)^2)^{w_3}(1-(t_3^+)^2)^{2w_3}\end{pmatrix}^{\frac{1}{2}}}{\begin{pmatrix}(1+(-0.6)(t_1^+)^2)^{w_1}(1+(-0.6)(t_1^+)^2)^{2w_1}+\\(1+(-0.6)(t_1^+)^2)^{w_1}(1+(-0.6)(t_2^+)^2)^{2w_2}+\\(1+(-0.6)(t_1^+)^2)^{w_1}(1+(-0.6)(t_3^+)^2)^{2w_3}+\\(1+(-0.6)(t_2^+)^2)^{w_2}(1+(-0.6)(t_2^+)^2)^{2w_2}+\\(1+(-0.6)(t_2^+)^2)^{w_2}(1+(-0.6)(t_3^+)^2)^{2w_3}+\\(1+(-0.6)(t_3^+)^2)^{w_3}(1+(-0.6)(t_3^+)^2)^{2w_3}\end{pmatrix}^{\frac{1}{2}}+(-0.6)\begin{pmatrix}(1-(t_1^+)^2)^{w_1}(1-(t_1^+)^2)^{2w_1}+\\(1-(t_1^+)^2)^{w_1}(1-(t_2^+)^2)^{2w_2}+\\(1-(t_1^+)^2)^{w_1}(1-(t_3^+)^2)^{2w_3}+\\(1-(t_2^+)^2)^{w_2}(1-(t_2^+)^2)^{2w_2}+\\(1-(t_2^+)^2)^{w_2}(1-(t_3^+)^2)^{2w_3}+\\(1-(t_3^+)^2)^{w_3}(1-(t_3^+)^2)^{2w_3}\end{pmatrix}^{\frac{1}{2}}}$$

$$=\frac{\begin{pmatrix}(1+(-0.6)(0.6)^2)^{0.2}(1+(-0.6)(0.6)^2)^{2(0.2)}+\\(1+(-0.6)(0.6)^2)^{0.2}(1+(-0.6)(0.5)^2)^{2(0.3)}+\\(1+(-0.6)(0.6)^2)^{0.2}(1+(-0.6)(0.4)^2)^{2(0.5)}+\\(1+(-0.6)(0.5)^2)^{0.3}(1+(-0.6)(0.5)^2)^{2(0.3)}+\\(1+(-0.6)(0.5)^2)^{0.3}(1+(-0.6)(0.4)^2)^{2(0.5)}+\\(1+(-0.6)(0.4)^2)^{0.5}(1+(-0.6)(0.4)^2)^{2(0.5)}\end{pmatrix}^{\frac{1}{2}}-\begin{pmatrix}(1-(0.6)^2)^{0.2}(1-(0.6)^2)^{2(0.2)}+\\(1-(0.6)^2)^{0.2}(1-(0.5)^2)^{2(0.3)}+\\(1-(0.6)^2)^{0.2}(1-(0.4)^2)^{2(0.5)}+\\(1-(0.5)^2)^{0.3}(1-(0.5)^2)^{2(0.3)}+\\(1-(0.5)^2)^{0.3}(1-(0.4)^2)^{2(0.5)}+\\(1-(0.4)^2)^{0.5}(1-(0.4)^2)^{2(0.5)}\end{pmatrix}^{\frac{1}{2}}}{\begin{pmatrix}(1+(-0.6)(0.6)^2)^{0.2}(1+(-0.6)(0.6)^2)^{2(0.2)}+\\(1+(-0.6)(0.6)^2)^{0.2}(1+(-0.6)(0.5)^2)^{2(0.3)}+\\(1+(-0.6)(0.6)^2)^{0.2}(1+(-0.6)(0.4)^2)^{2(0.5)}+\\(1+(-0.6)(0.5)^2)^{0.3}(1+(-0.6)(0.5)^2)^{2(0.3)}+\\(1+(-0.6)(0.5)^2)^{0.3}(1+(-0.6)(0.4)^2)^{2(0.5)}+\\(1+(-0.6)(0.4)^2)^{0.5}(1+(-0.6)(0.4)^2)^{2(0.5)}\end{pmatrix}^{\frac{1}{2}}+(-0.6)\begin{pmatrix}(1-(0.6)^2)^{0.2}(1-(0.6)^2)^{2(0.2)}+\\(1-(0.6)^2)^{0.2}(1-(0.5)^2)^{2(0.3)}+\\(1-(0.6)^2)^{0.2}(1-(0.4)^2)^{2(0.5)}+\\(1-(0.5)^2)^{0.3}(1-(0.5)^2)^{2(0.3)}+\\(1-(0.5)^2)^{0.3}(1-(0.4)^2)^{2(0.5)}+\\(1-(0.4)^2)^{0.5}(1-(0.4)^2)^{2(0.5)}\end{pmatrix}^{\frac{1}{2}}}$$

$$= \frac{\left(\begin{matrix}0.8642+\\0.8640+\\0.8611+\\0.8639+\\0.8610+\\0.8595\end{matrix}\right)^{\frac{1}{2}} - \left(\begin{matrix}0.7651+\\0.7696+\\0.7683+\\0.7719+\\0.7705+\\0.7699\end{matrix}\right)^{\frac{1}{2}}}{\left(\begin{matrix}0.8642+\\0.8640+\\0.8611+\\0.8639+\\0.8610+\\0.8595\end{matrix}\right)^{\frac{1}{2}} + (-0.6)\left(\begin{matrix}0.7651+\\0.7696+\\0.7683+\\0.7719+\\0.7705+\\0.7699\end{matrix}\right)^{\frac{1}{2}}}$$

$$= \frac{(5.1736)^{\frac{1}{2}} - (4.6153)^{\frac{1}{2}}}{(5.1736)^{\frac{1}{2}} + (-0.6)(4.6153)^{\frac{1}{2}}}$$

$$= 0.1281$$

Theorem 2. *(Idempotency property) Let* $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2,\dots,n)$ *be a set of BPNS. If* $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle$ *for all* i *and* $w_i = 1$ *are equal to* A, *then* $BPN - GWHHM^{p,q}(A_1, A_2, \dots, A_n) = A$.

Proof Since $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle (i = 1,2,\dots,n)$ and $(w_i = w_j = 1)$, then,

$$BPNS - GWHHM^{p,q}(A_1, A_2, \dots, A_n) = \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(w_iA_i)^p \otimes (w_jA_j)^q\right)\right]^{\frac{1}{p+q}}$$

$$= \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}A_i{}^p \otimes A_j{}^q\right)\right]^{\frac{1}{p+q}}$$

$$= \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}A^p \otimes A^q\right)\right]^{\frac{1}{p+q}}$$

$$= \left[\frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(A)^{p+q}\right)\right]^{\frac{1}{p+q}}$$

$$= \left[\frac{2}{n(n+1)}\left(\frac{n(n+1)}{2}\right)(A)^{p+q}\right]^{\frac{1}{p+q}}$$

$$= A$$

$$\Rightarrow BPN - GHHM^{p,q}(A_1, A_2, \dots, A_n) = A. \ \blacksquare$$

Theorem 3. *(Monotonicity property) For any two set of BPNS,* $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^- \rangle$ *and* $A_i^* = \langle t_i^{+*}, i_i^{+*}, f_i^{+*}, t_i^{-*}, i_i^{-*}, f_i^{-*} \rangle$, *such that* $A_i \le A_i^*$ *for all* $i = 1,2,\dots,n$. *If* $t^+ \le t^{+*}, i^+ \ge i^{+*}, f^+ \ge f^{+*}$ *and* $t^- \ge t^{-*}, i^- \le i^{-*}, f^- \le f^{-*}$, *then*
$BPN - GWHHM^{p,q}(A_1, A_2, \dots, A_n) \le BPN - GWHHM^{p,q}(A_1^*, A_2^*, \dots, A_n^*)$.

Proof If $t^+ \le t^{+*}, i^+ \ge i^{+*}, f^+ \ge f^{+*}$ and $t^- \ge t^{-*}, i^- \le i^{-*}, f^- \le f^{-*}$, then,

$$\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)(t_j^+)^2\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}} - \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^+)^2\right)^{pw_i}\left(1-(t_j^+)^2\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^+)^2\right)^{pw_i}\left(1+(\tau-1)\left(t_j^+\right)^2\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}} + (\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^+)^2\right)^{pw_i}\left(1-\left(t_j^+\right)^2\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}} \le$$

$$\frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^{+*})^2\right)^{pw_i}\left(1+(\tau-1)(t_j^{+*})^2\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}} - \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^{+*})^2\right)^{pw_i}\left(1-(t_j^{+*})^2\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(t_i^{+*})^2\right)^{pw_i}\left(1+(\tau-1)\left(t_j^{+*}\right)^2\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}} + (\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1-(t_i^{+*})^2\right)^{pw_i}\left(1-\left(t_j^{+*}\right)^2\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}},$$

Similarly,

$$\frac{\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^+)^{pw_i}(i_j^+)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(1-i_i^+)\right)^{pw_i}\left(1+(\tau-1)\left(1-i_j^+\right)\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}} + (\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^+)^{pw_i}(i_j^+)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}} \ge$$

$$\frac{\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^{+*})^{pw_i}(i_j^{+*})^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)(1-i_i^{+*})\right)^{pw_i}\left(1+(\tau-1)\left(1-i_j^{+*}\right)\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}} + (\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(i_i^{+*})^{pw_i}(i_j^{+*})^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}},$$

$$\sqrt{\frac{\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^+)^{pw_i}(f_j^+)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-f_i^+))^{pw_i}(1+(\tau-1)(1-f_j^+))^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}} + (\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^+)^{pw_i}(f_j^+)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}} \ge$$

$$\sqrt{\frac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^{+*})^{pw_i}(f_j^{+*})^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1-f_i^{+*}))^{pw_i}(1+(\tau-1)(1-f_j^{+*}))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(f_i^{+*})^{pw_i}(f_j^{+*})^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}},$$

$$-\sqrt{\frac{\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^{-}|^{pw_i}|t_j^{-}|^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)\left(1+t_i^{-}\right)\right)^{pw_i}\left(1+(\tau-1)\left(1+t_j^{-}\right)\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^{-}|^{pw_i}|t_j^{-}|^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}} \geq$$

$$-\sqrt{\frac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^{-*}|^{pw_i}|t_j^{-*}|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+t_i^{-*}))^{pw_i}(1+(\tau-1)(1+t_j^{-*}))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}|t_i^{-*}|^{pw_i}|t_j^{-*}|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}},$$

$$-\frac{\tau\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^{-}|^{pw_i}|i_j^{-}|^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}}{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+(\tau-1)\left(1+i_i^{-}\right)\right)^{pw_i}\left(1+(\tau-1)\left(1+i_j^{-}\right)\right)^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^{-}|^{pw_i}|i_j^{-}|^{qw_j}\right)^{\frac{2(p+q)}{n(n+1)}}} \leq$$

$$-\frac{\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^{-*}|^{pw_i}|i_j^{-*}|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(1+i_i^{-*}))^{pw_i}(1+(\tau-1)(1+i_j^{-*}))^{qw_j})^{\frac{2(p+q)}{n(n+1)}}+(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}|i_i^{-*}|^{pw_i}|i_j^{-*}|^{qw_j})^{\frac{2(p+q)}{n(n+1)}}},$$

And

$$\frac{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^{-})^2)^{pw_i}(1+(\tau-1)(f_j^{-})^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(f_i^{-})^2)^{pw_i}(1+\left(f_j^{-}\right)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^{-})^2)^{pw_i}(1+(\tau-1)(f_j^{-})^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+\left(f_i^{-}\right)^2)^{pw_i}(1+\left(f_j^{-}\right)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}} \leq$$

$$\frac{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^{-})^2)^{pw_i}(1+(\tau-1)(f_j^{-})^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(f_i^{-})^2)^{pw_i}(1+\left(f_j^{-}\right)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}{(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+(\tau-1)(f_i^{-})^2)^{pw_i}(1+(\tau-1)(f_j^{-})^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}-(\tau-1)(\sum_{i=1}^{n}\sum_{j=1}^{n}(1+\left(f_i^{-}\right)^2)^{pw_i}(1+\left(f_j^{-}\right)^2)^{qw_j})^{\frac{2(p+q)}{n(n+1)}}}.$$

Theorem 4. *(Boundedness property) For a collection of BPNS, let* $A_i = \langle t_i^+, i_i^+, f_i^+, t_i^-, i_i^-, f_i^-\rangle (i = 1,2,\dots,n)$ *where* $\tau > 0$*, and if*

$$A^- = \langle min(t^+), max(i^+), max(f^+), max(t^-), min(i^-), min(f^-)\rangle$$
$$A^+ = \langle max(t^{+*}), min(i^{+*}), min(f^{+*}), min(t^{-*}), max(i^{-*}), max(f^{-*})\rangle$$

Then, $BPN - GWHHM^{p,q}(A^-, A^-, \dots, A^-) \leq BPN - GWHHM^{p,q}(A_1, A_2, \dots, A_n) \leq BPN - GWHHM^{p,q}(A^+, A^+, \dots, A^+)$. Hence, $A^- \leq BPN - GWHHM^{p,q}(A_1, A_2, \dots, A_n) \leq A^+$.

MCDM Techniques

To address real-world MCDM problems, this section demonstrates the following comprehensive approach based on BPN-GWHHM operator. Let $G = \{G_1, G_2, \dots, G_m\}$ be a set of alternatives, $F = \{F_1, F_2, \dots, F_n\}$ be a set of criteria, a set of weights $w = \{w_1, w_2, \dots, w_n\}$ satisfying ($w_i \in [0,1]$ and $w_1 + w_2 + \dots + w_n = 1$).

Step 1. Construct a decision matrix employing BPNSs. Assume $A_{ij} = \langle t_{ij}^+, i_{ij}^+, f_{ij}^+, t_{ij}^-, i_{ij}^-, f_{ij}^-\rangle$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, where each column corresponds to an evaluation criteria and each row denotes an alternative:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix}. \quad (11)$$

Step 2. Normalize the decision matrix. The criteria have two types, categorized as either benefit criteria or cost criteria. Benefit criteria indicate that larger values are preferable (positive effects), while cost criteria denote that smaller values are preferred (negative effects). The corresponding equation is presented as follows:

$$A^{\text{q}} = \begin{cases} \left[t_{ij}^+, i_{ij}^+, f_{ij}^+, t_{ij}^-, i_{ij}^-, f_{ij}^-\right]_{m,n}, & if\ F_j\ is\ a\ benefit\ criteria \\ \left[f_{ij}^+, i_{ij}^+, t_{ij}^+, f_{ij}^-, i_{ij}^-, t_{ij}^-\right]_{m,n}, & if\ F_j\ is\ a\ cost\ criteria \end{cases} \quad (12)$$

Step 3. Calculate the aggregated BPNSs values for each alternative $A_i (i = 1,2,3)$ using the proposed operator. Each alternative can be expressed as follows:

$$A^{\natural} = \begin{bmatrix} A^{\natural}_{11} & A^{\natural}_{12} & \cdots & A^{\natural}_{1m} \\ A^{\natural}_{21} & A^{\natural}_{22} & \cdots & A^{\natural}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A^{\natural}_{n1} & A^{\natural}_{n2} & \cdots & A^{\natural}_{nm} \end{bmatrix}. \tag{13}$$

Step 4. Determine the ranking of the alternatives based on the score values by assuming that, $r = \langle \alpha_r^+, l_r^+, \beta_r^+, \alpha_r^-, l_r^-, \beta_r^- \rangle$ is under BPNSs information:

$$s(r) = \frac{1}{6}(\alpha_r^+ + 1 - l_r^+ + 1 - \beta_r^+ + 1 - \alpha_r^- + l_r^- + \beta_r^-) \tag{14}$$

The highest ranked alternative is the most preferred.
Step 5. Select the best alternative(s)

Numerical Example

This sub-section presents a practical example of applying the proposed model. The author generated numerical data illustrations to solve an MCDM problem. A financial consulting firm evaluates five companies $(G_1, G_2, G_3, G_4, G_5)$ to determine the most financially sustainable firm for long-term investment. The firm prioritized three financial criteria, F_1 (profitability): net profit margin and return on investment, F_2 (financial stability): debt-to-equity ratio and financial risk level, and F_3 (operational efficiency): cost management and asset utilization. F_1 and F_2 are considered benefit criteria, while F_3 is a cost criterion. The goal is to rank the firms based on the key financial performance indicators. The weight is stated as $w = (0.3, 0.5, 0.2)$.
Step 1. The decision matrix of BPNSs is shown in Table 1.
Step 2. Normalized decision matrix of BPNSs based on *Eq. (12)*.
Step 3. Aggregated the values by assuming $G_i(i = 1,2,3,4,5)(p = 1, q = 2, \tau = 0.4)$.
Step 4. Rank all the alternatives by the score value in Figure 1.
Step 5. Hence, the most suitable firm is G_2.

Table 1. Bipolar Pythagorean Neutrosophic Decision Matrix.

	F_1	F_2	F_3
G_1	(0.2, 0.7, 0.9, -0.4, -0.3, -0.1)	(0.6, 0.3, 0.2, -0.4, -0.5, -0.7)	(0.2, 0.6, 0.2, -0.4, -0.3, -0.2)
G_2	(0.4, 0.1, 0.2, -0.3, -0.2, -0.6)	(0.8, 0.1, 0.3, -0.5, -0.6, -0.1)	(0.2, 0.8, 0.1, -0.2, -0.4, -0.7)
G_3	(0.5, 0.7, 0.4, -0.3, -0.2, -0.8)	(0.2, 0.4, 0.7, -0.5, -0.5, -0.2)	(0.1, 0.5, 0.7, -0.3, -0.2, -0.6)
G_4	(0.3, 0.6, 0.5, -0.1, -0.4, -0.7)	(0.3, 0.5, 0.4, -0.3, -0.4, -0.3)	(0.2, 0.3, 0.1, -0.4, -0.5, -0.6)
G_5	(0.1, 0.5, 0.3, -0.2, -0.3, -0.5)	(0.4, 0.6, 0.2, -0.6, -0.4, -0.5)	(0.5, 0.6, 0.7, -0.3, -0.6, -0.2)

The ranking is determined, $G_2 \succ G_3 \succ G_1 \succ G_5 \succ G_4$ with G_2 identified as the best firm to invest in. G_2 represents superior financial performance, gaining from strategic market placement, innovative operational systems and strong growth momentum that boost its high investment worth. The ranking results are shown as visual representations in Figure 1.

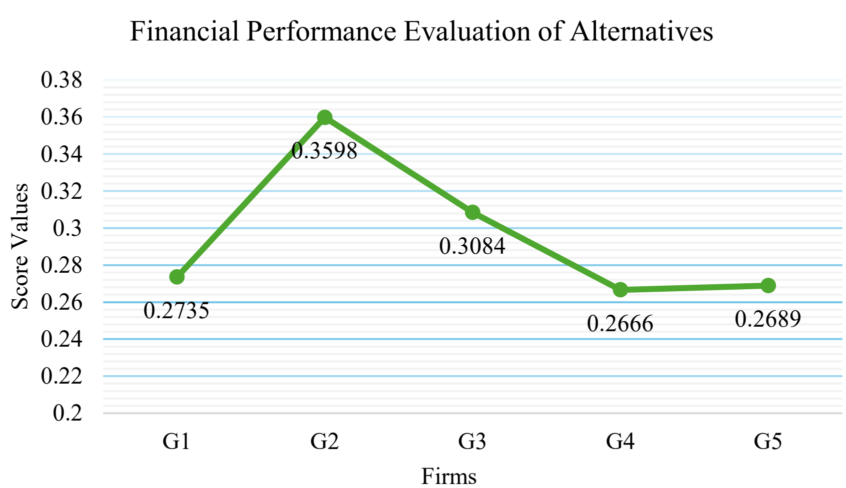

Figure 1. Score Value Comparisons of Firms using BPN-GWHHM Operator

Sensitivity Analysis

In this sub-section, different values of the parameter τ, p, and q are computed using BPN-GWHHM operator to evaluate and examine the influence of these variables in MCDM. Decision-makers may select alternatives based on their preferred criteria. Different possible values of parameter τ have been examined in Figure 2 below.

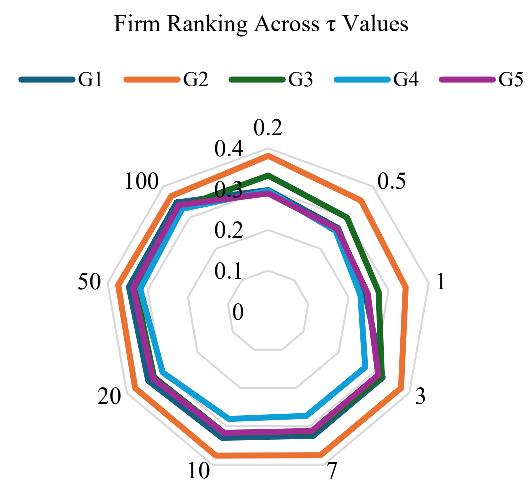

Figure 2. Firm Ranking Order across Different τ Values using BPN-GWHHM Operator

The radar chart in Figure 2 illustrates the effect of different parameter τ values on the ranking of firms. Line closer to the outer edge determines the highest performance scores, with the top-performing firm indicated by the farthest line. Firm G_2 consistently ranks first, achieving the highest score across increasing τ values, indicating strong return potential and marking it as a favorable investment target. In contrast, firms G_4 and G_5 attain the lowest scores, suggesting weaker performance and less favorable investment. Firms G_1 and G_3 show moderate performance with intermediate scores, making them suitable for balanced portfolios, though they do not emerge as top choices. Importantly, the variation in τ values have minimal impact on the overall ranking trend, highlighting the effectiveness and reliability of the BPN-GWHHM operator in financial decision-making. Figure 3 illustrates the ranking order on different parameters of p and q using proposed model.

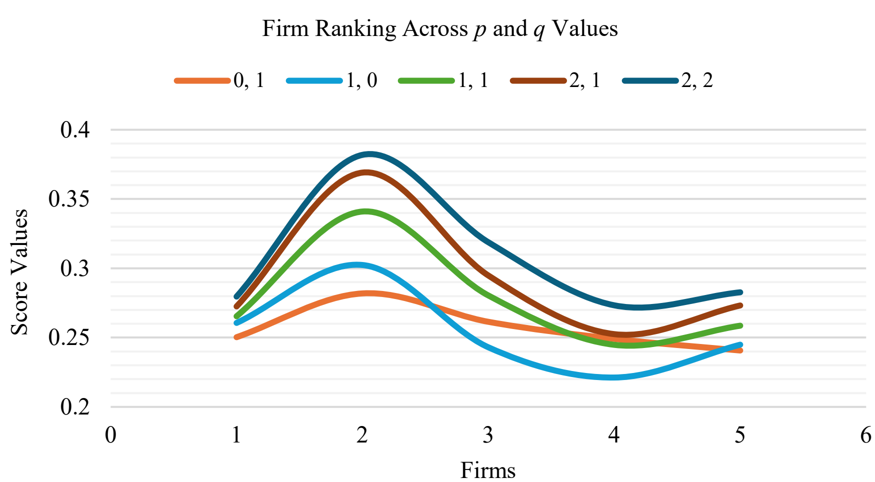

Figure 3. Ranking of Firm on Different p *and* q Values using BPN-GWHHM Operator

Figure 3 presents the ranking results based on varying parameter combinations of p and q using the proposed model. Higher parameter combinations like $p = 2$ and $q = 2$ result in higher score separation among firms, while lower parameters like $p = 0$ and $q = 1$ lead to more uniform, flattened scores. Firm G_2 maintains the top rank across all combinations, while firms G_4 and G_5 consistently ranking lowest. The score follows a bell-shaped pattern, peaking at G_2 and declining towards the lower-ranked firms. This differentiation becomes more pronounced with larger p and q values, offering clearer insight into each firm's performance. The findings reinforce the flexibility and stability of the proposed operator in managing uncertainty within MCDM problems.

Conclusions

Financial stability is crucial for a company's growth, resilience, and development potential. Firms may be interested in investing the innovation, R&D, and sustainability projects that promote long-term growth and competitiveness when they achieve and maintain excellent financial performance. To guarantee the business's short-term operation, strong financial performance and adaptability to pursue objectives, adjust to changing market circumstances, and provide long-term value for stakeholders are essential. The Bipolar Pythagorean Neutrosophic Hamacher Heronian Mean (BPN-GWHHM) operator offers a thorough and reliable approach of assessing a company's financial performance and addressing uncertainty effectively. Despite the robustness and effectiveness of the proposed BPN-GWHHM operator, certain limitations should be acknowledged. These include potential computational complexity when applied to large-scale datasets and scalability concerns in real-time decision environments. Addressing these limitations in future work could further enhance the model's practicality and adaptability. For future research, the proposed operator could also be explored in other decision-making domains beyond financial performance evaluation, including supplier selection, investment portfolio optimization, and strategic planning.

Conflicts of Interest

The author declares that there is no conflict of interest in this paper.

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