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RESEARCH ARTICLE

Energy and Spectrum of the Line Graphs of a Unit Graphs

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Abstract The study delves into an analysis of energy and spectral properties of the line graph, denoted as $L(G)$, derived from the unit graph G offering a comprehensive mathematical exploration. Then, we begin by defining key concepts such as unit graphs, line graphs, and their spectral characteristics. Our investigation involves the computation of eigenvalues and their corresponding energy values, demonstrating how the adjacency matrix influences spectral behaviour. Furthermore, we analyse how variations in graph parameters impact energy distribution and eigenvalue structure. Theoretical derivations are supported by computational results, revealing significant trends and relationships in graph energy and spectrum. By building upon existing literature and introducing new theoretical insights, this research contributes to the ongoing development of spectral graph theory, with potential applications in network analysis, mathematical chemistry, and physics. By analysing the adjacency matrices of line graphs derived from unit graphs, this paper investigates $L(G)$ which mean the line graphs of a unit graphs, where edges in G act as vertices in $L(G)$, computes eigenvalues, examines graph properties, and derives general expressions for energy and spectrum. This study focuses on integer modulo rings $\mathbb{Z}_n$, $L(G(\mathbb{Z}_n))$ for specific values of n.

Keywords: Unit graph, line graph, graph energy; graph spectrum.

Introduction

As the fundamental area in mathematics, graph theory provides effective methods in modeling as well as analyzing connections among discrete objects. It plays a crucial role in various scientific and engineering fields, including computer science, biology, chemistry, and physics. Graphs serve as representations of networks, social structures, molecular structures, and communication systems, making their study essential for understanding complex interactions.

Among the numerous types of graphs, unit graphs and their corresponding line graphs possess intriguing spectral properties that have gained significant attention in mathematical research. A unit graph is defined based on a specific set and a given adjacency condition, while its line graph is derived by transforming each edge into a vertex. Studying the spectral characteristics of these graphs, including their eigenvalues and energy, helps in understanding their structural behaviour and applications in various domains.

The concept of graph energy, originally introduced in mathematical chemistry, has been widely used to characterize the stability and reactivity of chemical compounds. This study extends the concept of the energy for the line graphs of unit graphs, $L(G)$ aiming to explore their mathematical properties and

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potential applications. Through computational and theoretical approaches, we examine the eigenvalue distributions and energy variations of these graphs, highlighting their significance in spectral graph theory.

Several past studies have explored the energy of graphs and their spectral properties. For instance, I. Gutman first introduced the concept of graph energy and established its connection to molecular stability [1]. Further research by Cvetkovic, Rowlinson, and Simic expanded the understanding of graph spectra, linking eigenvalues to structural properties [2]. More recent work has investigated the spectral properties of line graphs, particularly in unit graphs, revealing new relationships between eigenvalues and energy distributions. By building upon these studies, our work extends the current understanding of line graph energy in unit graphs and provides new theoretical insights into their spectral behaviour.

For example, the unit graphs, G where vertex representing elements of the set, where an edge is formed based on a predefined condition such as divisibility or adjacency in a network [3] [4] [5]. The line graph, $L(G)$ emerges by transforming each link in G into the vertex of $L(G)$, such that any pair of vertices of $L(G)$ are connected when their corresponding links of G are incident to the same vertex. By analysing the eigenvalues of the adjacency matrix of $L(G)$, we can compute its spectral energy and understand how its structure affects its spectral properties.

In other words, the concept of unit graphs arises from algebraic structures where vertices correspond to elements for ring, R and adjacency is described based on unit sum conditions. Expanding on this, $L(G(R))$ transforms edges into vertices, maintaining adjacency based on shared vertices in the original unit graph.

Several studies have explored the energy of graphs, such as the relationship between graph energy and the Sombor index based on vertex degrees [6]. Additionally, numerous correlations between graph energy and various graph properties, including clique number, Randić index, and the number of vertices and edges, as well as maximum and minimum degrees, have been discussed in [6].

This study aims to provide a deeper insight into the spectral properties and energy distribution of $L(G(R))$ associated to rings $\mathbb{Z}_n$, $L(G(\mathbb{Z}_n))$. By leveraging existing literature and presenting new theoretical results, we contribute to the advancement of spectral graph theory and its practical applications. The findings in this research may serve as a foundation for future studies on graph energy, spectral analysis, and network optimization.

Definitions and Preliminaries

This section contains the definitions and preliminaries related to this research.

The following is the definition of planar graph.

Definition 1 [7]. Planar graphs are those capable of being represented on a flat surface while ensuring no edges overlap.

In other words, the edges can be arranged so that they only intersect at their endpoints (vertices) [8]. A fundamental result in graph theory asserts that the graphs are planar precisely when it lacks the subgraph that is homeomorphic to K_5, the complete graph on five vertices, or $K_{3,3}$, the complete bipartite graph with two sets of three vertices. Planar graphs play a crucial role in topological graph theory and have applications in circuit design and geographic mapping.

The definition of the bipartite graphs is as follows.

Definition 2 [9]. Bipartite graphs are the graphs in which the vertex set is divisible into two non-overlapping subsets, U and V, ensuring that each edge connects a vertex from U to the vertex from V.

Simply put, vertices belonging to the same group are never connected. Any graph is bipartite precisely when it does not contain cycles of odd length. Bipartite graphs are widely used in modelling relationships in computer science, scheduling problems, and network flow optimization.

The following describes the definition for unit graphs.

Definition 3 [3]. Let $R = \mathbb{Z}_n$ be the ring of integers modulo n, and let $U(R)$ denote the set of units in R. The unit graph $G(R)$ is a simple undirected graph whose vertex set is R, and in which two distinct vertices $x, y \in R$ are adjacent if and only if their sum satisfies:

$$x + y \in U(R).$$

That is, an edge exists between x and y in $G(R)$ if and only if the sum $x + y$ is a unit in R.

The following describes the definition of a line graph.

Definition 4 [10]. The line graphs of unit graphs, $L(G)$ is constructed by transforming an edge in G into a vertex in $L(G)$, with connections are established by shared endpoints in G.

Understanding the spectral properties of a line graph involves computing its eigenvalues and determining key structural characteristics. The eigenvalues of $L(G)$ can be derived from those of the original graph G, leading to important observations regarding their distribution and impact on graph behavior. Several theorems and lemmas are employed to establish these spectral relationships.

Next, following is the definition of spectrum of the graph.

Definition 5 [11]. Spectrum of the graph, $Spec(G)$ comprises the eigenvalues of its adjacency matrix.

The adjacency matrix, $A(G)$ is a matrix whose size dimension equals the vertex count of G. The eigenvalues of this matrix provide critical information about the graph's structural properties, such as connectivity, stability, and symmetry. The spectral radius, which is the largest absolute eigenvalue, plays a vital role in determining the dynamic behaviour of networks modelled by graphs.

The following definition refers to the energy of graph, as it is used in our calculations.

Definition 6 [12]. Energy of the graph, $E(G)$ is a numerical invariant derived from its eigenvalues. It is defined as the total sum of their absolute values in the adjacency matrix:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

where λ_i represents the eigenvalues of $A(G)$. Graph energy has applications in chemistry, where it correlates with the π-electron energy associated with molecular structures, as well as in network science, where it aids in analysing stability and robustness.

Graph energy plays a significant role in mathematical chemistry and network science. The energy of a line graph is calculated using its adjacency matrix spectrum. Through analytical derivations and computational methods, we establish formulas for the energy of specific families of unit graphs and their corresponding line graphs. We also explore how graph parameters influence these energy values.

The adjacency matrix is described as:

Definition 7 [13]. The adjacency matrix of the graph G is the square matrix $A(G) = [a_{ij}]$, which each entry a_{ij} is determined based on vertex connectivity:

- $a_{ij} = 1$, when the vertices i and j are connected by the edge.
- $a_{ij} = 0$, in all other cases.

This matrix serves as a fundamental representation of a graph and is crucial for computing eigenvalues, which in turn determine the spectral energy of the graph.

The concept of a ring is described as follows:

Definition 8 [14]. Ring, R is the algebraic structure defined by the set along with two binary operations, commonly referred to as addition and multiplication, which adhere to specific properties:

- The set is closed under both addition and multiplication, meaning the result of these operations always belongs to the set.

- Addition is both associative and commutative, with an additive identity (zero) and the existence of additive inverses for every element.
- Multiplication satisfies associativity and follows the distributive property over addition.

Then, R play the significant role in graph theory, particularly in the study of unit graphs, where the components in R act as vertices, while adjacency conditions are based on R operations.

The concept of a unit in R is described as follows:

Definition 9 [15]. Unit of ring R is any component $u \in R$ possessing an inverse under multiplication in R.

This implies the existence of another component $v \in R$ for which:

$$uv = vu = 1.$$

Here 1 represents the identity element for multiplication in R. The collection containing all these units of R constitutes a multiplicative group, known as the group of units in R.

Then, the preliminary results of unit graph that are used in this paper are also explained.

Below is an example of $G(\mathbb{Z}_n)$, where $n = 2^2$.

Example 1.
Consider the ring of integers modulo 4, denoted by $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. The units in $\mathbb{Z}_4$, which are elements that possess a multiplicative inverse, are 1 and 3. That is, both 1 and 3 satisfy the condition $x \cdot y \equiv 1$ mod 4, and thus the group of units is $U(\mathbb{Z}_4) = \{1, 3\}$. According to Definition 3, the unit graph $G(\mathbb{Z}_4)$ is constructed by joining two distinct vertices x and y if and only if their sum $x + y$ belongs to the unit group $U(\mathbb{Z}_4)$. Using this criterion, the pairs of vertices that form edges are: $0 + 1 = 1 \in U(\mathbb{Z}_4)$, $0 + 3 = 3 \in U(\mathbb{Z}_4)$, $1 + 2 = 3 \in U(\mathbb{Z}_4)$, and $2 + 3 = 1 \in U(\mathbb{Z}_4)$. Therefore, the set of edges in the graph is $E(G(\mathbb{Z}_4))$ = {(0, 1), (0, 3), (1, 2), (2, 3)}. The resulting graph is connected, as there exists a path between every pair of vertices. It is also planar, since it can be drawn without any edge crossings. Furthermore, each vertex in $G(\mathbb{Z}_4)$ is incident to exactly two edges, making it a 2-regular graph. The corresponding diagram is provided in Figure 1 with a layout that avoids overlapping edges for clarity.

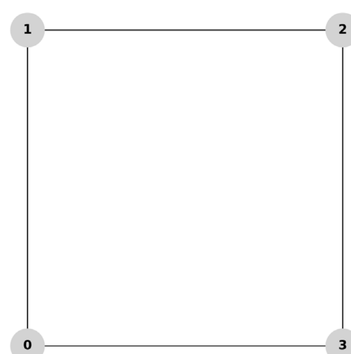

Figure 1 $G(\mathbb{Z}_4)$

In the following, an example of $G(\mathbb{Z}_6)$, illustrated for $n = 2(3)$.

Example 2.
Now consider the ring of integers modulo 6, given by $\mathbb{Z}_6$ = {0, 1, 2, 3, 4, 5}. The elements that have multiplicative inverses in $\mathbb{Z}_6$ are 1 and 5, since $1 \cdot 1 \equiv 1$ mod 6, and $5 \cdot 5 \equiv 1$ mod 6. Thus, the unit group in this case is $U(\mathbb{Z}_6) = \{1, 5\}$. In constructing the unit graph $G(\mathbb{Z}_6)$, two vertices x and y are adjacent if and only if $x + y \in U(\mathbb{Z}_6)$, as defined in Definition 3. Applying this rule, we find that the following vertex pairs satisfy the condition: $0 + 1 = 1$, $0 + 5 = 5$, $1 + 4 = 5$, $2 + 3 = 5$, $2 + 5 = 7 \equiv 1$ mod 6, and $3 + 4 = 7 \equiv 1$ mod 6, all of which lie in $U(\mathbb{Z}_6)$. Consequently, the edge set is $E(G(\mathbb{Z}_6))$ = {(0, 1), (0, 5), (1, 4), (2, 3), (2, 5), (3, 4)}. The resulting graph is connected and planar, and it is also 2-regular, as each vertex

has a degree of exactly two. Figure 2 presents the corresponding diagram of $G(\mathbb{Z}_6)$, drawn with non-overlapping edges for improved readability.

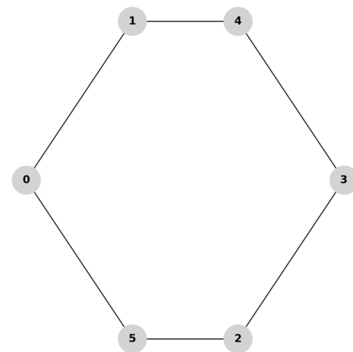

Figure 2 $G(\mathbb{Z}_6)$

Some Results and Discussions

This part presents the findings of this study, beginning with the construction of $L(G(\mathbb{Z}_n))$, followed by the calculation of its energy and spectrum.

Construction of $L(G(\mathbb{Z}_n))$

This subsection explains some concepts of the $L(G(R))$ providing multiple examples. The graphical representation of $L(G(\mathbb{Z}_n))$ is structured as follows:

Definition 10. Line graphs of unit graphs, $L(G(R))$ are formed by representing each edge in $G(R)$, as a vertex in $L(G(R))$. Two vertices of $L(G(R))$ are connected when their corresponding edges of $G(R)$ have a mutual vertex.

Below is an example of $L(G(\mathbb{Z}_4))$.

Example 3.
Consider the ring of the integer modulo 4, $\mathbb{Z}_4$. According to Definition 10, the vertices in $L(G(\mathbb{Z}_4))$ correspond to the edges in $G(\mathbb{Z}_4)$. Since $G(\mathbb{Z}_4)$ has four edges, $L(G(\mathbb{Z}_4))$ consists of four vertices: (0,1), (0,3), (1,2), and (2,3). Two vertices in $L(G(\mathbb{Z}_4))$ are connected when their corresponding edges of $G(\mathbb{Z}_4)$ have at least one mutual vertex.

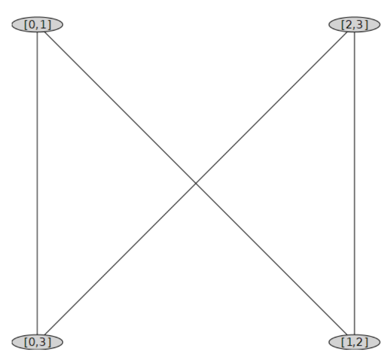

Figure 3 $L(G(\mathbb{Z}_4))$

Every pair of vertices in $L(G(\mathbb{Z}_4))$ is connected by at least one path, confirming that the graph is connected. According to Definition 1, $L(G(\mathbb{Z}_4))$ is planar. Then $L(G(\mathbb{Z}_4))$ is also a regular graph because each vertex has the same degree. Each vertex of $L(G(\mathbb{Z}_4))$ has degree 2. Thus, $L(G(\mathbb{Z}_4))$ is 2-regular.

In the following, an example of a line graph of unit graph is presented for ring $\mathbb{Z}_6$, $L(G(\mathbb{Z}_6))$.

Example 4.
Consider the ring of integers modulo 6, $\mathbb{Z}_6$. According to Definition 10, the vertex in $L(G(\mathbb{Z}_6))$ corresponding to the edge in $G(\mathbb{Z}_6)$. Since $G(\mathbb{Z}_6)$ has six edges, $L(G(\mathbb{Z}_6))$ consists of six vertices: (0,1), (0,5), (1,4), (2,5), (2,3), and (3,4). Two vertices in $L(G(\mathbb{Z}_6))$ are connected when their corresponding edges of $G(\mathbb{Z}_6)$ have at least one mutual vertex.

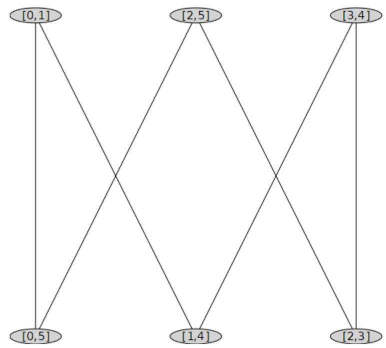

Figure 4 $L(G(\mathbb{Z}_6))$

Each pair of vertices in $L(G(\mathbb{Z}_6))$ has at least one path connecting them, making it a connected graph. According to Definition 1, it is planar. Then, it is also a regular graph because each vertex has the same degree. Each vertex of $L(G(\mathbb{Z}_6))$ has two degrees. Thus, $L(G(\mathbb{Z}_6))$ is 2-regular.

Energy and spectrum of $L(G(\mathbb{Z}_n))$

This section calculates the energy and spectrum of $L(G(\mathbb{Z}_n))$ for specific values of n. In this paper, the eigenvalue of the matrix is determined using Maple software.

Theorem 1. Energy of $L(G(\mathbb{Z}_n))$, when $n = 2^m$ and $m > 1$ is:

$$E(L(G(\mathbb{Z}_n))) = (n-2)^2.$$

Proof. Suppose that $n = 2^m$, where $m > 1$. According to Definition 2, an adjacency matrix of $L(G(\mathbb{Z}_{2^m}))$ represents as bipartite graph, where vertex sets *r* and *s*. Based on Definition 7, an adjacency matrix of $L(G(\mathbb{Z}_{2^m}))$ is structured as follows.

$$M\big(L(G(\mathbb{Z}_{2^m}))\big) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

where B is an $r \times s$ matrix. Therefore, the matrix $M\big(L(G(\mathbb{Z}_{2^m}))\big)$ of dimension $2^{2(m-1)}$ is formed by $2^{2(m-1)}$.

Case (I): Let $m = 2$, characteristic polynomial can be expressed as:

$$f(\lambda) = \lambda^2 (\lambda + 2)(\lambda - 2).$$

Setting $f(\lambda) = 0$, the non-zero eigenvalues of $L\big(G(\mathbb{Z}_4)\big)$ are 2 and -2. To compute the energy of $L\big(G(\mathbb{Z}_4)\big)$, sum the absolute values of these eigenvalues:

$$|-2| + |2| = 4.$$

Case (II): Let $m = 3$, characteristic polynomial can be written as:

$$f(\lambda) = (\lambda - 2)^6 (\lambda + 2)^9 (\lambda - 6).$$

Setting $f(\lambda) = 0$, the nonzero eigenvalues of the matrix include 2 with has multiplicity 6, -2 with multiplicity 9, and -6 appearing as a single eigenvalue. The energy of $L(G(\mathbb{Z}_8))$ is determined by summing the absolute values of these eigenvalues:

$$9\,|-2\,| + 6\,|\,2\,| + |-6\,| = 36.$$

Case (III): Let $m > 3$. In general, eigenvalues in $L(G(\mathbb{Z}_{2^m}))$ consist of:

$$(2m-4)^{n-2},$$

$$(-2)^{\frac{n^2}{2}-\frac{n}{2}+2m-mn-1},$$

$$n-2.$$

Therefore, energy of the $L(G(\mathbb{Z}_{2^m}))$ is determined by summing the absolute values of its eigenvalues:

$$|(n-2)| \;+\; (n-2)|(2m-4)| \;+\; (\frac{n^2}{2}-\frac{n}{2}+2m-mn-1)|-2| = (n-2)^2.$$

Next, according to Proposition 1, energy of $L(G(\mathbb{Z}_4))$ is determined.

Proposition 1. Energy of the $L(G(\mathbb{Z}_4))$ is calculated as 4.

Proof. From Example 3 the adjacencies between the vertices of $L(G(\mathbb{Z}_4))$ can be observed.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Let the characteristic polynomial be $f(\lambda) = \det(\lambda I - A)$.

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & -1 & 0 \\ -1 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & -1 \\ 0 & -1 & -1 & \lambda \end{bmatrix}$$

Thus,

$f(\lambda) \;=\; 0 \;=\; (\lambda-2)\,(\lambda+2)\,\lambda\,\lambda,$

$(\lambda-2)\,(\lambda+2)\,\lambda\,\lambda = 0,$

$\lambda = -2, 0, 0, 2.$

Hence,

$E\left(L(G(\mathbb{Z}_4))\right) = |-2| + |2|,$

$E\left(L(G(\mathbb{Z}_4))\right) = 4.$

Theorem 2. The spectrum of line graph of unit graph $L(G(\mathbb{Z}_{2^m})), m > 1$ is given by

$$Spec(L(G(\mathbb{Z}_{2^m}))) = \begin{pmatrix} n-2 & 2m-4 & -2 \\ 1 & n-2 & \frac{n^2}{2}-\frac{n}{2}+2m-mn-1 \end{pmatrix}.$$

Proof. Energy of $L(G(\mathbb{Z}_{2^m})), m \neq 1$ can be easily determined by spectrum,

$$\left(\frac{n^2}{2}-\frac{n}{2}+2m-mn-1\right)\,|-2| + \,|n-2| + \,(n-2)\,|2m-4| \,= (n-2)^2.$$

As a result, $E(L(G(\mathbb{Z}_{2^m}))) = (n-2)^2$.

Theorem 3. Let $n = 6$. The energy of the line graph $L(G(\mathbb{Z}_6))$ is 8, and the nonzero eigenvalues of its adjacent matrix are:

$$Spec(L(G(\mathbb{Z}_6))) = \{2, 1, 1, -1, -1, -2\}.$$

Proof. From Example 4, the adjacencies between the vertices of $L(G(\mathbb{Z}_6))$ can be observed and the adjacency matrix A is defined as:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

To compute the characteristic polynomial $f(\lambda) = \det(\lambda I - A)$ you subtract A from the 6×6 identity matrix scaled by λ, giving:

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & -1 & 0 & 0 & 0 \\ -1 & \lambda & 0 & 0 & -1 & 0 \\ -1 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & \lambda & -1 & -1 \\ 0 & -1 & 0 & -1 & \lambda & 0 \\ 0 & 0 & -1 & -1 & 0 & \lambda \end{bmatrix}$$

The characteristic polynomial of the adjacency matrix of $L(G(\mathbb{Z}_6))$ is given by:

$$f(\lambda) = (\lambda + 1)^2 \, (\lambda + 2) \, (\lambda - 2) \, (\lambda - 1)^2 \, .$$

Assuming $f(\lambda) = 0$, the nonzero eigenvalues of $L(G(\mathbb{Z}_6))$ are 2, 1, 1, -1, -1, -2. The energy of the graph is computed as the sum of the absolute values of its eigenvalues:

$$E\left(L(G(\mathbb{Z}_6))\right) = 2\,|\,1\,|\, + \,|\,2\,|\, + \,|\, - \,2\,|\, + \,2\,|\, - \,1\,|\, = \,8.$$

Theorem 4. For the line graph of the unit graph $L(G(\mathbb{Z}_{2p}))$, the energy is given by the expression

$$E\left(L(G(\mathbb{Z}_\mathrm{n}))\right) = n^2 - 6n + 4,$$

where $n = 2p$, where p is an odd prime, and $p > 3$.
Proof. The spectrum of line graph of unit graph $L(G(\mathbb{Z}_{2p}))$, $p > 3$ is provided by

$$Spec(L(G(\mathbb{Z}_{2\mathrm{p}}))) = \begin{pmatrix} n-4 & q-2 & q-4 & -2 \\ 1 & q-1 & q-1 & p^2 - n - p + 1 \end{pmatrix},$$

where $n = 2p$ and $q = p - 1$.
The energy of the $L(G(\mathbb{Z}_{2p}))$, where $p > 3$, can be easily determined by spectrum,

$$(q-1)\,|\,q-4\,| + (p^2 - n - p + 1)\,|-2\,| + |\,n-4\,| + (q-1)\,|\,q-2\,| \, = \, n^2 - 6n + 4.$$

Evaluating this expression yields:

$$E(L(G(\mathbb{Z}_{2\mathrm{p}}))) = n^2 - 6n + 4.$$

Then, energy of the $L(G(\mathbb{Z}_{10}))$ is then determined, as indicated in Proposition 2.

Proposition 2. Energy of the $L(G(\mathbb{Z}_{10}))$ is calculated as 44.

Proof. The adjacencies between the vertices of $L(G(\mathbb{Z}_{10}))$ can be observed.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

20 × 20 Matrix

Let the characteristic polynomial of the adjacency matrix of $L(G(\mathbb{Z}_{10}))$ is

$$f(\lambda) = (\lambda - 6)(\lambda - 3)^4(\lambda - 1)^4(\lambda + 2)^{11}.$$

The corresponding eigenvalues are $\lambda = 6$, $\lambda = 3$ with multiplicity 4, $\lambda = 1$ with multiplicity 5, and $\lambda = -2$ with multiplicity 11. Therefore, the energy of $L(G(\mathbb{Z}_{10}))$, defined as the sum of the absolute values of its eigenvalues, is calculated as:

$$E\left(L(G(\mathbb{Z}_{10}))\right) = |6| + 4|3| + 4|1| + 11|-2| = 6 + 12 + 4 + 22 = 44.$$

Summary

This study computes and generalizes the eigenvalues, energy, and spectrum of $L(G(\mathbb{Z}_n))$. The analysis considers two cases: $n = 2^m$ (where $m > 1$), $n = 6$, and $n = 2p$, with p is the odd primes, $p > 3$.

These findings reveal distinct patterns and spectral properties for different values of n, contributing to a deeper understanding of graph structures in algebraic settings. The results are summarized in Table 1, offering a concise reference for eigenvalues, energy, and spectrum across cases.

In Table 1, the results of the energy and spectrum of a line graph of unit graph are summarized.

Table 1. Summary for the eigenvalues, energy and spectrum of $L(G(\mathbb{Z}_n))$ for $n = 2^m, m > 1$ and $n = 2p, p$ are odd primes, $p > 3$

n	Eigenvalues	Energy	Spectrum
$n = 2^m$, $m > 1$.	$(2m-4)^{n-2}$ $(-2)^{\frac{n^2}{2}-\frac{n}{2}+2m-mn-1}$ $n-2$	$(\mathrm{n}-2)^2$	$\begin{pmatrix} n-2 & 2m-4 & -2 \\ 1 & n-2 & \frac{n^2}{2}-\frac{n}{2}+2m-mn-1 \end{pmatrix}$
$n = 6$	$2, 1, 1, -1, -1, -2$	8	$\begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}$

n	Eigenvalues	Energy	Spectrum
$n = 2p$, p be odd prime, $p > 3$.	$(p-4)^{(p-1)}$ $(-2)^{p^2-n-p+1}$ $n-4$ $(p-2)^{(p-1)}$	$n^2 - 6n + 4$	$\begin{pmatrix} n-4 & q-2 & q-4 & -2 \\ 1 & q-1 & q-1 & p^2-n-p+1 \end{pmatrix}$

Furthermore, significance in this finding is to find the energy pattern of $L(G(\mathbb{Z}_n))$, how it may be expressed with different n values, and whether it affects its subgraphs.

This paper provides a comprehensive study of the energy and spectral characteristics of $L(G(\mathbb{Z}_n))$. The results contribute to a deeper understanding of graph theory and its applications in mathematical and scientific research. Future work may involve extending these analyses to broader graph classes and exploring real-world applications.

Conflict of Interest

The authors confirm the absence of any competing interest related to this paper's publication.

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Appendix

Maple Coding

In this paper, utilizing Maple software, the eigenvalues of the adjacent matrix are computed.

The Maple coding is shown below.

For $\mathbb{Z}_4$.

The linear algebra command is stated initially.

with(LinearAlgebra):

The coding started with the declaration of matrix A.

$$A := \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Next, λI is generated.

$$\lambda I := \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

Next, $\lambda I - A$ is calculated.

$\lambda I - A$

The result is shown below.

$$\begin{bmatrix} \lambda & -1 & -1 & 0 \\ -1 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & -1 \\ 0 & -1 & -1 & \lambda \end{bmatrix}$$

The eigenvalues of the adjacency matrix are calculated.

Eigenvalues$(\lambda I - A)$

The result is shown below.

$$\begin{bmatrix} -2+\lambda \\ \lambda \\ \lambda \\ 2+\lambda \end{bmatrix}$$

From the result above, $(\lambda+2)\lambda\lambda(\lambda-2)$. Hence, the eigenvalues of matrix A are -2, 0, 0, and 2 for $f(\lambda) = 0$.

Then, the same coding is used for Z_{n}, for all values of n needed.