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RESEARCH ARTICLE

Enhancing Average Run Length Efficiency of the Exponentially Weighted Moving Average Control Chart under the SAR(1)$_L$ Model with Quadratic Trend

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Abstract This study proposes an explicit formula for finding the Average Run Length (ARL) of Exponentially Weighted Moving Average (EWMA) control charts when applied to seasonal autoregressive processes with a quadratic trend. The ARL values derived from the proposed explicit formula were evaluated for accuracy by comparing them with results from the numerical integral equation (NIE) approach utilizing the Midpoint rule. These methods were assessed using real-world applications in the medical field, along with comprehensive simulations. The results demonstrate that the proposed explicit formula and the NIE method closely align in terms of accuracy, with the explicit formula significantly enhancing computational efficiency compared to the NIE method. These findings suggest that the explicit formula is an effective tool for enhancing control chart performance across multiple disciplines.

Keywords: Average run length, explicit formulas, numerical integral equation, Banach's fixed-point theorem.

Introduction

Statistical process control is an essential technique for effective process monitoring, as it utilizes control charts to ensure consistency and quality. The Shewhart control chart, developed by Shewhart [1] in 1931, is the first recognized control chart for monitoring processes. In 1954, Page [2] introduced the Cumulative Sum (CUSUM) control chart, representing a major advancement in the field. The CUSUM chart is particularly effective at detecting small shifts in the process mean, offering greater sensitivity compared to the Shewhart chart. Similarly, the Exponentially Weighted Moving Average (EWMA) control chart, first developed by Roberts in 1959 [3], represents another important innovation. The EWMA chart provides robust capabilities for identifying gradual changes in process parameters. The objective of this research is to manage process variation through the application of statistical quality control tools. Traditionally, the effectiveness of control chart studies assumes that data follows a normal distribution. However, in many real-world applications, data often exhibits time-series characteristics.

The Average Run Length (ARL) is an important metric for assessing how effectively control charts detect shifts in a process. It indicates the average number of subgroups that need to be monitored before the process signals that it is out of control. ARL consists of two components: in-control ARL (ARL_0) and out-of-control ARL (ARL_1). Various methods have been proposed for estimating the ARL, including Monte Carlo simulations (MC), the Markov Chain approach (MCA), and the Integral Equation (IE) approach. While the MC technique is useful for validating analytical results, it often requires significant computational time. Roberts [3] utilized the MC method to compute the ARL for the EWMA control chart. Crowder [4] introduced a numerical procedure for calculating run length in an EWMA control chart based on the normal distribution, extending the approach to non-normal cases and one-sided EWMA charts. Lucas and Saccucci (1990) examined the EWMA control scheme for monitoring process means and proposed an MCA-based design procedure, including parameter values for detecting small shifts [5]. However, due to the limitations of the MC and MCA methods, researchers have explored alternative approaches, such

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as the integral equation method. Areepong [6] proposed analytical solutions for the average delay (AD) and ARL in EWMA control chart for exponential distribution observations. More recently, Mititelu *et al*. [7] applied the Fredholm integral equation method to derive explicit formulas for ARL in specialized control charts, including CUSUM and EWMA charts, offering a more computationally efficient alternative. The assumption of independent and identically distributed observations is typically the basis of control charts. But when processes show signs of autocorrelation, certain control charts are required. We can assess the ARL for these control charts using Integral Equation (IE) techniques. In 1997, Vanbrackle and Reynolds [8] proposed using the IE approach to determine the ARL of EWMA and CUSUM control charts for an AR(1) model with additional random errors. Subsequently, Busaba *et al*. [9] provided analytical ARL solutions for the CUSUM control chart under stationary AR(1) models. Petcharat *et al*. [10] further developed explicit ARL formulas for both EWMA and CUSUM control charts using a Moving Average (MA) model. Sunthornwat *et al*. [11] compared analytical and numerical ARL calculations for EWMA and CUSUM control charts. They introduced a method to estimate optimal parameters for EWMA and AFRIMA processes, demonstrating that analytical EWMA ARL is a viable alternative for evaluating chart efficiency due to its strong performance. Later, Phanyaem [12] formulated explicit and numerical integral equation (NIE) approaches to compute the ARL for the CUSUM control chart based on the SARX(P,r)$_L$ model. Their study employed the Fredholm integral equation alongside numerical techniques such as the midpoint rule, trapezoidal rule, Simpson's rule, and the Gaussian rule to approximate ARL values. Recently, Petcharat [13] proposed an explicit formula for the ARL of a CUSUM control chart to monitor the SAR(P)$_L$ model with a trend process, utilizing Banach's fixed-point theorem to ensure the existence and uniqueness of the solution. Furthermore, Peerajit [14] further compares analytical integral equations for ARL, derived from Banach's fixed-point theorem, with the NIE method for a fractionally integrated moving average with exogenous variables (FIMAX) model, where the underlying process follows exponential white noise. Sunthornwat *et al*. [15] propose an explicit formula for the ARL of an autoregressive process with a quadratic trend on a modified EWMA control chart, and compare its performance with that of the numerical integral equation method.

This paper presents an explicit formula for the ARL of EWMA control chart under the seasonal autoregressive model of order 1 with quadratic trend. This represents a novel contribution to the field, as it has not been previously explored. The ARL derived from the proposed method is compared with results obtained using numerical integral equation approaches. The structure of this paper is organized as follows: Section 2 provides a detailed description of the materials used in the study. Sections 3 and 4 outline the methodologies employed in the research. Section 5 describes the procedure for calculating the ARL. Section 6 presents the results obtained from the proposed method, while Section 7 offers concluding remarks and insights.

Materials and Methods

This section explains the seasonal autoregressive model of order 1 with a quadratic trend (SAR(1)$_L$). This model is used in the EWMA control chart for monitoring shifts in process mean. The final subsection assesses the ARL properties, which are important for evaluating the control chart's performance.

The Seasonal Autoregressive Model with Quadratic Trend

The SAR(1)$_L$ model with quadratic trend is a time series model that integrates autoregressive components with seasonality and a quadratic trend. The SAR(1)$_L$ model with a quadratic trend can be generalized as

$$Y_t = \mu + \phi Y_{t-L} + \beta_1 t + \beta_2 t^2 + \varepsilon_t, t = 1,2,\ldots \quad (1)$$

where μ is a constant, the initial values of Y_t are represented by Y_{t-L}, ϕ refers to the autoregressive coefficient parameters, β_1 is the linear coefficient parameters, β_2 is the quadratic coefficient parameters, ε_t is a white noise process assumed to be exponentially distributed.

The EWMA Control Chart

The EWMA control chart, introduced by Robert [3], is recognized for its superiority over the Shewhart control chart in detecting small to medium shifts in the process mean. The most common type of the EWMA control chart is based on a sequence of data.

$$E_t = (1-\lambda)E_{t-1} + \lambda Y_t; t = 1,2,\ldots, \quad (2)$$

where E_t is the EWMA statistic, $Y_t; t = 1,2,\ldots$ is the sequence of the SAR(1)$_L$ model with quadratic trend, and λ is an exponential smoothing parameter with of EWMA control chart with $0 < \lambda < 1.$

The upper and lower control limit of the EWMA control chart are defined as UCL and $LCL,$ respectively.

$$UCL = \mu_0 + L_{EWMA}\sqrt{\sigma_0^2 \frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2t})},$$

$$LCL = \mu_0 - L_{EWMA}\sqrt{\sigma_0^2 \frac{\lambda}{2-\lambda}(1-(1-\lambda)^{2t})},$$

whereμ_0 and σ_0^2 are the mean and variance of the in-control process, respectively.

In an EWMA control chart, the width coefficient of control chart (L_{EWMA}) is determined by the choice of λ and the target ARL0. An out-of-control signal occurs whenever the statistics $E_t$exceeds the UCLor falls below the LCL. EWMA control chart stopping time definition:

$$\tau_b = inf\{t > 0; E_t > b\},$$

where b represents the upper control limit. The ARL for the SAR(1)L model with quadratic trend with the initial value of the EWMA statistic;$E_0 = u$ is defined under the density function with a change-point at $\theta < \infty$, denoted by $\mathbb{E}_\infty(.)$.

$$ARL = H(u) = \mathbb{E}_\infty(\tau_b) < \infty.$$

Methodological of Explicit Formulas for ARL

This section identifie$H(u)$ as the ARL function for EWMA control chart under SAR(1)L model with quadratic trend. Analytical expressions are derived from the Fredholm integral equation of second kind. The $UCL$and$LCL$of the EWMA control chart are set to 0 and b, respectively. Let $\mathbb{P}_E$represent the probability measure and $\mathbb{E}_E$represent the expectation corresponding to E_0. The function $H(u)$ represented in the form of Fredholm integral equation of the second kind, is expressed as follows:

$$H(u) = 1 + \mathbb{E}_E[I\{0 < E_1 < b\}H(E_1)] + \mathbb{P}_E\{E_1 = 0\}H(0).$$

Consider the EWMA statistics,E_1under the assumption that the process is in-control state. It can be expressed as follows:

$$0 \le (1-\lambda)u + \lambda\mu + \lambda\phi Y_{t-L} + \lambda\beta_1 t + \lambda\beta_2 t^2 \le b. \quad (3)$$

In a situation where the process is out-of-control, it can be expressed as follows:

$$(1-\lambda)u + \lambda\mu + \lambda\phi Y_{t-L} + \lambda\beta_1 t + \lambda\beta_2 t^2 > b \text{ or } (1-\lambda)u + \lambda\mu + \lambda\phi Y_{t-L} + \lambda\beta_1 t + \lambda\beta_2 t^2 < 0.$$

In this paper, we are interested in the$H(u)$ function to the following Champ and Rigdon's method [16].

$$H(u) = 1 + \int_0^b H(E_1) f(\varepsilon_1) d\,\varepsilon_1. \quad (4)$$

Let the variable $z$be defined as $z = (1-\lambda)u + \lambda\mu + \lambda\phi Y_{t-L} + \lambda\beta_1 t + \lambda\beta_2 t^2$. Equation (4) can be transformed into Equation (5) by changing the integration variable.

$$H(u) = 1 + \int_0^b H\{(1-\lambda)u + \lambda\mu + \lambda\phi Y_{t-L} + \lambda\beta_1 t + \lambda\beta_2 t^2\} f(z) dz.$$

$$= 1 + \frac{1}{\lambda}\int_0^b H(z)\, f\left(\frac{z-(1-\lambda)u}{\lambda} - \mu - \phi Y_{t-L} - \beta_1 t - \beta_2 t^2\right) dz. \quad (5)$$

In order to obtain the following integral equation:

$$H(u) = 1 + \frac{1}{\lambda\alpha}\int_0^b H(z)\, e^{-\frac{z}{\lambda\alpha}}.\, e^{\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\mu + \phi Y_{t-L} + \beta_1 t + \beta_2 t^2}{\alpha}} dz. \quad (6)$$

Proofs of Existence and Uniqueness

In this section, we prove Banach's Fixed Point Theorem, which guarantees the existence and uniqueness of the solutions to the integral equation.

Definition 1. A **fixed point** of a mapping $T: X \to X$ of a set Xinto itself is a $x \in X$which is mapped onto itself, that is $Tx = x$.

Definition 2. Let (X, d) be a metric space. A mapping $T: X \to X$ is called a **contraction** on X if there exists a positive constant $q < 1$ such that $\|T(H_1) - T(H_2)\|_\infty \le q\|H_1 - H_2\|$ for all $H_1, H_2 \in X$.

Theorem 1: Banach's Fixed Point Theorem

Let (X, d) be a metric space and let $T: X \to X$ be a contraction mapping, meaning there exists a constant $0 \le k < 1$ such that

$$d(T(H_1(u)), T(H_2(u))) \le kd(H_1(u), H_2(u)) \text{ for all } H_1(u), H_2(u) \in X.$$

Then the operator Thas a unique fixed point $H(u) \in X$such that

$$T(H(u)) = H(u).$$

Proof

We consider the integral equation

$$H(u) = 1 + \frac{1}{\lambda\alpha}\int_0^b H(z)\, e^{-\frac{z}{\lambda\alpha}}.\, e^{\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\mu + \phi Y_{t-L} + \beta_1 t + \beta_2 t^2}{\alpha}} dz.$$

Step 1: Define an operator T working on functions H as follows:

$$T(H(u)) = 1 + \frac{1}{\lambda\alpha}\int_0^b H(z)\,K(u,z)dz,$$

where $K(u,z) = e^{-\frac{z}{\lambda\alpha}}e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}$.

Step 2: Define the Banach space as the space of continuous fuctions $H: [0,b] \to \mathbb{R}$ with the supremum norm:

$$\|H\|_\infty = \sup_{u\in[0,b]} |H(u)|.$$

Step 3: Since X is a Banach space, we can use Banach's Fixed Point Theorem if T is a contraction mapping. For any two functions $H_1, H_2 \in X$, we compute:

$$T(H_1(u) - T(H_2(u)) = \frac{1}{\lambda\alpha}\int_0^b (H_1(z) - H_2(z))K(u,z)dz.$$

Taking norms on both sides,

$$\|T(H_1) - T(H_2)\|_\infty \le \frac{1}{\lambda\alpha}\sup_{u\in[0,b]}\int_0^b |H_1(z) - H_2(z)|\,|K(u,z)|dz.$$

Since $K(u,z)$ is bounded by some constant M, we get:

$$\|T(H_1) - T(H_2)\|_\infty \le \frac{Mb}{\lambda\alpha}\|H_1 - H_2\|_\infty.$$

If we choose parameters such that $k = \frac{Mb}{\lambda\alpha} < 1$, then T is a contraction mapping.

Step 4: Since T is a contraction in a Banach space, Banach's Fixed Point Theorem ensures a unique function $H(u)$ exists such that $T(H(u)) = H(u)$, confirming the solution's existence and uniqueness for the integral equation.

In this section, we will analyze the integral equations presented in Equation (6) to derive an explicit formula for the ARL of the EWMA control chart. This chart is based on the SAR(1)$_L$ model with quadratic trend, and our approach employs the Fredholm integral equation.
We consider the given integral equation:

$$H(u) = 1 + \frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}}{\lambda\alpha}\int_0^b H(z)\,e^{-\frac{z}{\lambda\alpha}}dz.$$

This equation is in the form of a Fredholm integral equation of the second kind, which has the general form:

$$H(u) = f(u) + \lambda\int_0^b K(u,z)H(z)\,dz.$$

Since $f(u) = 1, K(u,z) = e^{-\frac{z}{\lambda\alpha}}e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}$ and the parameter λ is set to $\frac{1}{\lambda\alpha}$.
Let $C(u)$ denote the coefficient function given by:

$$C(u) = e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}$$

By introducing the notation $C(u)$, the function $H(u)$ can be expressed as:

$$H(u) = 1 + \frac{C(u)}{\lambda\alpha}\int_0^b H(z)\,e^{-\frac{z}{\lambda\alpha}}dz, \quad 0 \le u \le b. \tag{7}$$

By defining $k = \int_0^b H(z)e^{-\frac{z}{\lambda\alpha}}dz$ to represent the integral equation of $H(z)$, Equation (7) can be rewritten as:

$$H(u) = 1 + \frac{C(u)}{\lambda\alpha}k. \tag{8}$$

The next step is to derive the value of k, we proceed as follows:

$$\begin{aligned}
k &= \int_0^b H(z)e^{-\frac{z}{\lambda\alpha}}dz,\\
&= \int_0^b \left(1 + \frac{C(z)}{\lambda\alpha}k\right)e^{-\frac{z}{\lambda\alpha}}dz,\\
&= \int_0^b e^{-\frac{z}{\lambda\alpha}}dz + \int_0^b \frac{C(z)}{\lambda\alpha}k\,e^{-\frac{z}{\lambda\alpha}}dz,\\
&= \int_0^b e^{-\frac{z}{\lambda\alpha}}dz + \frac{k}{\lambda\alpha}\int_0^b e^{\frac{(1-\lambda)v}{\lambda\alpha}+\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}.e^{-\frac{z}{\lambda\alpha}}dz,\\
&= \int_0^b e^{-\frac{z}{\lambda\alpha}}dz + \frac{k}{\lambda\alpha}.e^{\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}\int_0^b e^{\frac{(1-\lambda)z}{\lambda\alpha}-\frac{z}{\lambda\alpha}}dz,\\
&= -\lambda\alpha(e^{-\frac{b}{\lambda\alpha}} - 1) + \frac{k}{\lambda\alpha}.e^{\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}\int_0^b e^{-\frac{z}{\alpha}}dz,\\
&= -\lambda\alpha(e^{-\frac{b}{\lambda\alpha}} - 1) - \frac{k}{\lambda}.e^{\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}.(e^{\frac{-b}{\alpha}} - 1).
\end{aligned}$$

As a result of the proof, the obtained value of k is given by:

$$k = \frac{-\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)}{1+\frac{1}{\lambda}.e^{\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}.(e^{-\frac{b}{\alpha}}-1)}.$$

By substituting the obtained value of k into Equation (8), we can rewrite the integral equation as:

$$H(u) = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}(e^{-\frac{b}{\lambda\alpha}}-1)}{\lambda e^{-\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}+(e^{-\frac{b}{\alpha}}-1)}.$$

Therefore, the explicit formulas for the ARL of the EWMA control chart for the SAR(1)$_L$ model with a quadratic trend are as follows:

$$ARL_{ExplicitFormula} = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}(e^{-\frac{b}{\lambda\alpha}}-1)}{\lambda e^{-\frac{\mu+\phi Y_{t-L}+\beta_1 t+\beta_2 t^2}{\alpha}}+(e^{-\frac{b}{\alpha}}-1)}. \quad (9)$$

where α is a parameter of exponential white noise, b is upper control limit, Y_{t-L} are the initial values of SAR(1)$_L$ model with quadratic trend, ϕ refers to the autoregressive coefficient parameters, β_1 is the linear coefficient parameters, β_2 is the quadratic coefficient parameters.

Methodological of Numerical Integral Equation for ARL

In this section, we will explain how to calculate the numerical integration of the ARL of the EWMA control chart for the SAR(1)$_L$ model with quadratic trend, considering exponential distribution for the white noise processes. We consider the integral equation

$$H(u) = 1 + \frac{1}{\lambda}\int_0^b H(z)\, f\left(\frac{z-(1-\lambda)u}{\lambda} - \mu - \phi Y_{t-L} - \beta_1 t - \beta_2 t^2\right)dz.$$

Using the Midpoint Rule, we approximate the integral as:

$$\int_0^b W(z)f(z)dz \approx \sum_{j=1}^{m} w_j f(a_j),$$

where $a_j = \frac{b}{m}\left(j-\frac{1}{2}\right); j = 1,2,\ldots,m$ is the midpoint of each subinterval and $w_j = \frac{b}{m}$ is the step size.

Therefore, we transform the integral equation into a numerical integral equation using the Midpoint Rule. Rewriting Equation (5) in a structured numerical form as follows:

$$\widetilde{H}(u) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j\, \widetilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)u}{\lambda} - \mu - \phi Y_{t-L} - \beta_1 t - \beta_2 t^2\right). \quad (10)$$

This equation can now be solved iteratively for $\widetilde{H}(u)$, using numerical methods as follows:

$$\widetilde{H}(a_1) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j\, \widetilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_1}{\lambda} - \mu - \phi Y_{t-L} - \beta_1 t - \beta_2 t^2\right),$$
$$\widetilde{H}(a_2) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j\, \widetilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_2}{\lambda} - \mu - \phi Y_{t-L} - \beta_1 t - \beta_2 t^2\right),$$
$$\vdots$$
$$\widetilde{H}(a_m) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j\, \widetilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_m}{\lambda} - \mu - \phi Y_{t-L} - \beta_1 t - \beta_2 t^2\right),$$

or as a matrix $\boldsymbol{H}_{m\times 1} = \boldsymbol{1}_{m\times 1} + \boldsymbol{R}_{m\times m}\boldsymbol{H}_{m\times 1}$,

where $\boldsymbol{H}_{m\times 1} = \begin{pmatrix}\widetilde{H}(a_1)\\ \widetilde{H}(a_2)\\ \vdots \\ \widetilde{H}(a_m)\end{pmatrix}$ $\boldsymbol{1}_{m\times 1} = \begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$, $[\boldsymbol{R}]_{ij} \approx \frac{1}{\lambda} w_j f\left(\frac{a_j-(1-\lambda)a_i}{\lambda} - \mu - \phi Y_{t-L} - \beta_1 t - \beta_2 t^2\right)$,

and $\boldsymbol{I}_m = \mathrm{diag}(1,1,\ldots,1)$. If $(\boldsymbol{I}_m - \boldsymbol{R}_{m\times m})^{-1}$ there exist $\boldsymbol{H}_{m\times 1} = (\boldsymbol{I}_m - \boldsymbol{R}_{m\times m})^{-1}\boldsymbol{1}_{m\times 1}$.

Therefore, the numerical integral equation for the ARL of the EWMA control chart for the SAR(1)$_L$ model with quadratic trend are as follows:

$$ARL_{NIE} \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j\, \widetilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)u}{\lambda} - \mu - \phi Y_{t-L} - \beta_1 t - \beta_2 t^2\right). \quad (11)$$

The Procedure for Calculating the ARL

In this section, we will explain the procedure for calculating the ARL of the EWMA control chart for the SAR(1)$_L$ model with quadratic trend using the explicit formula and numerical integral equation method.

Procedure for Calculating ARL Using Explicit Formulas

Step 1: Create observations based on a seasonal autoregressive model with quadratic trend, denoted as SAR(1)$_L$ by (Y_t).

- The white noise process is assumed to be in-control state and follows an exponential distribution with a known mean, $\alpha = \alpha_0$= 1.
- Set the initial values for the SAR(1)$_L$ model to Y_{t-L}= 1.
- Set the values for the autoregressive coefficient: ϕ = 0.10, 0.50, and 0.80, respectively.
- Set the values for the linear coefficient parameter: β_1= 0.20, 0.30.
- Set the values for the quadratic coefficient parameter: β_2= 0.30, 0.40.

Step 2: Calculate the statistics for the EWMA control chart using the variable $E_t = (1-\lambda)E_{t-1} + \lambda Y_t$.

- Set the initial values for the EWMA statistics: $E_0 = u$ = 1.
- Set the values for the exponential smoothing parameter as follows: λ = 0.05, 0.10 and 0.15, respectively.

Step 3. Determination of the EWMA control chart based on a SAR(1)$_L$ model with quadratic trend.

- Establish the desired in-control ARL; ARL$_0$. Set ARL$_0$ to 370 and 500, respectively.
- Set the UCL; b. Calculate the ARL$_0$ using the explicit formulas provided in equation (9). If the calculated ARL$_0$ matches the desired ARL$_0$, then conclude this step and proceed to the steps for identifying an out-of-control state. If the calculated ARL$_0$ does not match the desire value, adjust the upper control limit and repeat the steps (1) - (2).

Step 4. Calculate the ARL$_1$ for any shift size using the explicit formulas provided in equation (9).

- Consider a state that is out-of-control; specify the shift size in the mean of the exponential parameter $(\alpha = \alpha_1)$ where $\alpha_1 = (1+\delta)\alpha_0$; δ = 0.01, 0.03, 0.05, 0.10, 0.20, 0.30, 0.40, 0.50, 1.00, 1.50 and 2.00, respectively.

Step 5. Repeat steps (1) – (4) using the new parameter to compute the ARL$_1$ for an out-of-control state, and record the computational time for the first out-of-control signal.

Procedure for Calculating ARL Using Numerical Integral Equation

Step 1: Create observations based on a seasonal autoregressive model with quadratic trend, denoted as SAR(1)$_L$ by (Y_t).

- The white noise process is assumed to be in-control state and follows an exponential distribution with a known mean $(\alpha = \alpha_0)$ = 1.
- Set the initial values for the SAR(1)$_L$ model to Y_{t-L}= 1.
- Set the values for the autoregressive coefficient: ϕ = 0.10, 0.50, and 0.80, respectively.
- Set the values for the linear coefficient parameter: β_1= 0.20, 0.30.
- Set the values for the quadratic coefficient parameter: β_2= 0.30, 0.40.

Step 2: Calculate the statistics for the EWMA control chart using the variable $E_t = (1-\lambda)E_{t-1} + \lambda Y_t$.

- Set the initial values for the EWMA statistics: $E_0 = u = 1$.
- Set the values for the exponential smoothing parameter as follows: λ = 0.05, 0.10 and 0.15, respectively.

Step 3. Determination of the EWMA control chart based on SAR(1)$_L$ model with quadratic trend model.

- Establish the desired in-control ARL; ARL$_0$. Set ARL$_0$ to 370 and 500, respectively.
- Set the upper control limit (UCL); b.
- Set the number of division point to m = 500.
- Calculate the ARL$_0$ using the numerical integral equation method provided in equation (11). If the calculated ARL$_0$ matches the desired ARL$_0$, then conclude this step and proceed to the steps for identifying an out-of-control state. If the calculated ARL$_0$ does not match the desire value, adjust the upper control limit and repeat the steps (1) - (2).

Step 4. Calculate the ARL$_1$ for any shift size using the numerical integral equation provided in equation (11).

- Consider a state that is out-of-control; specify the shift size in the mean of the exponential parameter $(\alpha = \alpha_1)$ where $\alpha_1 = (1+\delta)\alpha_0$; δ = 0.01, 0.03, 0.05, 0.10, 0.20, 0.30, 0.40, 0.50, 1.00, 1.50 and 2.00, respectively.

Step 5. Repeat steps (1) – (4) using the new parameter to compute the ARL$_1$ for an out-of-control state, and record the computational time for the first out-of-control signal.

Criteria for Evaluating the Efficiency of Methods

The Absolute Percentage Difference Of ARL

The absolute percentage difference between the ARL obtained from explicit formulas and the results of NIE measures the relative discrepancy between these two methods. This metric is essential for evaluating the consistency and reliability of methods used to calculate ARL, ensuring accurate decision-making in process control and quality monitoring. It is calculated using the following formula:

$$\text{AbsolutePercentageDifference} = \left|\frac{\text{ARL}_{ExplicitFormula} - \text{ARL}_{\text{NIE}}}{\text{ARL}_{ExplicitFormula}}\right| \times 100\%, \quad (12)$$

where $\mathrm{ARL}_{ExplicitFormula}$ represents the ARL calculated using an explicit formula method
$\mathrm{ARL}_{\mathrm{NIE}}$ represents the ARL calculated using numerical integration method.

Computational Time

The computational time (or time used) refers to the duration needed for an algorithm to calculate the ARL. This paper compares two distinct computational methods and highlights the importance of measuring computational time to evaluate efficiency. This criterion is essential for determining which method performs calculations more quickly. The results of this paper are expressed in seconds.

Numerical Results

Suppose that the process goes out-of-control with respect to mean and $\alpha = \alpha_1 = (1+\delta)\alpha_0$ be the out-of-control parameter with δ time change in α_0. Under the assumption that the white noise data follows an exponential distribution with mean α_0.

In Tables 1 - 3 present the ARL values derived using both the explicit formula and the NIE methods. These methods effectively detected small shifts in the mean, with the shift size (δ) values of 0.01, 0.02, 0.03, 0.04, 0.05, 0.10, 0.30, 0.50, 1.00, 1.50, and 2.00, respectively. These tables show the consistency of ARL from both methods based on $\mathrm{SAR}(1)_4$ models with quadratic trend on an EWMA control chart, with ARL_0 = 370. The results show that the ARL_1 values obtained from the explicit formula and the NIE methods were similar and rapidly declined as the shift size decreased. The absolute percentage difference between the ARL values obtained using the explicit formula and the NIE method is less than 0.00001. However, the computational time for the explicit formula is 0.001 seconds, whereas the NIE method requires 2.157 seconds. Furthermore, these ARL values were significantly affected by changes in the exponential smoothing parameter(λ), which was set at 0.05, 0.10, or 0.15, significantly affected these values. For example, in Table 1 presents the ARL_1 values for the $\mathrm{SAR}(1)_4$ model with a quadratic trend, the parameter are set as follows:ϕ_1= 0.1,β_1= 0.2,β_2= 0.3, andλ is set to 0.05, 0.10, and 0.15, respectively. The corresponding δ include 0.01, 0.02, 0.03, 0.04, 0.05, 0.10, 0.30, 0.50, 1.00, 1.50, and 2.00). The results for these shift sizes are as follows:

- Forλ = 0.05: 301.948, 203.454, 139.217, 57.476, 12.728, 4.082, 1.975, 1.358, 1.010, 1.001, 1.000.
- Forλ = 0.10: 333.060, 271.424, 222.881, 140.435, 62.520, 31.596, 17.725, 10.863, 2.485, 1.453, 1.199.
- Forλ = 0.15: 337.264, 282.224, 238.244, 161.300, 82.955, 47.911, 30.189, 20.368, 5.547, 2.846, 1.985.

It can be seen that usingλ = 0.05 allows ARL_1 values to detect a small shift in the mean more quickly than with$\lambda$of 0.10 and 0.15. In Table 2-3 demonstrates that the EWMA control chart is more effective with λ set to 0.05 compared to the other values of λ, which are 0.10 and 0.15. Similarly, Tables 4–6 display the ARL values calculated using both the explicit formula and the NIE methods. These methods effectively identified small shifts in the mean, with shift sizes ranging from 0.01 to 2.00 in increments of 0.01. The tables also demonstrate the consistency of ARL values obtained from both methods, based on $\mathrm{SAR}(1)_4$ models with quadratic trend on an EWMA control chart, where the ARL_0 = 500. The findings indicate that the ARL_1 values derived from both the explicit formula and the NIE methods were comparable and showed a rapid decline as the shift size decreased. The absolute percent difference between the ARL values from the two methods is less than 0.00001. Notably, the explicit formula required only 0.001 seconds for computation, while the NIE method took 2.044 seconds. Additionally, variations in λ set at 0.05, 0.10, or 0.15, had a significant impact on these ARL values. For example, in Table 4 presents the ARL_1 values for the $\mathrm{SAR}(1)_4$ model with quadratic trend, the parameter are set as follows:ϕ_1= 0.1,β_1= 0.2,β_2= 0.3, and λ is set to 0.05, 0.10, and 0.15, respectively. The corresponding δ include 0.01, 0.02, 0.03, 0.04, 0.05, 0.10, 0.30, 0.50, 1.00, 1.50, and 2.00). The results for these shift sizes are as follows:

- Forλ = 0.05: 408.346, 275.031, 188.082, 77.443, 16.8774, 5.171, 2.319, 1.484, 1.014, 1.002, 1.000.
- Forλ = 0.10: 450.094, 366.582, 300.841, 189.261, 83.944, 42.207, 23.505, 14.263, 2.992, 1.608, 1.267.
- Forλ = 0.15: 452.601, 373.298, 311.228, 205.582, 102.347, 57.893, 35.941, 23.969, 6.270, 3.121, 2.126.

It can be seen that using λ = 0.05 allows ARL_1 values to detect a small shift in the mean more quickly than withλ of 0.10 and 0.15.

The comparative efficiency of the proposed explicit formula is illustrated in the graphs presented in Figures 1 and 2. Three distinct exponential smoothing parameters—0.05, 0.10, and 0.15—are illustrated in the graph to demonstrate their respective effects. In Figure 1 displays a comparative graph of ARL_1 values for various $\mathrm{SAR}(1)_4$ model with quadratic trend based on an EWMA control chart (ARL_0 = 300). When the EWMA control chart is set with λ = 0.05, it demonstrates improved effectiveness in the early detection of out-of-control processes across all magnitudes of shift size. The analysis indicates that the ARL_1 value is consistently lowest across all models whenλ is set to 0.05, in comparison to the values of 0.10 and 0.15. This suggests that using an EWMA control chart withλ = 0.05 enhances sensitivity in detecting changes in the process mean more effectively. Figure 2 presents a comparison of ARL_1 values for different $SAR(1)_4$ models with quadratic trend, as shown on an EWMA control chart (ARL_0 = 500). The graph illustrates the effects of threeλ: 0.05, 0.10, and 0.15. The results indicate that whenλ is set to 0.05, the ARL_1 value is the lowest among all models, in contrast to λ values of 0.10 and 0.15. This suggests that using an EWMA control chart withλ = 0.05 enhances the ability to detect changes in the process mean more effectively.

Table 1. Comparison of ARL derived from explicit formulas and NIE methods for two $SAR(1)_4$ model with quadratic trend on an EWMA control chart (ARL_0 = 370, ϕ = 0.10 and -0.10,β_1= 0.20,β_2= 0.30)

	$SAR(1)_4$ model with quadratic trend (ϕ = 0.10, β_1= 0.20, β_2= 0.30)				$SAR(1)_4$ model with quadratic trend (ϕ = -0.10, β_1= 0.20, β_2= 0.30)			
λ = 0.05	b = 0.00000005674				b = 0.0000000693			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	370.056	0.001	370.056	2.106	370.043	0.001	370.043	2.137
0.01	301.948	0.001	301.948	2.152	302.534	0.001	302.534	2.184
0.03	203.454	0.001	203.454	2.153	204.630	0.001	204.630	2.137
0.05	139.217	0.001	139.217	2.121	140.535	0.001	140.535	2.153
0.10	57.476	0.001	57.476	2.137	58.510	0.001	58.510	2.168
0.20	12.728	0.001	12.728	2.200	13.125	0.001	13.125	2.169
0.30	4.082	0.001	4.082	2.153	4.227	0.001	4.227	2.153
0.40	1.975	0.001	1.975	2.169	2.032	0.001	2.032	2.184
0.50	1.358	0.001	1.358	2.168	1.382	0.001	1.382	2.121
1.00	1.010	0.001	1.010	2.122	1.011	0.001	1.011	2.184
1.50	1.001	0.001	1.001	2.137	1.001	0.001	1.001	2.169
2.00	1.000	0.001	1.000	2.121	1.000	0.001	1.000	2.199
λ = 0.10	b = 0.0024185				b = 0.002962			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	370.046	0.001	370.046	2.138	370.092	0.001	370.092	2.106
0.01	333.060	0.001	333.060	2.215	333.798	0.001	333.798	2.137
0.03	271.424	0.001	271.424	2.153	273.128	0.001	273.128	2.200
0.05	222.881	0.001	222.881	2.168	225.151	0.001	225.151	2.153
0.10	140.435	0.001	140.435	2.121	143.146	0.001	143.146	2.168
0.20	62.520	0.001	62.520	2.153	64.714	0.001	64.714	2.169
0.30	31.596	0.001	31.596	2.184	33.110	0.001	33.110	2.137
0.40	17.725	0.001	17.725	2.168	18.752	0.001	18.752	2.169
0.50	10.863	0.001	10.863	2.184	11.571	0.001	11.571	2.168
1.00	2.485	0.001	2.485	2.122	2.646	0.001	2.646	2.184
1.50	1.453	0.001	1.453	2.184	1.513	0.001	1.513	2.137
2.00	1.199	0.001	1.199	2.168	1.228	0.001	1.228	2.216
λ = 0.15	b = 0.05016				b = 0.06245			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	370.006	0.001	370.006	2.106	370.074	0.001	370.074	2.122
0.01	337.264	0.001	337.264	2.168	338.869	0.001	338.869	2.153
0.03	282.224	0.001	282.224	2.231	285.953	0.001	285.953	2.199
0.05	238.244	0.001	238.244	2.200	243.217	0.001	243.217	2.152
0.10	161.300	0.001	161.300	2.153	167.306	0.001	167.306	2.138
0.20	82.955	0.001	82.955	2.200	88.079	0.001	88.079	2.168
0.30	47.911	0.001	47.911	2.215	51.712	0.001	51.712	2.215
0.40	30.189	0.001	30.189	2.152	32.977	0.001	32.977	2.152
0.50	20.368	0.001	20.368	2.106	22.446	0.001	22.446	2.169
1.00	5.547	0.001	5.547	2.152	6.206	0.001	6.206	2.200
1.50	2.846	0.001	2.846	2.169	3.152	0.001	3.152	2.246
2.00	1.985	0.001	1.985	2.153	2.161	0.001	2.161	2.152

Table 2. Comparison of ARL derived from explicit formulas and NIE methods for two $SAR(1)_4$ model with quadratic trend on an EWMA control chart (ARL_0 = 370, ϕ = 0.50 and -0.50, β_1 = 0.30, β_2 = 0.40)

	SAR(1)$_4$ model with quadratic trend (ϕ = 0.50, β_1 = 0.30, β_2 = 0.40)				SAR(1)$_4$ model with quadratic trend (ϕ = -0.50, β_1 = 0.30, β_2 = 0.40)			
λ = 0.05	b = 0.0000000312				b = 0.0000000847			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	370.772	0.001	370.772	2.844	370.290	0.001	370.290	2.453
0.01	300.746	0.001	300.746	2.860	303.334	0.001	303.334	2.828
0.03	200.333	0.001	200.333	2.812	205.957	0.001	205.957	2.657
0.05	135.584	0.001	135.584	2.828	141.964	0.001	141.964	2.734
0.10	54.582	0.001	54.582	2.828	59.605	0.001	59.605	2.578
0.20	11.632	0.001	11.632	2.828	13.544	0.001	13.544	2.609
0.30	3.688	0.001	3.688	2.813	4.382	0.001	4.382	2.829
0.40	1.823	0.001	1.823	2.812	2.094	0.001	2.094	2.578
0.50	1.293	0.001	1.293	2.813	1.409	0.001	1.409	2.593
1.00	1.008	0.001	1.008	2.797	1.013	0.001	1.013	2.750
1.50	1.001	0.001	1.001	2.797	1.001	0.001	1.001	2.657
2.00	1.000	0.001	1.000	2.812	1.000	0.001	1.000	2.531
λ = 0.10	b = 0.001321				b = 0.003629			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	370.238	0.001	370.238	2.891	370.057	0.001	370.057	2.844
0.01	331.160	0.001	331.160	2.844	334.466	0.001	334.466	2.937
0.03	266.636	0.001	266.636	2.843	274.787	0.001	274.787	2.969
0.05	216.434	0.001	216.434	2.860	227.401	0.001	227.401	2.750
0.10	132.765	0.001	132.765	2.828	145.889	0.001	145.889	2.047
0.20	56.466	0.001	56.466	2.859	66.981	0.001	66.981	2.813
0.30	27.518	0.001	27.518	2.844	34.699	0.001	34.699	2.062
0.40	15.016	0.001	15.016	2.828	19.843	0.001	19.843	2.516
0.50	9.030	0.001	9.030	2.813	12.331	0.001	12.331	2.812
1.00	2.093	0.001	2.093	2.812	2.825	0.001	2.825	2.735
1.50	1.314	0.001	1.314	2.844	1.580	0.001	1.580	2.859
2.00	1.133	0.001	1.133	2.844	1.262	0.001	1.262	2.875
λ = 0.15	b = 0.0265				b = 0.0780999			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	370.042	0.001	370.042	2.075	370.092	0.001	370.092	2.594
0.01	332.870	0.001	332.870	2.390	340.487	0.001	340.487	2.578
0.03	271.948	0.001	271.948	2.344	289.831	0.001	289.831	2.687
0.05	224.743	0.001	224.743	2.312	248.463	0.001	248.463	2.797
0.10	145.579	0.001	145.579	2.313	173.805	0.001	173.805	2.735
0.20	70.239	0.001	70.239	2.547	93.804	0.001	93.804	2.703
0.30	38.804	0.001	38.804	2.422	56.048	0.001	56.048	2.938
0.40	23.676	0.001	23.676	2.625	36.205	0.001	36.205	2.734
0.50	15.605	0.001	15.605	2.329	24.879	0.001	24.879	2.985
1.00	4.118	0.001	4.118	2.515	7.000	0.001	7.000	2.718
1.50	2.200	0.001	2.200	2.328	3.526	0.001	3.526	2.766
2.00	1.618	0.001	1.618	2.328	2.379	0.001	2.379	2.641

Table 3. Comparison of ARL derived from explicit formulas and NIE methods for two SAR(1)$_4$ model with quadratic trend on an EWMA control chart (ARL$_0$ = 370, ϕ = 0.80 and -0.80, β_1= 0.20,β_2= 0.30)

	SAR(1)$_4$ model with quadratic trend (ϕ = 0.80, β_1 = 0.20, β_2= 0.30)				SAR(1)$_4$ model with quadratic trend (ϕ = -0.80, β_1 = 0.20, β_2= 0.30)			
λ = 0.05	b = 0.000000028179				b = 0.00000013955			
δ	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used
0.00	370.092	0.001	370.092	2.703	370.034	0.001	370.034	2.813
0.01	299.899	0.001	299.899	2.750	304.624	0.001	304.624	2.734
0.03	199.388	0.001	199.388	2.672	208.820	0.001	208.820	2.719
0.05	134.699	0.001	134.699	2.859	145.261	0.001	145.261	2.797
0.10	53.999	0.001	53.999	2.797	62.288	0.001	62.288	2.828
0.20	11.437	0.001	11.437	2.796	14.625	0.001	14.625	2.765
0.30	3.622	0.001	3.622	2.875	4.793	0.001	4.793	2.813
0.40	1.798	0.001	1.798	2.870	2.261	0.001	2.261	2.812
0.50	1.283	0.001	1.283	2.843	1.483	0.001	1.483	2.610
1.00	1.007	0.001	1.007	2.860	1.016	0.001	1.016	2.656
1.50	1.001	0.001	1.001	2.750	1.002	0.001	1.002	2.719
2.00	1.000	0.001	1.000	2.844	1.000	0.001	1.000	2.781
λ = 0.10	b = 0.001194				b = 0.006053			
δ	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used
0.00	370.056	0.001	370.056	2.718	370.078	0.001	370.078	2.625
0.01	330.654	0.001	330.654	2.750	336.255	0.001	336.255	2.734
0.03	265.696	0.001	265.696	2.969	279.095	0.001	279.095	2.766
0.05	215.258	0.001	215.258	2.641	233.242	0.001	233.242	2.734
0.10	131.458	0.001	131.458	2.781	153.087	0.001	153.087	2.594
0.20	55.489	0.001	55.489	2.875	73.090	0.001	73.090	2.859
0.30	26.882	0.001	26.882	2.750	39.081	0.001	39.081	2.625
0.40	14.604	0.001	14.604	2.844	22.913	0.001	22.913	2.641
0.50	8.756	0.001	8.756	2.750	14.509	0.001	14.509	2.750
1.00	2.038	0.001	2.038	2.734	3.371	0.001	3.371	2.719
1.50	1.296	0.001	1.296	2.609	1.793	0.001	1.793	2.828
2.00	1.124	0.001	1.124	2.750	1.370	0.001	1.370	2.703
λ = 0.15	b = 0.023874				b = 0.140474			
δ	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used
0.00	370.035	0.001	370.035	2.641	370.102	0.001	370.102	2.547
0.01	332.146	0.001	332.146	2.687	344.936	0.001	344.936	2.562
0.03	270.306	0.001	270.306	2.750	300.805	0.001	300.805	2.703
0.05	222.629	0.001	222.629	2.828	263.639	0.001	263.639	2.719
0.10	143.208	0.001	143.208	2.750	193.458	0.001	193.458	2.734
0.20	68.412	0.001	68.412	2.813	112.231	0.001	112.231	2.688
0.30	37.536	0.001	37.536	2.719	70.595	0.001	70.595	2.844
0.40	22.790	0.001	22.790	2.640	47.353	0.001	47.353	2.781
0.50	14.968	0.001	14.968	2.531	33.462	0.001	33.462	2.875
1.00	3.936	0.001	3.936	2.485	9.958	0.001	9.958	2.797
1.50	2.120	0.001	2.120	2.765	4.949	0.001	4.949	2.875
2.00	1.574	0.001	1.574	2.719	3.216	0.001	3.216	2.718

Table 4. Comparison of ARL derived from explicit formulas and NIE methods for two $SAR(1)_4$ model with quadratic trend on an EWMA control chart (ARL_0 = 500, ϕ = 0.10 and -0.10, β_1= 0.20,β_2= 0.30)

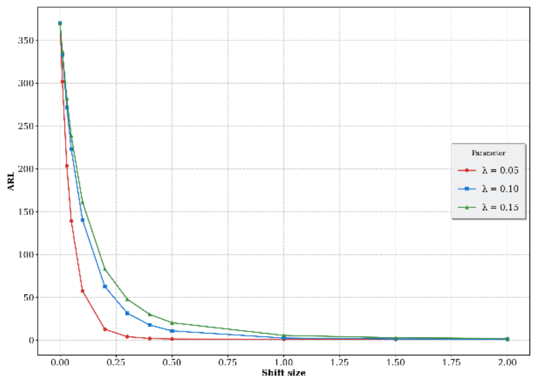
	$SAR(1)_4$ model with quadratic trend (ϕ = 0.10, β_1= 0.20, β_2= 0.30)				$SAR(1)_4$ model with quadratic trend (ϕ = -0.10, β_1= 0.20, β_2= 0.30)			
λ = 0.05	b = 0.0000000768				b = 0.000000093765			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	500.533	0.001	500.533	1.969	500.326	0.001	500.326	2.062
0.01	408.346	0.001	408.346	2.047	408.985	0.001	408.985	2.125
0.03	275.031	0.001	275.031	1.968	276.518	0.001	276.518	2.094
0.05	188.082	0.001	188.082	2.032	189.795	0.001	189.795	2.079
0.10	77.443	0.001	77.443	2.015	78.813	0.001	78.813	2.140
0.20	16.874	0.001	16.874	2.016	17.405	0.001	17.405	2.156
0.30	5.171	0.001	5.171	1.984	5.367	0.001	5.367	2.204
0.40	2.319	0.001	2.319	2.016	2.396	0.001	2.396	2.109
0.50	1.484	0.001	1.484	2.000	1.517	0.001	1.517	2.093
1.00	1.014	0.001	1.014	2.015	1.015	0.001	1.015	2.125
1.50	1.002	0.001	1.002	2.031	1.002	0.001	1.002	2.125
2.00	1.000	0.001	1.000	2.016	1.000	0.001	1.000	2.078
λ = 0.10	b = 0.003234				b = 0.003965			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	500.225	0.001	500.225	2.062	500.408	0.001	500.408	2.047
0.01	450.094	0.001	450.094	2.063	451.219	0.001	451.219	2.093
0.03	366.582	0.001	366.582	2.078	369.016	0.001	369.016	2.110
0.05	300.841	0.001	300.841	2.093	304.036	0.001	304.036	2.062
0.10	189.261	0.001	189.261	2.047	193.031	0.001	193.031	2.063
0.20	83.944	0.001	83.944	2.063	86.970	0.001	86.970	2.140
0.30	42.207	0.001	42.207	2.125	44.286	0.001	44.286	2.157
0.40	23.505	0.001	23.505	2.047	24.912	0.001	24.912	2.140
0.50	14.263	0.001	14.263	2.032	15.231	0.001	15.231	2.141
1.00	2.992	0.001	2.992	2.062	3.212	0.001	3.212	2.047
1.50	1.608	0.001	1.608	2.000	1.689	0.001	1.689	2.109
2.00	1.267	0.001	1.267	2.078	1.306	0.001	1.306	2.094
λ = 0.15	b = 0.056611				b = 0.07042			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	500.605	0.001	500.605	1.984	500.092	0.001	500.092	2.110
0.01	452.601	0.001	452.601	2.016	454.574	0.001	454.574	2.093
0.03	373.298	0.001	373.298	2.078	378.569	0.001	378.569	2.094
0.05	311.228	0.001	311.228	2.063	318.312	0.001	318.312	2.047
0.10	205.582	0.001	205.582	2.062	213.946	0.001	213.946	2.125
0.20	102.347	0.001	102.347	2.094	109.143	0.001	109.143	2.140
0.30	57.893	0.001	57.893	2.078	62.770	0.001	62.770	2.094
0.40	35.941	0.001	35.941	2.047	39.436	0.001	39.436	2.110
0.50	23.969	0.001	23.969	2.063	26.530	0.001	26.530	2.141
1.00	6.270	0.001	6.270	2.047	7.049	0.001	7.049	2.125
1.50	3.121	0.001	3.121	2.109	3.476	0.001	3.476	2.125
2.00	2.126	0.001	2.126	2.094	2.329	0.001	2.329	2.109

Table 5. Comparison of ARL derived from explicit formulas and NIE methods for two SAR(1)$_4$ model with quadratic trend on an EWMA control chart (ARL$_0$ = 500, ϕ = 0.50 and -0.50, β_1= 0.30,β_2= 0.40)

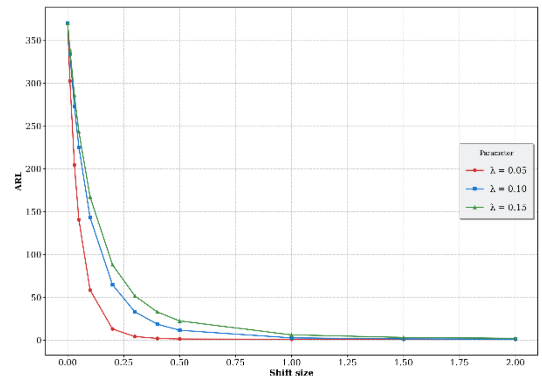
	SAR(1)$_4$ model with quadratic trend (ϕ = 0.50, β_1= 0.30, β_2= 0.40)					SAR(1)$_4$ model with quadratic trend (ϕ = -0.50, β_1= 0.30, β_2= 0.40)			
λ = 0.05	b = 0.000000042165					b = 0.0000001145			
δ	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used		ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used
0.00	500.726	0.001	500.726	2.062		500.218	0.001	500.218	2.047
0.01	406.090	0.001	406.090	2.141		409.705	0.001	409.705	2.125
0.03	270.387	0.001	270.387	2.125		278.067	0.001	278.067	2.094
0.05	182.883	0.001	182.883	2.141		191.560	0.001	191.560	2.109
0.10	73.413	0.001	73.413	2.109		80.224	0.001	80.224	2.109
0.20	15.369	0.001	15.369	2.125		17.958	0.001	17.958	2.079
0.30	4.633	0.001	4.633	2.141		5.572	0.001	5.572	2.109
0.40	2.112	0.001	2.112	2.109		2.478	0.001	2.478	2.093
0.50	1.397	0.001	1.397	2.140		1.553	0.001	1.553	2.172
1.00	1.010	0.001	1.010	2.125		1.017	0.001	1.017	2.125
1.50	1.001	0.001	1.001	2.141		1.002	0.001	1.002	2.125
2.00	1.000	0.001	1.000	2.109		1.000	0.001	1.000	2.141
λ = 0.10	b = 0.001763					b = 0.004861			
δ	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used		ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used
0.00	500.350	0.001	500.350	2.047		500.158	0.001	500.158	2.031
0.01	447.354	0.001	447.354	2.125		451.959	0.001	451.959	2.109
0.03	359.900	0.001	359.900	2.094		371.155	0.001	371.155	2.109
0.05	291.906	0.001	291.906	2.140		307.014	0.001	307.014	2.125
0.10	178.702	0.001	178.702	2.141		196.727	0.001	196.727	2.047
0.20	75.658	0.001	75.658	2.109		90.047	0.001	90.047	2.062
0.30	36.644	0.001	36.644	2.094		46.445	0.001	46.445	2.063
0.40	19.820	0.001	19.820	2.093		26.394	0.001	26.394	2.172
0.50	11.773	0.001	11.773	2.156		16.263	0.001	16.263	2.093
1.00	2.463	0.001	2.463	2.141		3.455	0.001	3.455	2.078
1.50	1.420	0.001	1.420	2.125		1.780	0.001	1.780	2.063
2.00	1.177	0.001	1.177	2.140		1.351	0.001	1.351	2.078
λ = 0.15	b = 0.0299163					b = 0.088049			
δ	ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used		ARL$_{Explicit}$	Time Used	ARL$_{NIE}$	Time Used
0.00	500.182	0.001	500.182	2.063		500.724	0.001	500.724	2.375
0.01	445.310	0.001	445.310	2.140		457.658	0.001	457.658	2.266
0.03	357.397	0.001	357.397	2.125		384.958	0.001	384.958	2.390
0.05	291.035	0.001	291.035	2.141		326.553	0.001	326.553	2.234
0.10	183.349	0.001	183.349	2.125		223.526	0.001	223.526	2.266
0.20	85.493	0.001	85.493	2.187		117.024	0.001	117.024	2.359
0.30	46.267	0.001	46.267	2.141		68.500	0.001	68.500	2.312
0.40	27.825	0.001	27.825	2.172		43.585	0.001	43.585	2.500
0.50	18.133	0.001	18.133	2.093		29.596	0.001	29.596	2.406
1.00	4.587	0.001	4.587	2.094		8.001	0.001	8.001	2.422
1.50	2.371	0.001	2.371	2.125		3.916	0.001	3.916	2.234
2.00	1.704	0.001	1.704	2.110		2.582	0.001	2.582	2.203

Table 6. Comparison of ARL derived from explicit formulas and NIE methods for two SAR(1)$_4$ model with quadratic trend on an EWMA control chart (ARL_0 = 500, ϕ = 0.80 and -0.80, β_1 = 0.20, β_2 = 0.30)

	SAR(1)$_4$ model with quadratic trend (ϕ = 0.80, β_1 = 0.20, β_2 = 0.30)				SAR(1)$_4$ model with quadratic trend (ϕ = -0.80, β_1 = 0.20, β_2 = 0.30)			
λ = 0.05	b = 0.000000038145				b = 0.0000001889			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	500.628	0.001	500.628	2.031	500.539	0.001	500.539	2.844
0.01	405.610	0.001	405.610	2.110	411.997	0.001	411.997	2.953
0.03	269.551	0.001	269.551	2.093	282.312	0.001	282.312	2.484
0.05	181.984	0.001	181.984	2.078	196.277	0.001	196.277	2.734
0.10	72.744	0.001	72.744	2.141	83.961	0.001	83.961	2.547
0.20	15.129	0.001	15.129	2.094	19.443	0.001	19.443	2.594
0.30	4.550	0.001	4.550	2.047	6.134	0.001	6.134	5.703
0.40	2.080	0.001	2.080	2.078	2.706	0.001	2.706	2.100
0.50	1.384	0.001	1.384	2.063	1.654	0.001	1.654	2.562
1.00	1.010	0.001	1.010	2.078	1.022	0.001	1.022	2.015
1.50	1.001	0.001	1.001	2.078	1.003	0.001	1.003	2.609
2.00	1.000	0.001	1.000	2.094	1.001	0.001	1.001	2.203
λ = 0.10	b = 0.00159345				b = 0.00814605			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	500.183	0.001	500.183	2.031	500.643	0.001	500.643	2.047
0.01	446.734	0.001	446.734	2.094	454.847	0.001	454.847	2.578
0.03	358.669	0.001	358.669	2.094	377.455	0.001	377.455	2.813
0.05	290.341	0.001	290.341	2.078	315.372	0.001	315.372	2.156
0.10	176.944	0.001	176.944	2.109	206.851	0.001	206.851	2.641
0.20	74.340	0.001	74.340	2.110	98.555	0.001	98.555	2.844
0.30	35.785	0.001	35.785	2.140	52.522	0.001	52.522	2.703
0.40	19.264	0.001	19.264	2.156	30.642	0.001	30.642	2.062
0.50	11.404	0.001	11.404	2.141	19.269	0.001	19.269	2.735
1.00	2.390	0.001	2.390	2.109	4.204	0.001	4.204	2.813
1.50	1.395	0.001	1.395	2.078	2.072	0.001	2.072	2.687
2.00	1.165	0.001	1.165	2.047	1.499	0.001	1.499	2.703
λ = 0.15	b = 0.0269605				b = 0.1577605			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	500.544	0.001	500.544	2.063	500.058	0.001	500.058	3.953
0.01	444.497	0.001	444.497	2.078	464.159	0.001	464.159	2.890
0.03	355.158	0.001	355.158	2.094	401.668	0.001	401.668	3.047
0.05	288.117	0.001	288.117	2.109	349.527	0.001	349.527	4.063
0.10	180.154	0.001	180.154	2.094	252.440	0.001	252.440	3.031
0.20	83.150	0.001	83.150	2.141	142.812	0.001	142.812	3.390
0.30	44.692	0.001	44.692	2.063	88.131	0.001	88.131	2.860
0.40	26.746	0.001	26.746	2.078	58.229	0.001	58.229	3.172
0.50	17.369	0.001	17.369	2.094	40.639	0.001	40.639	3.328
1.00	4.377	0.001	4.377	2.094	11.597	0.001	11.597	2.859
1.50	2.280	0.001	2.280	2.079	5.603	0.001	5.603	3.578
2.00	1.653	0.001	1.653	2.078	3.560	0.001	3.560	3.500



(A) ϕ = 0.10, β_1= 0.20,β_2= 0.30



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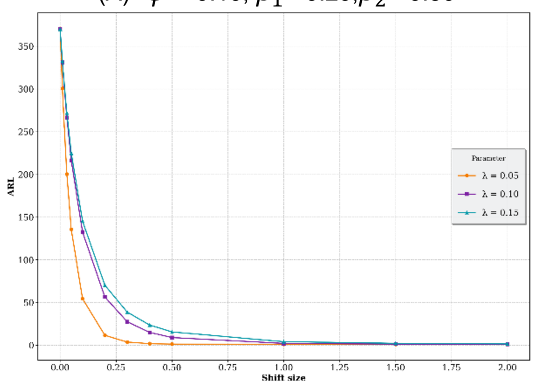
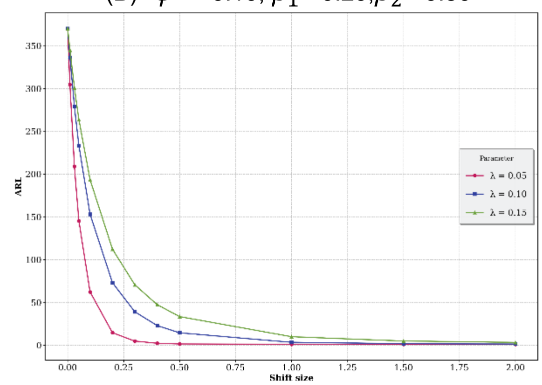
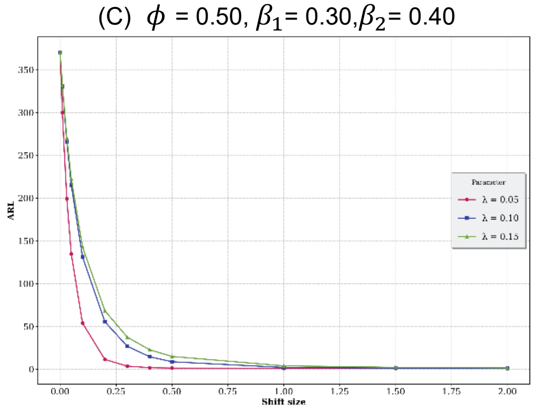


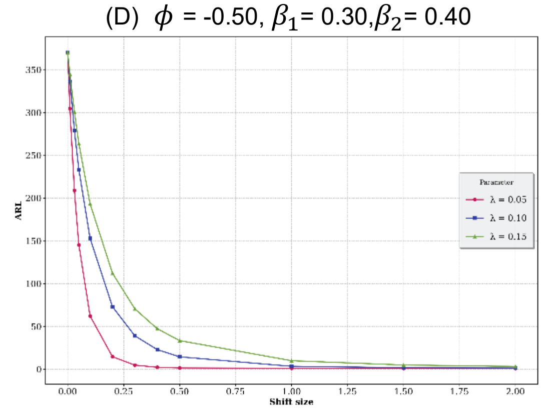
(C) ϕ = 0.50, β_1= 0.30,β_2= 0.40



(D) ϕ = -0.50, β_1= 0.30,β_2= 0.40

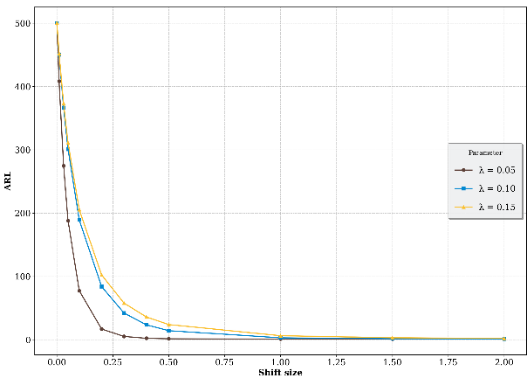


(E) ϕ = 0.80, β_1= 0.20,β_2= 0.30

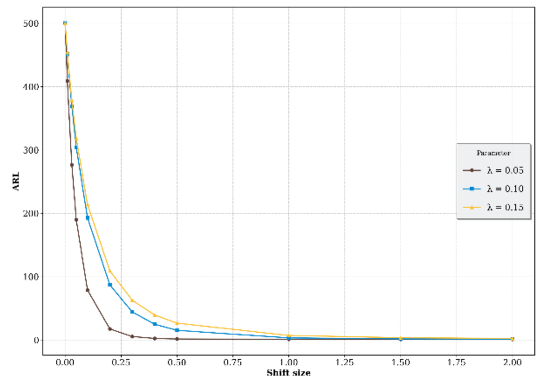


(F) ϕ = -0.80, β_1= 0.20,β_2= 0.30

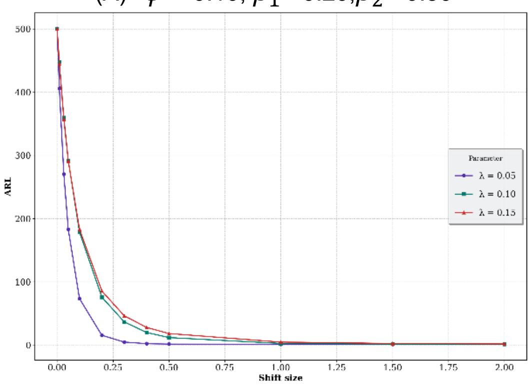
Figure 1. ARL curves of EWMA control chart for various SAR(1)$_4$ model with quadratic trend at (A) ϕ = 0.10,β_1= 0.20,β_2= 0.30, (B) ϕ = -0.10, β_1= 0.20,β_2= 0.30, (C) ϕ = 0.50, β_1= 0.30,β_2= 0.40, (D) ϕ = -0.50, β_1= 0.30,β_2= 0.40, (E) ϕ = 0.80, β_1= 0.20,β_2= 0.30 and (F) ϕ = -0.80, β_1= 0.20,β_2= 0.30 when ARL_0 = 370



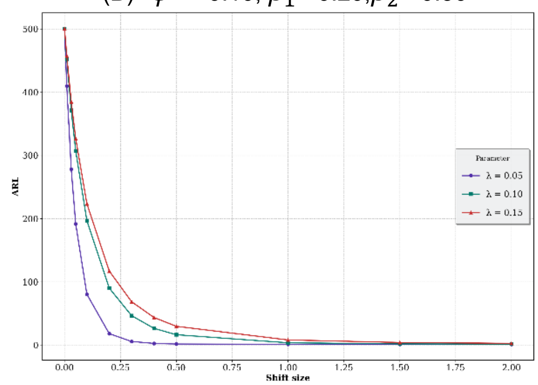
(A) ϕ = 0.10, β_1= 0.20,β_2= 0.30



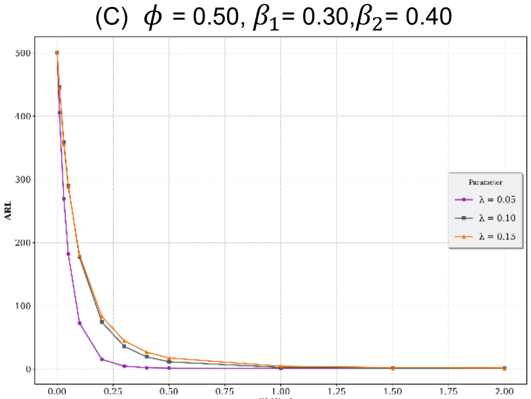
(B) ϕ = -0.10, β_1= 0.20,β_2= 0.30



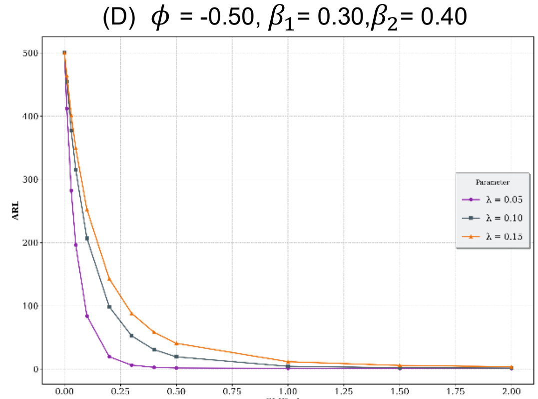
(C) ϕ = 0.50, β_1= 0.30,β_2= 0.40



(D) ϕ = -0.50, β_1= 0.30,β_2= 0.40



(E) ϕ = 0.80, β_1= 0.20,β_2= 0.30



(F) ϕ = -0.80, β_1= 0.20,β_2= 0.30

Figure 2. ARL curves of EWMA control chart for various $SAR(1)_4$ model with quadratic trend at (A) ϕ = 0.10,β_1= 0.20,β_2= 0.30, (B) ϕ = -0.10, β_1= 0.20,β_2= 0.30, (C) ϕ = 0.50, β_1= 0.30,β_2= 0.40, (D) ϕ = -0.50, β_1= 0.30,β_2= 0.40, (E) ϕ = 0.80, β_1= 0.20,β_2= 0.30 and (F) ϕ = -0.80, β_1= 0.20,β_2= 0.30 when ARL_0 = 500

Performance Evaluation of EWMA Control Charts in Detecting Shifts in $SAR(1)_L$ Model Quadratic Trend

This section provides a comparative performance evaluation of EWMA control chart when applied to two different models: the $SAR(1)_L$ model with a quadratic trend component and the $SAR(1)_L$ model without this quadratic trend. The analysis aims to clarify how quadratic trends affect the ARL performance of EWMA control charts. The comparative framework carefully analyzes how incorporating quadratic trend components impacts the sensitivity and detection capabilities of EWMA control charts used for monitoring seasonal autoregressive processes. By comparing the ARL profiles in both scenarios, we can quantify how much quadratic trends affect the control chart's ability to accurately and promptly detect shifts in the process. This analysis highlights how deterministic quadratic patterns affect the performance of EWMA control charts used with a stochastic $SAR(1)_L$ process.

Tables 7 and 8 present the ARL values derived using both the explicit formula and the NIE methods. These tables show the consistency of ARL from both methods based on $SAR(1)_{12}$ models with and without quadratic trend on an EWMA control chart, with ARL_0 = 370 and 500, respectively. These methods effectively detected small shifts in the mean, with the shift size (δ) = 0.01, 0.02, 0.03, 0.04, 0.05, 0.10, 0.30, 0.50, 1.00, 1.50, and 2.00 and various the exponential smoothing parameter(λ), 0.05, 0.10 and 0.15. The results show that the absolute percentage difference between the ARL values obtained using the explicit formula and the NIE method is less than 0.000001. Furthermore, the ARL from the $SAR(1)_{12}$ model with quadratic trend was smaller than that from the $SAR(1)_{12}$ model without quadratic trend. This indicates that quadratic trends can enhance the effectiveness of control charts. The EWMA control chart withλ = 0.05 demonstrated superior performance in early detection of process change.

Real-World Application

The explicit formulas of ARL for EWMA control charts was applied to monitor pneumonia disease among patients admitted to Siriraj Hospital in Bangkok, Thailand. The research incorporates monthly observations obtained during the period January 2019-December 2024. Subsequently, the researchers estimated a time series model through the maximum likelihood estimation. The coefficient parameters of $SAR(1)_{12}$ model with quadratic trend were obtained as follows:$\mu = 161.7624$, $\phi = 0.342961, \beta_1 = -1.849275$ and $\beta_2 = 0.0312841$. According to the derived parameters, the prediction model can be structured as follows:

$$Y_t = 161.7624 + 0.3429617Y_{t-12} - 1.849275t + 0.0312841t^2 + \varepsilon_t, t = 1,2,\ldots$$

The ARL values for the $SAR(1)_{12}$ model with quadratic trend on the EWMA control chart are compared in terms of ARL using the explicit formulas and NIE method. The results of this comparison are shown in Table 9. The results show that the ARL_1 values obtained from the explicit formula and the NIE methods were similar and rapidly declined as the shift size decreased. The absolute percentage difference between the ARL values obtained using the explicit formula and the NIE method is approximate 4.94001x-10-7. However, the computational time for the explicit formula is 0.001 seconds, whereas the NIE method requires 2 - 3 seconds.

Table 7. Comparison of ARL derived from explicit formulas and NIE methods for $SAR(1)_{12}$ model with quadratic trend and without quadratic trend on an EWMA control chart at ARL_0 = 370

$SAR(1)_{12}$ model with quadratic trend (ϕ = 0.30, β_1 = 1.00, β_2= 0.20)						$SAR(1)_{12}$ model without quadratic trend (ϕ = 0.30)			
λ = 0.05, b = 0.00000002307						λ = 0.05, b = 0.0000000766			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used		$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	**370.076**	0.001	**370.076**	2.145		370.099	0.001	370.099	2.447
0.01	**299.294**	0.001	**299.294**	2.225		302.879	0.001	302.879	2.578
0.03	**198.227**	0.001	**198.227**	2.394		205.255	0.001	205.255	2.613
0.05	**133.426**	0.001	**133.426**	2.440		141.222	0.001	141.222	2.256
0.10	**53.042**	0.001	**53.042**	2.341		59.044	0.001	59.044	2.441
0.20	**11.095**	0.001	**11.095**	2.309		13.331	0.001	13.331	2.544
0.30	**3.504**	0.001	**3.504**	2.494		4.303	0.001	4.303	2.603
0.40	**1.754**	0.001	**1.754**	2.493		2.062	0.001	2.062	2.162
0.50	**1.265**	0.001	**1.265**	2.256		1.395	0.001	1.395	2.535
1.00	**1.007**	0.001	**1.007**	2.441		1.012	0.001	1.012	2.613
1.50	**1.001**	0.001	**1.001**	2.525		1.001	0.001	1.001	2.387
2.00	**1.000**	0.001	**1.000**	2.440		1.000	0.001	1.000	2.403

	SAR(1)$_{12}$ model with quadratic trend (ϕ = 0.30, β_1 = 1.00, β_2= 0.20)				SAR(1)$_{12}$ model without quadratic trend (ϕ = 0.30)			
	(ϕ = 0.30, β_1 = 1.00, β_2= 0.20)				λ = 0.10, b = 0.003278			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	**370.628**	0.001	**370.628**	2.247	370.038	0.001	370.038	2.575
0.01	**330.479**	0.001	**330.479**	2.423	334.099	0.001	334.099	2.466
0.03	**264.492**	0.001	**264.492**	2.494	273.928	0.001	273.928	2.590
0.05	**213.462**	0.001	**213.462**	2.257	226.250	0.001	226.250	2.434
0.10	**129.208**	0.001	**129.208**	2.421	144.496	0.001	144.496	2.466
0.20	**53.722**	0.001	**53.722**	2.234	65.830	0.001	65.830	2.659
0.30	**25.718**	0.001	**25.718**	2.243	33.891	0.001	33.891	2.512
0.40	**13.848**	0.001	**13.848**	2.393	19.287	0.001	19.287	2.700
0.50	**8.255**	0.001	**8.255**	2.456	11.943	0.001	11.943	2.506
1.00	**1.939**	0.001	**1.939**	2.443	2.733	0.001	2.733	2.622
1.50	**1.262**	0.001	**1.262**	2.325	1.546	0.001	1.546	2.534
2.00	**1.108**	0.001	**1.108**	2.240	1.244	0.001	1.244	2.203
	λ = 0.15, b = 0.019401				λ = 0.15, b = 0.06979			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	**370.029**	0.001	**370.029**	2.345	370.037	0.001	370.037	2.462
0.01	**330.716**	0.001	**330.716**	2.425	339.628	0.001	339.628	2.525
0.03	**267.083**	0.001	**267.083**	2.414	287.832	0.001	287.832	2.494
0.05	**218.505**	0.001	**218.505**	2.440	245.769	0.001	245.769	2.579
0.10	**138.645**	0.001	**138.645**	2.540	170.465	0.001	170.465	2.340
0.20	**64.961**	0.001	**64.961**	2.445	90.847	0.001	90.847	2.456
0.30	**35.170**	0.001	**35.170**	2.424	53.799	0.001	53.799	2.504
0.40	**21.150**	0.001	**21.150**	2.154	34.526	0.001	34.526	2.709
0.50	**13.798**	0.001	**13.798**	2.236	23.610	0.001	23.610	2.593
1.00	**3.610**	0.001	**3.610**	2.444	6.583	0.001	6.583	2.425
1.50	**1.978**	0.001	**1.978**	2.325	3.329	0.001	3.329	2.325
2.00	**1.495**	0.001	**1.495**	2.340	2.264	0.001	2.264	2.478

Table 8. Comparison of ARL values for EWMA control chart at ARL_0 = 500: SAR(1)$_{12}$ model with quadratic trendand without quadratic trend using explicit formulas and NIE methods

	SAR(1)$_{12}$ model with quadratic trend (ϕ = 0.80,β_1 = 2.00,β_2= 0.50)				SAR(1)$_{12}$ model without quadratic trend (ϕ = 0.80)			
	λ = 0.05, b = 0.000000005156				λ = 0.05, b = 0.00000006282			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	**500.012**	0.001	**500.012**	2.862	500.068	0.001	500.068	2.323
0.01	**397.188**	0.001	**397.188**	2.741	407.162	0.001	407.162	2.450
0.03	**254.042**	0.001	**254.042**	2.525	273.186	0.001	273.186	2.494
0.05	**165.340**	0.001	**165.340**	2.441	186.137	0.001	186.137	2.540
0.10	**60.743**	0.001	**60.743**	2.609	75.996	0.001	75.996	2.545
0.20	**11.111**	0.001	**11.111**	2.525	16.339	0.001	16.339	2.340
0.30	**3.235**	0.001	**3.235**	2.541	4.979	0.001	4.979	2.345
0.40	**1.609**	0.001	**1.609**	2.409	2.245	0.001	2.245	2.560
0.50	**1.197**	0.001	**1.197**	2.440	1.453	0.001	1.453	2.503
1.00	**1.004**	0.001	**1.004**	2.325	1.013	0.001	1.013	2.441
1.50	**1.000**	0.001	**1.000**	2.541	1.001	0.001	1.001	2.420
2.00	**1.000**	0.001	**1.000**	2.309	1.000	0.001	1.000	2.440

	SAR(1)12 model with quadratic trend (ϕ = 0.80,β_1 = 2.00,β_2= 0.50)				SAR(1)12 model without quadratic trend (ϕ = 0.80)			
	λ = 0.10, b = 0.0002142				λ = 0.10, b = 0.00264			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	**500.065**	0.001	**500.065**	2.237	500.150	0.001	500.150	2.450
0.01	**437.351**	0.001	**437.351**	2.235	449.072	0.001	449.072	2.425
0.03	**337.198**	0.001	**337.198**	2.224	364.250	0.001	364.250	2.494
0.05	**262.615**	0.001	**262.615**	2.250	297.754	0.001	297.754	2.545
0.10	**146.389**	0.001	**146.389**	2.251	185.619	0.001	185.619	2.541
0.20	**52.888**	0.001	**52.888**	2.219	81.055	0.001	81.055	2.555
0.30	**22.616**	0.001	**22.616**	2.244	40.245	0.001	40.245	2.453
0.40	**11.163**	0.001	**11.163**	2.193	22.193	0.001	22.193	2.593
0.50	**6.263**	0.001	**6.263**	2.236	13.368	0.001	13.368	2.250
1.00	**1.505**	0.001	**1.505**	2.251	2.796	0.001	2.796	2.141
1.50	**1.118**	0.001	**1.118**	2.235	1.537	0.001	1.537	2.130
2.00	**1.043**	0.001	**1.043**	2.260	1.232	0.001	1.232	2.542
	λ = 0.15, b = 0.0035221				λ = 0.15, b = 0.045635			
δ	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used	$ARL_{Explicit}$	Time Used	ARL_{NIE}	Time Used
0.00	**500.001**	0.001	**500.001**	2.050	500.076	0.001	500.076	2.222
0.01	**422.489**	0.001	**422.489**	2.122	449.786	0.001	449.786	2.124
0.03	**310.531**	0.001	**310.531**	2.350	367.557	0.001	367.557	2.454
0.05	**235.265**	0.001	**235.265**	2.340	303.982	0.001	303.982	2.540
0.10	**129.085**	0.001	**129.085**	2.441	197.539	0.001	197.539	2.545
0.20	**50.073**	0.001	**50.073**	2.149	96.126	0.001	96.126	2.509
0.30	**23.959**	0.001	**23.959**	2.124	53.541	0.001	53.541	2.494
0.40	**13.219**	0.001	**13.219**	2.293	32.872	0.001	32.872	2.593
0.50	**8.135**	0.001	**8.135**	2.366	21.745	0.001	21.745	2.556
1.00	**2.100**	0.001	**2.100**	2.161	5.614	0.001	5.614	2.444
1.50	**1.350**	0.001	**1.350**	2.134	2.825	0.001	2.825	2.425
2.00	**1.159**	0.001	**1.159**	2.145	1.958	0.001	1.958	2.440

Table 9. Comparison of ARL derived from explicit formulas and NIE methods for SAR(1)12 model with quadratic trend on an EWMA control chart with λ = 0.05 and b = 0.011125

SAR(1)12 model with quadratic trend (μ = 161.7624, ϕ = 0.342961, β_1= -1.849275, β_2= 0.0312841)					
δ	$ARL_{Explicit\ Formulas}$	Time Used	ARL_{NIE}	Time Used	Absolute Percentage Difference
0.00	370.7977116444	0.001	370.7977085932	2.812	8.22875×10−7
0.01	341.2097266337	0.001	341.2097238819	2.766	8.06483×10−7
0.03	290.2658716751	0.001	290.2658694253	2.765	7.75083×10−7
0.05	248.3839378180	0.001	248.3839359665	2.750	7.45419×10−7
0.10	172.2830242304	0.001	172.2830230624	2.797	6.77954×10−7
0.20	90.5140314454	0.001	90.5140309325	2.796	5.66654×10−7
0.30	52.3628141150	0.001	52.3628138643	2.356	4.78775×10−7
0.40	32.7314969870	0.001	32.7314968534	2.750	4.08170×10−7
0.50	21.8045923548	0.001	21.8045922785	2.797	3.49926×10−7
1.00	5.5346954708	0.001	5.5346954614	2.828	1.69838×10−7
1.50	2.7291001754	0.001	2.7291001731	2.921	8.42769×10−8
2.00	1.8795413030	0.001	1.8795413022	2.907	4.25636×10−8

Conclusions

This paper proposed a method for computing the ARL for $SAR(1)_L$ models with a quadratic trend on an EWMA control chart using explicit formulas. Its effectiveness was examined across various exponential smoothing parameter and shift size values. Additionally, the performance of the EWMA control chart was demonstrated using two different models: the $SAR(1)_L$ model with a quadratic trend component and the $SAR(1)_L$ model without quadratic trend. In conclusion, the finding of this study shows that the proposed explicit formula effectively computes ARL values for $SAR(1)_4$ and $SAR(1)_{12}$ models with a quadratic trend using EWMA control charts. There is an excellent agreement between the ARL values obtained from the explicit formula and the NIE method, with an absolute percentage difference of less than 0.000001. Moreover, the results indicate that an exponential smoothing parameterλ = 0.05 consistently yields superior detection performance by providing the lowest ARL values across all magnitudes of shift, thereby improving the sensitivity of the control chart. Nevertheless, the explicit formula provides a significant advantage in computational efficiency, requiring only 0.001 seconds compared to 2 - 3 seconds for the NIE method. Furthermore, including a quadratic trend in the $SAR(1)_{12}$ model results in reduced ARL values compared to models without a quadratic trend, highlighting the effect of trend components on control chart performance. Notably, this study focuses only on $SAR(1)_L$ models with a quadratic trend on an EWMA control chart. Future research could investigate the potential of this method in other interesting models.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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