MJFAS
Malaysian Journal of Fundamental and Applied Sciences

RESEARCH ARTICLE

Laplacian Spectrum and Laplacian Energy of the Zero Divisor Graph for $\mathbb{Z}_{3\alpha}$

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Abstract The Laplacian energy of a graph refers to the total absolute differences between its eigenvalues and the graph's mean degree. These eigenvalues are derived from the Laplacian matrix, L which is defined as a square matrix with entries $L_{ii} = d_i$ (the vertex degree of v_i), $L_{ij} = -1$ if two distinct vertices are adjacent and otherwise $L_{ij} = 0$. The zero divisor graph of a commutative ring R is the graph in which all nonzero zero divisors of the ring form the vertices where two distinct vertices are adjacent if and only if they are commute and equal to zero. This paper constructs the zero divisor graphs of $\mathbb{Z}_{3\alpha}$ where $\alpha = 5, 7, 11, 13$ and 17 by using Python software. Subsequently, the general formula for the Laplacian spectrum and Laplacian energy are derived from the constructed graph, with an example provided to illustrate the main theorems.

Keywords: Laplacian matrix, Laplacian eigenvalues, Laplacian energy, zero divisor graph.

Introduction

Graph energy, introduced by Gutman in 1978, is a concept that links graph theory and chemistry. It is calculated as the sum of the positive eigenvalues of the adjacency matrix [1]. For a long time, graph energy has been a topic of study, and Gutman and Zhou [2] introduced the idea of Laplacian energy (LLE). Since then, a lot of research has been done on LLE. Its uses have been investigated and expanded by numerous researchers, especially in the context of graph theory such as Hameed *et al*. [3] studied the LLE of Laplacian integrals graph and they found the lower and upper bounds for the LLE. Yalçın [4] determined the LLE of r-Uniform Hypergraphs, with the derived bounds being influenced by key hypergraph properties such as degree and pair-degree. Dsouza *et al*. [5] introduced the LLE of partial complement of a graph and investigated properties associated with their partial complement Laplacian eigenvalues. Furthermore, Mutlu Varlıoğlu and Büyükköse [6] explored on LLE of the power graphs of finite cyclic groups and Bhat *et al*. [7] derived general formulas for the signless Laplacian energy and Laplacian energy of orbit graphs for the dihedral and quaternion groups.

Moreover, previous scholars have studied rings and graphs extensively [8-12]. This paper considers the zero divisor graph (ZEDG) as the main graph. Beck [13] initially introduced the idea of ZEDG of a commutative ring where he focuses on colorization of commutative rings in 1988 and the ZEDG is defined as the graph which the vertices are all elements of the ring. In 1999, Anderson and Livingston [14] subsequently presented an updated version of the ZEDG of commutative rings. The definition given forward by Anderson and Livingston will be used in this paper.

Numerous researchers have shown interest in the research of graph of rings. For instance, Rather *et al*.

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Received: 27 March 2024
Accepted: 3 Oct. 2025

[15] looked into the distance signless Laplacian eigenvalues of ZEDG of commutative rings, whereas Mönius [16] examined the eigenvalues and adjacency matrices of ZEDG of finite commutative rings. In 2023, Mondal *et al*. [17] computed various topological indices for the ZEDG of commutative rings and in the same year also Semil *et al*. [10] established a general formula to compute the first Zagreb index of the ZEDG associated with the commutative ring $\mathbf{Z}_{p^k}$. Furthermore, a general formula for determining the number of zero divisors of the finite ring was discovered by Zaid *et al*. [18]. Kumar and Prakash [19] studied the Roman domination number of zero divisor graphs, providing bounds and generalizations for various ring structures and Pirzada and Altaf [20] analyzed cliques in extended zero divisor graphs, identifying rings for cliques up to order six.

In this paper, the ZEDG of $\mathbf{Z}_{3\alpha}$ for $\alpha = 5,7,11,13$ and 17 are constructed by using Python software. After that, based on the ZEDG, the general formulas for the Laplacian spectrum (LSpect) and LLE for $\mathbf{Z}_{3\alpha}$ are determined.

Preliminaries

This section presents several fundamental ideas and definitions related to graph theory, ring theory and energy that are used to find the main results.

Definition 1 [21] Zero Divisors of a Ring
When two nonzero elements a and b in a ring multiply together to produce zero, they are referred as zero divisors. The zero divisors of ring R are collectively expressed as $Z(R)$.

The following definition states the ZEDG of commutative rings, which will be the main focus of this paper. The construction of this graph is needed, in order to determine the Laplacian matrix.

Definition 2 [14] Zero Divisor Graph (ZEDG) of Commutative Rings
The ZEDG of a commutative ring R, $\Gamma(Z(R))$, is the graph where the vertices are all elements of zero divisors of R and two distinct vertices a and b are adjacent if and only if $ab = ba = 0$.

Definition 3 [22] Vertex Degree
The edges' number incident to the vertex t is referred to its degree, denoted as $\mathrm{d}(t)$.

To compute the LSpect of the ZEDG of the commutative rings, the Laplacian matrix must first be determined. Then, its eigenvalues are obtained from the characteristic polynomial. The following definition defines the Laplacian matrix of a graph.

Definition 4 [23] Laplacian Matrix
The Laplacian matrix, L of a graph G with vertex set $\{t_1, t_2, ..., t_n\}$ is defined as $L_{ii} = d_i$ (the degree of vertex t_i), $L_{ij} = -1$ if t_i and t_j are adjacent where $i, j = \{1, 2, ..., n\}$ which n is the total number of vertices in the graph , and $L_{ij} = 0$ otherwise.

The following is an example demonstrates the Laplacian matrix of a simple graph.

Example 1 A graph Γ that consists of a set of vertices of $\{G, H, I, J, K, L\}$ is shown in Figure 1 and its Laplacian matrix, $L(\Gamma)$ is determined.

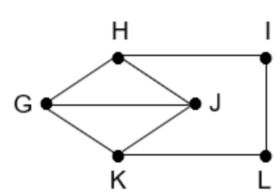

Figure 1 A simple graph consist of six vertices and eight edges

Based on the graph Γ in Figure 1, the vertices $\{G,H,I,J,K,L\}$ are used to label the columns and rows of the Laplacian matrix $L(\Gamma)$. By Definition 4, for any adjacent vertices t_i and t_j, the element at the intersection of column j and row i (as well as at the symmetric position, column i and row j) in $L(\Gamma)$ is -1. Diagonal entries correspond to the degree of the vertex t_i, while entries are 0 if t_i and t_j are not adjacent. Then, $\mathrm{d}(G)=3, \mathrm{d}(H)=3, \mathrm{d}(I)=2, \mathrm{d}(J)=3, \mathrm{d}(K)=3$ and $\mathrm{d}(L)=2$. Therefore, the Laplacian matrix is as follows:

$$L(\Gamma)=\begin{array}{c} \\ G \\ H \\ I \\ J \\ K \\ L \end{array}\begin{array}{c} \begin{array}{cccccc} G & H & I & J & K & L \end{array} \\ \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

Definition 5 [24] Block Matrix
A block matrix is a matrix divided into smaller submatrices, known as blocks. It can be represented as

$$T=\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C and D are individual submatrices forming the larger matrix. A block matrix is created by dividing a matrix horizontally and vertically, resulting in four distinct sections, each considered a block.

Definition 6 [22] Characteristic Polynomial
Let B be an $n \times n$ matrix. The characteristic polynomial of B is given by $\det(B-\lambda I)$, which is a polynomial of degree n in the complex variable λ. The characteristic equation of B is defined as $\det(B-\lambda I)=0$.

Definition 7 [23] Laplacian Eigenvalues
The graph's Laplacian eigenvalues are the eigenvalues of the Laplacian matrix, L. The eigenvalues are always non-negative, with the smallest eigenvalue being 0. The set of the eigenvalues is also known as the matrix's spectrum.

Example 2 illustrates the characteristic polynomial of the Laplacian matrix that has been found in Example 1. Its eigenvalues are also determined in the example.

Example 2 Refer to the Laplacian matrix presented in Example 1. The characteristic polynomial is calculated using the Laplacian matrix, as stated below.

$$\begin{aligned} f(L,\lambda) &= \det(L-\lambda I) \\ &= \begin{vmatrix} 3-\lambda & -1 & 0 & -1 & -1 & 0 \\ -1 & 3-\lambda & -1 & -1 & 0 & 0 \\ 0 & -1 & 2-\lambda & 0 & 0 & -1 \\ -1 & -1 & 0 & 3-\lambda & -1 & 0 \\ -1 & 0 & 0 & -1 & 3-\lambda & -1 \\ 0 & 0 & -1 & 0 & -1 & 2-\lambda \end{vmatrix} \\ &= \lambda^6-16\lambda^5+98\lambda^4-284\lambda^3+384\lambda^2-192\lambda \end{aligned}$$

From the characteristic equation $\det(L-\lambda I)=0$, the eigenvalues are obtained as $\lambda_1=0, \lambda_2=1.27, \lambda_3=2, \lambda_4-\lambda_5=4, \lambda_6=4.73$. Therefore, the LSpect is $\{0,1.27,2,4,4,4.73\}$.

Definition 8 [2] Laplacian Energy

The LLE of a graph, Γ is the sum of the absolute differences between its eigenvalues and the graph's average degree. For a graph with n vertices, the Laplacian matrix's eigenvalues are $\lambda_1, \lambda_2, ..., \lambda_n$, and $\bar{r}$ represents the graph's average degree. The LLE of a graph is calculated as:

$$LLE(\Gamma) = \sum_{i=1}^{n} \left| \lambda_i - \bar{r} \right|,$$

where $\bar{r} = \frac{2m}{n}$ and m represents the number of edges of the graph.

Next, the concept of block matrix is defined in Lemma 1 and it is used to prove the main theorem.

Lemma 1 [25] Let U, V, W and X be matrices, with U being an invertible. The block matrix T is defined as

$$T = \begin{bmatrix} U & V \\ W & X \end{bmatrix}.$$

The determinant of T can be expressed as

$$\det(T) = \det(U) \cdot \det(X - WU^{-1}V),$$

where the term $X - WU^{-1}V$ is called the Schur complement of U in T.

The Python Syntax for Constructing the Zero Divisor Graph

In this section, the Python syntax to construct the ZEDG of commutative ring $\mathbf{Z}_{3^\alpha}$ for $\alpha = 5, 7, 11, 13$ and 17 is presented as follows.

```
import networkx as nx
import matplotlib.pyplot as plt
import math

# Function to calculate gcd
def gcd(a, b):
    while b:
        a, b = b, a % b
    return a

# Function to find zero divisors in Z_n
def zero_divisors(n):
    return [a for a in range(1, n) if gcd(a, n) > 1]

# Function to create the zero divisor graph
def create_zero_divisor_graph(n):
    G = nx.Graph()
    zero_divs = zero_divisors(n)

    # Add vertices (zero divisors)
    G.add_nodes_from(zero_divs)

    # Add edges
    for i in range(len(zero_divs)):
        for j in range(i + 1, len(zero_divs)):
            if (zero_divs[i] * zero_divs[j]) % n == 0:
                G.add_edge(zero_divs[i], zero_divs[j])

    return G
```

```
# Main program
n = 3*q
G = create_zero_divisor_graph(n)

# Draw the graph
plt.figure(figsize=(4, 4))
nx.draw(G, with_labels=True, node_color='lightblue', node_size=250,
        font_size=10, font_weight='bold', edge_color='black')
plt.title(f"Zero Divisor Graph of Z_{n}", fontsize=10)
plt.show()
```

The Python outputs are shown in Figure 2 until Figure 6 for $\mathbf{Z}_{3\alpha}$ where $\alpha = 5,7,11,13,17$.

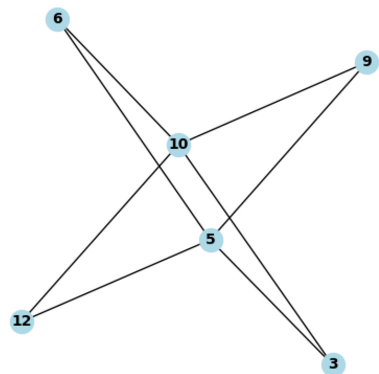

Figure 2 Python output for $\mathbf{Z}_{15}$

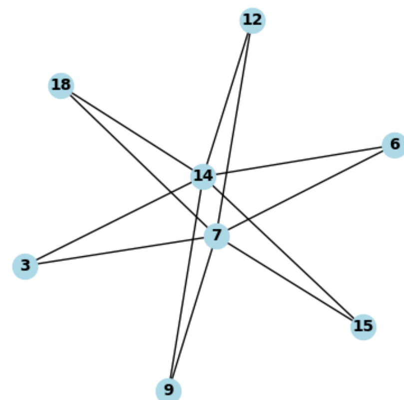

Figure 3 Python output for $\mathbf{Z}_{21}$

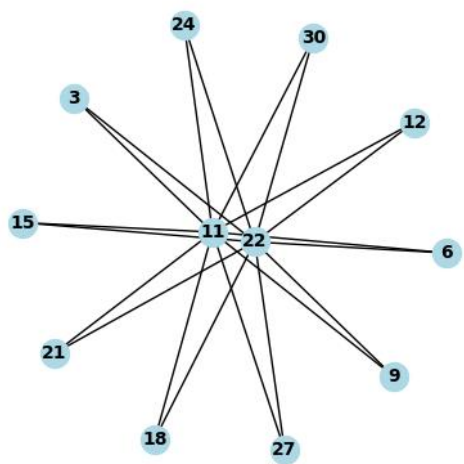

Figure 4 Python output for $\mathbf{Z}_{33}$

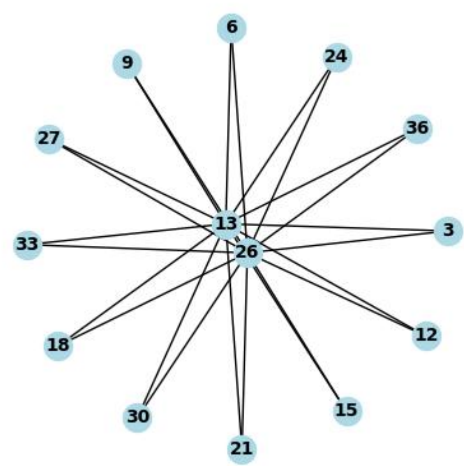

Figure 5 Python output for $\mathbf{Z}_{39}$

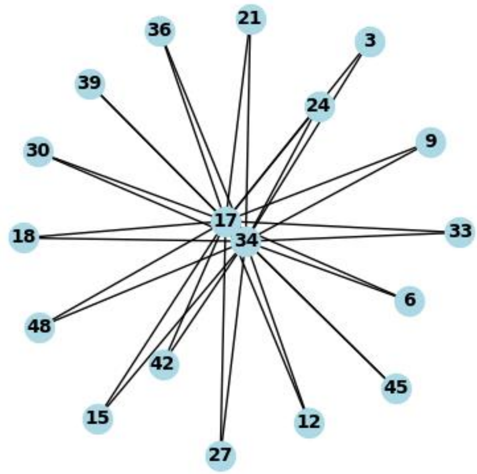

Figure 6 Python output for $\mathbf{Z}_{51}$

Main Results

In this section, the general formula of LSpect and LLE of the ZEDG for $\mathbf{Z}_{3\alpha}$ where $\alpha = 5, 7, 11, 13$ and 17 are presented. The examples are also given at the last part of the section to illustrate the theorems.

Theorem 1 shows the general formula of LSpect of the ZEDG for $\mathbf{Z}_{3\alpha}$ where it is proved by using the concept of block matrix and determinant of a square matrix which both concepts are presented in Lemma 1 and Lemma 2, respectively.

Lemma 2 Let $n \times n$ matrix be defined as $M = \begin{bmatrix} f & g & \cdots & g \\ g & f & \cdots & g \\ \vdots & \vdots & \ddots & \vdots \\ g & g & \cdots & f \end{bmatrix}$ where n is the total number of eigenvalues. The determinant of M is $\det(M) = (f + g(n-1))(f-g)^{n-1}$.

Proof Let I and J be $n \times n$ matrix. Matrix M can be expressed as $M = fI + g(J - I)$, where I is the identity matrix and J is the matrix of all ones. The eigenvalue of matrix J are n (with multiplicity 1) and 0 (with multiplicity $n-1$). Subtracting the eigenvalues of I which is 1 from J resulting in $n-1$ and -1 (with multiplicity $n-1$). Thus, for the eigenvalue $n-1$, substituting it into $M = f\lambda_I + g\lambda_{(J-I)}$ gives $\lambda_1 = f + g(n-1)$ and for eigenvalue -1, will be $\lambda_2 = f - g$. Since the matrix's determinant is the product of its eigenvalues, hence, $\det(M) = \lambda_1 \cdot \lambda_2^{\,n-1} = (f + g(n-1))(f-g)^{n-1}$.

Theorem 1 Let Γ be the ZEDG for commutative ring $\mathbf{Z}_{3\alpha}$. The LSpect of $\Gamma(\mathbf{Z}_{3\alpha})$ is $\left\{(0),(2)^{\alpha-2},(\alpha-1),(\alpha+1)\right\}$ where α is a prime and greater than three.

Proof Let the zero divisors of $\Gamma(\mathbf{Z}_{3\alpha})$ be divided into two sets: $R = \{3, 6, \ldots, 3(\alpha-1)\}$, the $\alpha-1$ multiples of 3, and $S = \{\alpha, 2\alpha\}$, the two multiples of α. In $\Gamma(\mathbf{Z}_{3\alpha})$, each vertex in S is adjacent to all vertices in R, and each vertex in R is adjacent to both vertices in S, forming a complete bipartite graph $K_{2,\alpha-1}$. The Laplacian matrix of $\Gamma(\mathbf{Z}_{3\alpha})$ is

$$L=\begin{bmatrix}(\alpha-1) & 0 & -1 & -1 & \cdots & -1\\ 0 & (\alpha-1) & -1 & -1 & \cdots & -1\\ -1 & -1 & 2 & 0 & \cdots & 0\\ -1 & -1 & 0 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & 0 & 0 & \cdots & 2\end{bmatrix}.$$

Consequently, to find the spectrum, the Laplacian matrix L is further structured into a block matrix as mentioned in Lemma 1,

$$L=\begin{bmatrix}(\alpha-1)I_2 & -J_{2,\alpha-1}\\ -J_{\alpha-1,2} & 2I_{\alpha-1}\end{bmatrix}$$

where J is a matrix of all ones with order $2\times(\alpha-1)$ and $(\alpha-1)\times 2$. To compute the determinant, let

$$L-\lambda I=\begin{bmatrix}(\alpha-1-\lambda)I_2 & -J_{2,\alpha-1}\\ -J_{\alpha-1,2} & (2-\lambda)I_{\alpha-1}\end{bmatrix}$$

By Lemma 1, $\det(L-\lambda I)=\det((\alpha-1-\lambda)I_2)\cdot\det\left((2-\lambda)I_{\alpha-1}-\frac{2}{\alpha-1-\lambda}J_{\alpha-1}\right)$, where

$$\det\left((\alpha-1-\lambda)I_2\right)=0$$
$$\lambda=\alpha-1,$$

For $\det\left((2-\lambda)I_{\alpha-1}-\frac{2}{\alpha-1-\lambda}J_{\alpha-1}\right)=0$, can be written as

$$\begin{vmatrix}2-\lambda-\frac{2}{\alpha-1-\lambda} & -\frac{2}{\alpha-1-\lambda} & \cdots & -\frac{2}{\alpha-1-\lambda}\\ -\frac{2}{\alpha-1-\lambda} & 2-\lambda-\frac{2}{\alpha-1-\lambda} & \cdots & -\frac{2}{\alpha-1-\lambda}\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{2}{\alpha-1-\lambda} & -\frac{2}{\alpha-1-\lambda} & \cdots & 2-\lambda-\frac{2}{\alpha-1-\lambda}\end{vmatrix}=0.$$

By Lemma 2, $n=\alpha-1$ since the order of the matrix is $(\alpha-1)\times(\alpha-1)$. Hence,

$$\left(2-\lambda-\frac{2}{\alpha-1-\lambda}(\alpha-2)\right)=0 \tag{1}$$

$$\left(2-\lambda-\frac{2}{\alpha-1-\lambda}+\frac{2}{\alpha-1-\lambda}\right)^{\alpha-2}=0 \tag{2}$$

From **(1)**,

$$\left(2-\lambda-\frac{2}{\alpha-1-\lambda}(\alpha-2)\right)=0$$
$$2-\lambda+\frac{(2-2\alpha)}{\alpha-1-\lambda}=0$$
$$(2-\lambda)(\alpha-1-\lambda)=2\alpha-2$$
$$\lambda^2-(1+\alpha)\lambda=0$$
$$\lambda\left[\lambda-(1+\alpha)\right]=0$$
$$\lambda=0$$
$$\lambda=\alpha+1$$

From **(2)**,

$$\left(2-\lambda-\frac{2}{\alpha-1-\lambda}+\frac{2}{\alpha-1-\lambda}\right)^{\alpha-2}=0$$
$$(2-\lambda)^{\alpha-2}=0$$
$$\lambda=2$$

Therefore, the LSpect of $\Gamma(\mathbf{Z}_{3\alpha})$ has eigenvalues of 0 (with multiplicity 1), 2 (with multiplicity $\alpha-2$), $\alpha-1$ and $\alpha+1$. Thus, the LSpect of $\Gamma(\mathbf{Z}_{3\alpha})$ is $\left\{(0),(2)^{\alpha-2},(\alpha-1),(\alpha+1)\right\}$.

Consequently, the theorem for general formula of LLE of the ZEDG for $\mathbf{Z}_{3\alpha}$ is computed from Theorem 1.

Theorem 2 Let Γ be the ZEDG for commutative ring $\mathbf{Z}_{3\alpha}$. The LLE of $\Gamma(\mathbf{Z}_{3\alpha})$ is $\frac{4(\alpha^2-3\alpha+4)}{1+\alpha}$ where α are primes and greater than three.

Proof By Definition 8 and Theorem 1, for $K_{2,\alpha-1}$, $m=2(\alpha-1)$ and $n=2+(\alpha-1)$, hence,

$$LLE(\Gamma(\mathbf{Z}_{3\alpha}))=\left|0-\frac{2(2\alpha-2)}{1+\alpha}\right|+\left|2-\frac{2(2\alpha-2)}{1+\alpha}\right|^{\alpha-2}+\left|(\alpha-1)-\frac{2(2\alpha-2)}{1+\alpha}\right|+\left|(\alpha+1)-\frac{2(2\alpha-2)}{1+\alpha}\right|$$
$$=\frac{4\alpha-4}{1+\alpha}+\left(\frac{4\alpha-4}{1+\alpha}-2\right)(\alpha-2)+\left(\frac{(\alpha-1)(1+\alpha)-(4\alpha-4)}{1+\alpha}\right)+\left(\frac{(\alpha+1)(1+\alpha)-(4\alpha-4)}{1+\alpha}\right)$$
$$=\frac{4\alpha-4}{1+\alpha}+\left(\frac{\alpha^2-4\alpha+3}{1+\alpha}\right)+\left(\frac{\alpha^2-2\alpha+5}{1+\alpha}\right)+\left(\frac{2\alpha-6}{1+\alpha}\right)(\alpha-2)$$
$$=\frac{2\alpha^2-2\alpha+4}{1+\alpha}+\left(\frac{(2\alpha-6)(\alpha-2)}{1+\alpha}\right)$$
$$=\frac{4\alpha^2-12\alpha+16}{1+\alpha}$$
$$=\frac{4\left(\alpha^2-3\alpha+4\right)}{1+\alpha}.$$

Example 3 Let Γ be the ZEDG of $\mathbf{Z}_{3(5)}$ that is $\mathbf{Z}_{15}$. Then, from Figure 2 and Definition 4, the Laplacian matrix of ZEDG for commutative ring of order 15 is

$$L(\Gamma)=\begin{bmatrix}2 & -1 & 0 & 0 & -1 & 0\\ -1 & 4 & -1 & -1 & 0 & -1\\ 0 & -1 & 2 & 0 & -1 & 0\\ 0 & -1 & 0 & 2 & -1 & 0\\ -1 & 0 & -1 & -1 & 4 & -1\\ 0 & -1 & 0 & 0 & -1 & 2\end{bmatrix}$$

By Definition 6, the characteristic polynomial is found as follows.

$$f(L,\lambda)=\det(L-\lambda I)$$
$$=\begin{vmatrix} 2-\lambda & -1 & 0 & 0 & -1 & 0 \\ -1 & 4-\lambda & -1 & -1 & 0 & -1 \\ 0 & -1 & 2-\lambda & 0 & -1 & 0 \\ 0 & -1 & 0 & 2-\lambda & -1 & 0 \\ -1 & 0 & -1 & -1 & 4-\lambda & -1 \\ 0 & -1 & 0 & 0 & -1 & 2-\lambda \end{vmatrix}$$
$$=\lambda^6-16\lambda^5+96\lambda^4-272\lambda^3+368\lambda^2-192\lambda$$

From the characteristic equation $\det(L-\lambda I)=0$, the eigenvalues are obtained as $\lambda_1=0,\lambda_2=2,\lambda_3=2,\lambda_4=2,\lambda_5=4$ and $\lambda_6=6$. Therefore, the LSpect is $\{0,2,2,2,4,6\}$ and by Definition 8, $m=8$, $n=6$, the LLE is

$$LLE(\Gamma(\mathbf{Z}_{15}))=\left|0-\frac{8}{3}\right|+\left|2-\frac{8}{3}\right|+\left|2-\frac{8}{3}\right|+\left|2-\frac{8}{3}\right|+\left|4-\frac{8}{3}\right|+\left|6-\frac{8}{3}\right|$$
$$=9.3333.$$

By Theorem 1 and Theorem 2, for $q=5$, the LSpect is $\left\{(0),(2)^3,(4),(6)\right\}$ and LLE is $\frac{4\left(5^2-3(5)+4\right)}{1+5}=9.3333$ which same as manually calculated.

Example 4 Let Γ be the ZEDG of $\mathbf{Z}_{3(7)}$ that is $\mathbf{Z}_{21}$. Then, from Figure 3 and Definition 6, the Laplacian matrix of ZEDG for commutative ring of order 21 is

$$L(\Gamma)=\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 6 & -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 6 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

By Definition 6, the characteristic polynomial is found as follows.

$$f(L,\lambda)=\det(L-\lambda I)$$
$$=\begin{vmatrix} 2-\lambda & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2-\lambda & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 6-\lambda & -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2-\lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2-\lambda & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 6-\lambda & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 2-\lambda & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 2-\lambda \end{vmatrix}$$
$$=\lambda^8-24\lambda^7+228\lambda^6-1120\lambda^5+3120\lambda^4-4992\lambda^3+4288\lambda^2-1536\lambda$$

From the characteristic equation $\det(L-\lambda I)=0$, the eigenvalues are obtained as $\lambda_1=0,\lambda_2=2,\lambda_3=2,\lambda_4=2,\lambda_5=2,\lambda_6=2,\lambda_7=6$ and $\lambda_8=8$. Therefore, the LSpect is $\{0,2,2,2,2,2,6,8\}$ and by Definition 8, $m=12$, $n=8$, the LLE is

$$\begin{aligned} LE(\Gamma(\mathbf{Z}_{21})) &= |0-3|+|2-3|+|2-3|+|2-3|+|2-3|+|2-3|+|6-3|+|8-3| \\ &= 16. \end{aligned}$$

By Theorem 1 and Theorem 2, for $q=7$, the LSpect is $\{(0),(2)^5,(6),(8)\}$ and LLE is $\frac{4\left(7^2-3(7)+4\right)}{1+7}=16$ which same as manually calculated.

Conclusion

In this paper, the Python syntax is applied to construct the ZEDG of $\mathbf{Z}_{3\alpha}$, which provides a computational approach to visualize and verify the graph structure. Then, the general formulas of LSpect and LLE of zero divisor graph for commutative ring $\mathbf{Z}_{3\alpha}$ are determined. Subsequently, the examples to illustrate the theorems are also stated. In future research, the LSpect and LLE of the zero divisor graph for a commutative ring $\mathbf{Z}_{\alpha\beta}$ can be generalized. In addition. Other types of energies can also be applied to the study.

Conflicts of Interest

The authors of this research state that they have no conflicts of interest with regard to this publication.

Acknowledgment

The Ministry of Higher Education Malaysia (MOHE) provided funding for this project under Fundamental Research Grant Scheme - Early Career Researcher (FRGS-EC/1/2024/STG06/UITM/02/16) and the first author would like to thank the MOHE for the scholarship under MyBrain 2.0.

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