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RESEARCH ARTICLE

On the Edge Decomposition of the Complete Multipartite Graph and Certain Generalized Petersen Graphs into Locally Irregular Subgraphs

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Abstract The least number of colors to decompose a graph G into a locally irregular edge coloring is the locally irregular chromatic index and denoted by $\chi'_{\mathrm{irr}}(G)$. In 2015, Baudon *et al*. showed that, for complete bipartite graph $K_{p,q}$, . $\chi'_{\mathrm{irr}}(K_{p,q})=2$. In this paper, we determine the locally irregular chromatic index for complete multipartite graphs. In 2023, B. Lužar *et al*. conjectured that, for generalized Petersen graphs $P(n,k)$ with girth at least 5 (except $P(7,2)$ and $P(11,2)$), $\chi'_{\mathrm{irr}}(P(n,k))=2$. Here, we partially verify the conjecture for $P(n,k)$ and determine the exact values for . $\chi'_{\mathrm{irr}}(P(n,1))$ .and. $\chi'_{\mathrm{irr}}(P(n,2))$. The exact values of $\chi'_{\mathrm{irr}}(P(n,k))$ are also determined for when n and k satisfies certain conditions.

Keywords: Locally irregular, subcubic graph, complete multipartite graph, generalized Petersen graph.

Introduction

Graph decomposition allows us to simplify large graphs with many vertices and edges into smaller, more manageable graphs. Some notable applications such as parallel computing [1, 2], logistics [3], dynamic programming [4] and bioinformatics [5, 6]. For comprehensive reviews of graph decomposition and their applications, we refer the interested reader to these surveys [7, 8].

A graph is *locally irregular* if any two adjacent vertices have different degrees. A k - *locally irregular edge-decomposition* (k -liec for short) of a graph G is a collection of locally irregular subgraphs $G_1, G_2, \ldots, G_k$ of G such that each G_i , $i = 1,2,\ldots,k$ is locally irregular and edge disjoint. The least number k such that G admits a k -liec is called *locally irregular chromatic index* . A graph G is *decomposable* if G admits a locally irregular edge-coloring. If G is not decomposable, G is *exceptional* and $\chi'_{\mathrm{irr}}(G)=\infty$.

Determining whether a given graph admits a locally irregular edge coloring with two colors is known to be NP-complete, even when restricted to planar graphs that have maximum degree at most six [9]. This places the problem within the broader family of computationally hard coloring problems, for which exact solutions are often intractable for large instances [10]. Although NP-complete problems are theoretically equivalent (there exists a bijective reduction between any two NP-complete problems [11]), their practical difficulties can vary depending on the structure of instances and available algorithms, even within a single NP-complete domain, such as SAT [12]. As a result, studying diverse NP-complete problems is essential for understanding their computational behavior and advancing solution methods.

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Here, we list some known results regarding the locally irregular edge coloring problem. In this paper, all graphs are considered to be simple, undirected and finite unless stated otherwise.

A *cactus* is a connected graph in which any two simple cycles have at most one vertex in common.

Theorem 1.1. *[13] A connected graph is exceptional, if and only if it is (1) a path of odd-length, (2) a cycle of add length, or (3) a special subclass of cacti,* T *.*

By Theorem 1.1, all exceptional graphs necessarily have odd number of edges and the complete set of exceptional graphs characterized in [13]. The family of graphs T, are defined inductively as follows:

- the triangle, $K_3 \in \mathrm{T}$;
- if $G \in \mathrm{T}$, then another graph $H \in \mathrm{T}$ can be obtained by first selecting either an even-length path or an odd-length path such that an end vertex of that path is identified with a vertex of a triangle. Then, we identify an end vertex, u belonging to either one of the graphs mentioned above, where $deg(u) = 1$ with a vertex $v \in V(G)$, belonging to a triangle in G, where $deg(v) = 2$. The resulting graph H will then also belong to T.

The only locally irregular path with size at least 1 is the path P_3. Therefore, it follows that a decomposable path must be of even-length. Similar arguments hold for the case of cycles. For the case of the family of graphs T, it was proven by contradiction that there cannot exist a minimal counterexample of $G \in \mathrm{T}$ such that G is decomposable. Conjecture 1.2 was verified for some other graph classes in recent years, see [14], [15].

Conjecture 1.2. *[13] (Local Irregularity Conjecture) Every decomposable graph G admits a 3-locally irregular decomposition,* $\chi'_{\mathrm{irr}}(G) \le 3$ *.*

Bensmail *et al*. [15] determined bounds of $\chi'_{\mathrm{irr}}(G)$ for bipartite graphs of even size with $\chi'_{\mathrm{irr}}(G) \le 9$ as well as bipartite graphs in general that are not odd-length paths with $\chi'_{\mathrm{irr}}(G) \le 10$. The bounds for bipartite graphs in general is then lowered such that $\chi'_{\mathrm{irr}}(G) \le 7$ [16].

In [13], it was shown that the locally irregular chromatic index of any complete graph K_n, where $n \ge 4$, is 3. The 3-liec of any complete graph, K_n, where $n \ge 4$, was obtained by first considering a 3-liec of the graph K_4, say with color 1, 2 and 3. Then by adding vertices where all of its incident edges are colored with the same color alternatively, say 1 and 2, we are able to obtain the 3-liec of $K_5, K_6, \ldots$ and so on. Additionally, it was also shown in [13] that if $K_{p,q}$ is a complete bipartite graph with $q \ge 2$ (or $p \ge 2$), then there exists its locally irregular 2-edge coloring, or even a locally irregular 1-edge coloring if $p \ne q$. In the same paper, the authors have shown that there exists a locally irregular 2-edge coloring of every regular bipartite graph G with minimum degree ≥ 3.

A *multipartite* graph (also known as a k*-partite* graph) is a graph whose vertices can be partitioned into $k > 1$ independent partite sets such that no edge connects vertices belonging to the same partite set. A *complete multipartite graph*, denoted as $K_{p_1,p_2,\ldots,p_k}$ is a graph where every pair of vertices not belonging to the same set are connected by an edge.

In 2021, Sedler and Škrekovski have proven that Conjecture 1.2 does not hold for the bow-tie graph B (Figure 1), which is a cactus that requires four colors for a locally irregular edge-coloring [17]. This shows that Conjecture 1.2 does not hold for all finite graphs in general. Then Lei *et al*. [18] showed that if G is a decomposable cactus, then $\chi'_{\mathrm{irr}}(G) \le 4$, and if G is a decomposable cactus without nontrivial cut edges, then $\chi'_{\mathrm{irr}}(G) \le 3$. Recently in [9], it is further proved that B is the only cactus such that $\chi'_{\mathrm{irr}}(G) > 3$. They believe that the bow-tie graph B is the only counter example to the local irregularity conjecture not just in the realm of cactus graphs but for all graphs in general. This led to the following modification of the local irregularity conjecture:

Conjecture 1.4. *[19] (Local Irregularity Conjecture) Every decomposable graph G, except for the bow-tie graph B admits a 3-locally irregular decomposition,* $\chi'_{\text{irr}}(G) \le 3$.

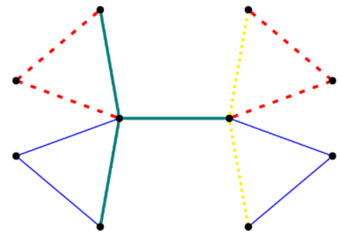


Figure 1. The bow-tie, *B* and its 4-liec

The bow-tie graph, *B* is illustrated in Figure 1. Recently, B. Lužar *et al*. [20] proved that decomposable claw-free graph with maximum degree 3, all cycle permutation graphs, and all the generalized Petersen graphs admit a locally irregular coloring with at most 3 colors. In the same paper, the authors proposed the following conjecture.

Conjecture 1.5. *[20] (Conjecture 5.3) For every generalized Petersen graph G with girth at least 5, with the exception of $P(7,2)$ and $P(11,2)$, we have that* $\chi'_{irr}(G) = 2$.

In this paper, we determine the locally irregular chromatic index for complete multipartite graph (see Theorems 2.2 and 2.4). Here, we also partially verify Conjecture 1.5 for $P(n,2)$ (Theorem 2.16), determine $\chi'_{\text{irr}}(P(n,1))$ (Theorem 2.11) for all $n \ge 3$, and $\chi'_{\text{irr}}(P(n,k))$ (Theorems 2.5, 2.6 and 2.7) for even n or odd k.

Given an edge coloring of a graph G using c colors, let G_i denote the subgraph of G induced by the edges of color i, for $i = 1,2,...,c$. Let $\deg_i(x)$ denote the degree of a vertex x in G_i. Throughout the paper, we say two vertices have the *same color degree* if they have the same number of edges of same color. Particularly, two vertices have *the same color degree* k if they have k edges of the same color. For any vertex $u \in G$, we let $\deg_i(u) = 0$ if $u \notin G_i$.

Results

Theorem 2.1. *[21] If G is complete k-partite graph different from $K_{1,1,\ldots,1}$, then* $\chi'_{\text{irr}}(G) \le 2$.

Theorem 2.1 was recently published. This result is similar to the upcoming Theorem 2.4. In [21], the authors ordered the partite sets in a non-increasing order and first established a coloring for k = 3. For $k \ge 4$, they extended the coloring inductively by assigning blue to all edges connected to the newly added partite set when *k* is even and red when *k* is odd.

Independently, we developed a different proof in which the partite sets are ordered in a non-decreasing order. Instead of coloring separate partite sets, we assign one color to the edges connected to half (by floor function) of the partite sets and the other color to the remaining half. We then demonstrate that, for each induced subgraph, the degree of any vertex in a given partite set is strictly greater than the degree of the vertices in the next set. Our proof follows a significantly different approach from [21], and we felt that it is worthwhile to show it.

Complete Multipartite Graphs Where the Cardinality of Each of the Partite Sets is Not Equal to 1.

The following theorem characterizes all complete multipartite graphs in which none of the partite sets is of size 1.

Theorem 2.2. Let G be the complete multipartite graph $K_{p_1,p_2,\ldots,p_m}$ with $p_1 \ge 2$ and $p_{i+1} \ge p_i$ for all $i = 1,2,\ldots,m-1$. Then

$$\chi'_{\text{irr}}(G)=\begin{cases}1, & \text{if } p_i\neq p_{i+1} \text{ for all } i;\\ 2, & \text{if } p_i=p_{i+1} \text{ for some } i\in\{1,2,\ldots,m-1\}\end{cases}$$

Proof. Suppose $p_i \neq p_{i+1}$ for all . Let x and y be some vertices of the partite sets of size p_{i_0} and p_{j_0}, respectively, where $i_0 \neq j_0$. Note that $\deg(x)=\sum_{1\le i\le m, i\neq i_0} p_i$ and $\deg(y)=\sum_{1\le i\le m, i\neq j_0} p_i$. This implies that $\deg(x)-\deg(y)=p_{j_0}-p_{i_0}\neq 0$. So, two non-adjacent vertices are of different degree. Hence, G is locally irregular and $\chi'_{\text{irr}}(G)=1$.

Suppose $p_{i_0}=p_{i_0+1}$ for some $i_0\in\{1,2,\ldots,m-1\}$. If every edge in G is assigned with 1 color, then the vertices in i_0-th partite and the vertices in (i_0+1)-th partite have the same degree. Since each of the vertices in i_0-th partite is adjacent to all the vertices in (i_0+1)-th partite, G is not locally irregular. Hence, $\chi'_{\text{irr}}(G)\geq 2$.

Now, to show that $\chi'_{\text{irr}}(G)=2$, it is sufficient to show that there exists a 2-liec φ for G. For each $i\in\{1,2,\ldots,m\}$, let P_i be the set of vertices in i-th partite and $p_i=|P_i|$. We let $P_i=T_i^1\cup T_i^2$ where $|T_i^1|=\left\lceil \frac{p_i}{2}\right\rceil$ and $|T_i^2|=\left\lfloor \frac{p_i}{2}\right\rfloor$.

The edge coloring φ will be defined as follows:

- For each P_i with $i=1,2,\ldots,m-1$ and for all $j>i$, all the edges of the form u_1v where $u_1\in T_i^1$ and $v\in P_j$ will be colored with 1 and all the edges of the form u_2v where $u_2\in T_i^2$ and $v\in P_j$ will be colored with 2.

Let G_r be the subgraph induced by the edges with color r under φ in G. Let $\deg_r(v_i)$ be the degree of $v_i\in P_i$ in G_r. Here, we claim that in each G_r for $r=1,2$, we have $\deg_r(v_i)>\deg_r(v_{i+1})$ for $v_i\in P_i$ and $v_{i+1}\in P_{i+1}$ where $1\le i\le m-1$. Suppose $v_i^r\in T_i^r$. Clearly, $\deg_1(v_1^1)=\sum_{2\le s\le m} p_s$ and $\deg_1(v_1^2)=0$. Also, $\deg_1(v_m)=\sum_{1\le s<m}\left\lceil \frac{p_s}{2}\right\rceil$ for every $v_m\in P_m$. In fact, for $i=1,2,\ldots,m$ and each P_i in G_1,

$$\deg_1(v_i^1)=\sum_{1\le s<i}\left\lceil \frac{p_s}{2}\right\rceil+\sum_{m\ge s>i} p_s;$$
$$\deg_1(v_i^2)=\sum_{1\le s<i}\left\lceil \frac{p_s}{2}\right\rceil.$$

Therefore, if $i\in\{1,2,\ldots,m-1\}$, then

$$\begin{aligned}\deg_1(v_{i+1}^1)&=\sum_{1\le s<i+1}\left\lceil \frac{p_s}{2}\right\rceil+\sum_{m\ge s>i+1} p_s\\ &=\left(\sum_{1\le s<i}\left\lceil \frac{p_s}{2}\right\rceil+\left\lceil \frac{p_i}{2}\right\rceil\right)+\left(\sum_{m\ge s>i} p_s-p_{i+1}\right)\\ &=\deg_1(v_i^1)+\left\lceil \frac{p_i}{2}\right\rceil-p_{i+1}\\ &<\deg_1(v_i^1),\end{aligned}$$

where the last inequality follows from $\left\lceil \frac{p_i}{2}\right\rceil-p_{i+1}<\left(\frac{p_i}{2}+1\right)-p_i=\frac{2-p_i}{2}\le 0$. Similarly,

$$\begin{aligned}\deg_1(v_{i+1}^2) &= \sum_{1\le s<i+1}\left\lceil \frac{p_s}{2}\right\rceil \\ &= \sum_{1\le s<i}\left\lceil \frac{p_s}{2}\right\rceil + \left\lceil \frac{p_i}{2}\right\rceil \\ &< \sum_{1\le s<i}\left\lceil \frac{p_s}{2}\right\rceil + p_{i+1} \\ &\le \sum_{1\le s<i}\left\lceil \frac{p_s}{2}\right\rceil + \sum_{m\ge s>i} p_s = \deg_1(v_i^1).\end{aligned}$$

Note that, in G_1, a vertex in T_{i+1}^1 is not adjacent to any vertices in T_i^2 and a vertex in T_{i+1}^2 is not adjacent to any vertices in T_i^2. Hence, $\deg_1(v_i) > \deg_1(v_{i+1})$ for all $1 \le i \le m-1$. This means G_1 is a locally irregular graph.

Next, for $i = 1, 2, \ldots, m$ and each P_i in G_2,

$$\deg_2(v_i^1) = \sum_{1\le s<i}\left\lfloor \frac{p_s}{2}\right\rfloor;$$

$$\deg_2(v_i^2) = \sum_{1\le s<i}\left\lfloor \frac{p_s}{2}\right\rfloor + \sum_{m\ge s>i} p_s.$$

So, for each $i \in \{1, 2, \ldots, m-1\}$, we have:

$$\begin{aligned}\deg_2(v_{i+1}^2) &= \sum_{1\le s<i+1}\left\lfloor \frac{p_s}{2}\right\rfloor + \sum_{m\ge s>i+1} p_s \\ &= \left(\sum_{1\le s<i}\left\lfloor \frac{p_s}{2}\right\rfloor + \left\lfloor \frac{p_i}{2}\right\rfloor\right) + \left(\sum_{m\ge s>i} p_s - p_{i+1}\right) \\ &= \deg_2(v_i^2) + \left\lfloor \frac{p_i}{2}\right\rfloor - p_{i+1} < \deg_2(v_i^2),\end{aligned}$$

where the last inequality follows from $\left\lfloor \frac{p_i}{2}\right\rfloor - p_{i+1} \le \left(\frac{p_i}{2}\right) - p_i = \frac{-p_i}{2} < 0$. Similarly,

$$\begin{aligned}\deg_2(v_{i+1}^1) &= \sum_{1\le s<i+1}\left\lfloor \frac{p_s}{2}\right\rfloor \\ &= \sum_{1\le s<i}\left\lfloor \frac{p_s}{2}\right\rfloor + \left\lfloor \frac{p_i}{2}\right\rfloor \\ &< \sum_{1\le s<i}\left\lfloor \frac{p_s}{2}\right\rfloor + p_{i+1} \\ &\le \sum_{1\le s<i}\left\lfloor \frac{p_s}{2}\right\rfloor + \sum_{m\ge s>i} p_s = \deg_2(v_i^2).\end{aligned}$$

Again, in G_2, a vertex in T_{i+1}^1 is not adjacent to any vertices in T_i^1 and a vertex in T_{i+1}^2 is not adjacent to any vertices in T_i^1. Hence, $\deg_2(v_i) > \deg_2(v_{i+1})$ for all $1 \le i \le m-1$. Therefore, G_2 is locally irregular.

□

Complete multipartite graphs where the cardinality of at least one (but not all) of the partite sets is equal to 1

Suppose G and H are two graphs. The *join of* G and H, denoted by $G+H$, is the graph with vertex set and edge set:

$$V(G+H)=V(G)\cup V(H)$$
$$E(G+H)=E(G)\cup E(H)\cup\{xy \mid x\in V(G), y\in V(H)\}$$

Here, we assume that $V(H)$ and $V(G)$ are disjoint. For the join of a graph G with a single vertex x, we write $G+x$.

Lemma 2.3. *Let G be a graph with $\chi'_{\text{irr}}(G)=r\geq 1$ and x be an isolated vertex where $x\notin V(G)$. Let $G=G_1\cup G_2\cup\cdots\cup G_r$ be an edge decomposition of G into locally irregular and edge disjoint subgraphs. If there exists r_0 such that the maximum degree of G_{r_0} is at most $|V(G)|-2$, then $\chi'_{\text{irr}}(G+x)\leq r$ and the equality holds if $r\in\{1,2\}$.*

Proof. Note that G is a subgraph of $G+x$. If $r=1$, then it is clear that $\chi'_{\text{irr}}(G+x)\geq\chi'_{\text{irr}}(G)=1$. Suppose $r=2$ and there exists a coloring φ of $G+x$ such that $\chi'_{\text{irr}}(G+x)=1$. This implies that $G+x$ is locally irregular in single color. However, there exist two vertices $u,v\in G$ such that $deg(u)=deg(v)$, a contradiction. Hence $\chi'_{\text{irr}}(G+x)\geq r$ if $r=1,2$.

Here, we show that $\chi'_{\text{irr}}(G+x)\leq r$. The edges in G will be colored according to the edge decomposition of G into the union of edge disjoint subgraphs $G_1,G_2,\ldots,G_r$. Now, each edge ux in $G+x$ with $u\in V(G)$ will be colored with r_0. Based on this edge coloring, G can be decomposed into edge disjoint subgraphs $G_{1'},G_{2'},\ldots,G_{r'}$ where $G_{i'}=G_i$ for all $i\neq r_0$ and $G_{r_0'}$ is the subgraph of $G+x$ with vertex set $V(G)\cup\{x\}$ and edge set $E(G_{r_0})\cup\{ux : u\in V(G)\}$. Since each G_i is locally irregular, it is sufficient to show that $G_{r_0'}$ is locally irregular. We shall denote the degree of a vertex u in the subgraph G_{r_0} by $\deg_{r_0}(u)$ and the degree of a vertex v in the subgraph $G_{r_0'}$ by $\deg_{r_0'}(v)$. Note that if $u\in V(G_{r_0})$, then $\deg_{r_0'}(u)=\deg_{r_0}(u)+1$. Furthermore, $\deg_{r_0'}(x)=|V(G)|$.

Suppose $G_{r_0'}$ is not locally irregular. Then there are two adjacent vertices of the same degree in $G_{r_0'}$, i.e., $\deg_{r_0'}(u)=\deg_{r_0'}(v)$ for some vertices $u,v\in V(G_{r_0'})$ where u and v are adjacent to each other. If $u,v\notin\{x\}$, then $\deg_{r_0'}(u)=\deg_{r_0'}(v)$ implies that $\deg_{r_0}(u)=\deg_{r_0}(v)$, which is not possible as G_{r_0} is locally irregular. Suppose one of the vertices in u,v is x. We may assume that $v=x$. Then $\deg_{r_0'}(u)=\deg_{r_0'}(x)=|V(G)|$. This implies that $\deg_{r_0}(u)=|V(G)|-1>|V(G)|-2$, a contradiction. Hence, $G_{r_0'}$ is locally irregular. ∎

Theorem 2.4. *Let G be the complete m-partite graph $K_{p_1,p_2,\ldots,p_m}$ with $p_1\geq 2, p_{i+1}\geq p_i$ for all $i=1,2,\ldots,m-1$ for $m\geq 2$. Then*

$$\chi'_{\text{irr}}(K_n+G)=\begin{cases}1, & \text{if } p_i\neq p_{i+1} \text{ for all } i \text{ and } n=1;\\ 2, & \text{otherwise.}\end{cases}$$

Proof. It is clear that K_n+G is the complete $(m+n)$-partite graph $K_{1,\ldots,1,p_1,p_2,\ldots,p_m}$.

Suppose $n=1$. By Theorem 2.2,

$$\chi'_{\text{irr}}(G)=\begin{cases}1, & \text{if } p_i \neq p_{i+1} \text{for all } i;\\ 2, & \text{if } p_i = p_{i+1} \text{ for some } i\in\{1,2,\ldots,m-1\}.\end{cases}$$

Since $p_1 \geq 2$, the maximum degree of G is at most $|V(G)|-2$. This implies that the maximum degree of any subgraph of G is at most $|V(G)|-2$. It follows from Lemma 2.3 that

$$\chi'_{\text{irr}}(G+x)=\begin{cases}1, & \text{if } p_i \neq p_{i+1} \text{ for all } i;\\ 2, & \text{otherwise.}\end{cases}$$

Suppose $n\geq 2$. If all edges of K_n+G are colored by one color, then it is clear that the n vertices in K_n have the same degree. Hence $\chi'_{\text{irr}}(K_n+G)\geq 2$. Note that $K_n+G=(K_{n-1}+G)+x$ for some $x\in V(K_n)$. We may assume by induction on n that $\chi'_{\text{irr}}(K_{n-1}+G)\leq 2$. If $\chi'_{\text{irr}}(K_{n-1}+G)=1$, then $\chi'_{\text{irr}}(K_n+G)=2$. Suppose $\chi'_{\text{irr}}(K_{n-1}+G)=2$. Let $K_{n-1}+G=G_1\cup G_2$ be an edge decomposition of $K_{n-1}+G$ into locally irregular and edge disjoint subgraphs. If the maximum degree of G_1 is at most $|V(K_{n-1}+G)|-2$, then by Lemma 2.3, $\chi'_{\text{irr}}(K_n+G)=2$. Suppose the maximum degree of G_1 is $|V(K_{n-1}+G)|-1$. Let $u\in V(G_1)$ be the vertex that has the maximum degree, i.e., $\deg_1(u)=|V(K_{n-1}+G)|-1$. Clearly, $u\in V(K_{n-1})$ and it is adjacent to all other vertices in $K_{n-1}+G$. Now, if G_2 has a vertex v of degree $|V(K_{n-1}+G)|-1$, then v is adjacent to u in G_2. This means the edge vu appears in G_2 and G_1, a contradiction. So, the maximum degree of G_2 is at most $|V(K_{n-1}+G)|-2$. By Lemma 2.3, $\chi'_{\text{irr}}(K_n+G)=2$.

□

Generalized Petersen Graphs

Let $n\geq 3$ and $k\geq 1$ be integers such that $1\leq k\leq \left\lfloor \frac{n-1}{2}\right\rfloor$. The generalized Petersen graph $P(n,k)$ is a graph on $2n$ vertices with vertex set:

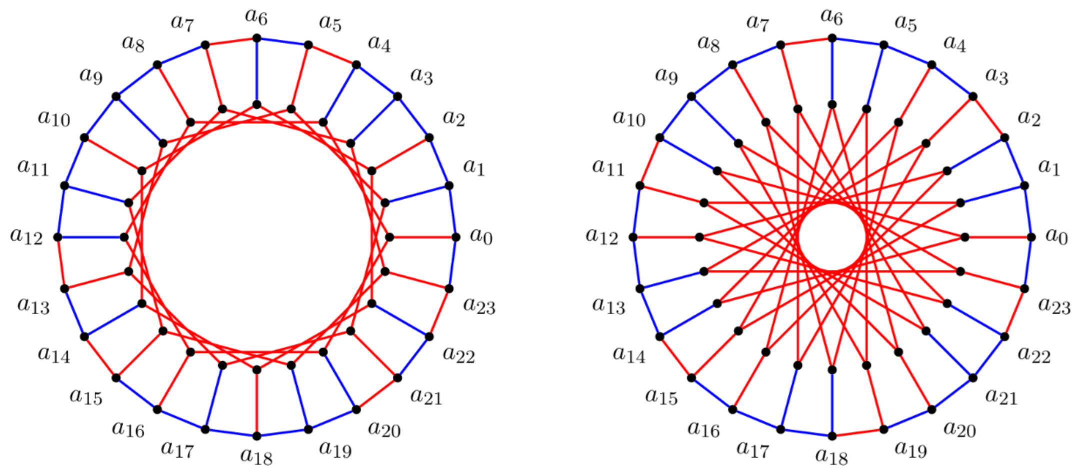
$$V(P(n,k))=\{a_i,b_i : i=0,1,2,\ldots,n-1\}$$

and edge set:

$$E(P(n,k))=\{a_ia_{i+1},a_ib_i,b_ib_{i+k} : i=0,1,2,\ldots,n-1\},$$

where subscripts are taken modulo n. See Figure 2 for examples of $P(24,4)$ and $P(24,10)$. Let $D_1=\{a_i : i=0,1,2,\ldots,n-1\}$ and $D_2=\{b_i : i=0,1,2,\ldots,n-1\}$. The subgraph induced by D_1 is the *outer rim* while the subgraph induced by D_2 is the *inner rim*. A *spoke* of $P(n,k)$ is an edge of the form a_ib_i for some $0\leq i\leq n-1$. We shall use these notations in this section. Furthermore, the colorings will be depicted in all diagrams in the rest of the paper as follows:

- Edges assigned with color 1 is shown in *red* lines.
- Edges assigned with color 2 is shown in *blue* lines.
- Edges which are yet to be colored are shown in *black* lines.
- Edges that are not to be colored by either color 1 or color 2 will be overlayed with an X as either of the colors assigned will result in the existence of two neighboring vertices having same degree of the same color.



(a) P(24,4) (b) P(24,10)

Figure 2. Examples of generalized Petersen graphs

First, we note here that $\chi'_{\text{irr}}(P(n,k)) \geq 2$. Indeed, if all the edges of $P(n,k)$ is of single color, $P(n,k)$ is not locally irregular as all the vertices in the outer rim are of the same degree. It is also to be noted that two graphs, $P(n,k)$ and $P(n,l)$ are isomorphic if $\gcd(n,k) = \gcd(n,l) = 1$ and $kl \equiv \pm 1 (\bmod n)$, see [22]. We begin with the following theorem.

Theorem 2.5. *If* $\dfrac{n}{\gcd(k,n)}$ *is even, then* $\chi'_{\text{irr}}(P(n,k)) = 2$.

Proof. Let $\lambda = \gcd(k,n)$ and $m = \dfrac{n}{\lambda}$. We will begin by partitioning the spokes $a_i b_i$ into m sets, say $T_0, T_1, T_2, \ldots, T_{m-1}$. For each $0 \leq j \leq m-1$, let $T_j = \{a_{j\lambda} b_{j\lambda}, a_{j\lambda+1} b_{j\lambda+1}, \ldots, a_{(j+1)\lambda-1} b_{(j+1)\lambda-1}\}$. Note that λ is the number of m-cycles in the inner rim of $P(n,k)$. With this partition in mind, we will color the edges $P(n,k)$ using $\varphi : E(P(n,k)) \mapsto \{1,2\}$ as follows:

(1) All the edges in the inner rim, $b_i b_{i+k}$ are colored with 1 for $0 \leq i \leq n-1$.
(2) For each $0 \leq j \leq m-1$ where *j* is even and $0 \leq l \leq \lambda - 1$, the edges $a_{j\lambda+l} b_{j\lambda+l} \in T_j$ are colored with 1 if l is even and 2 otherwise.
(3) For each $0 \leq j \leq m-1$ where *j* is odd and $0 \leq l \leq \lambda - 1$, the edges $a_{j\lambda+l} b_{j\lambda+l} \in T_j$ are colored with 2 if l is even and 1 otherwise.
(4) If λ is odd, color all the outer rim edges $a_0 a_1 \ldots a_{n-1} a_0$ with color 2.
(5) If λ is even and for j is even, color the outer rim edges $a_{i-1} a_i$ with color 2 for all a_i where $a_i b_i \in T_j$.
(6) If λ is even and for j is odd, color the outer rim edges $a_{i-1} a_i$ with color 1 if $\varphi(a_i b_i) = 1$ and color 2 if $\varphi(a_i b_i) = 2$ for $a_i b_i \in T_j$.

Figure 2 depicts the examples of $P(24,4)$ and $P(24,10)$ using φ.

Now, consider the subgraph induced by edges of color 1, G_1. The inner rim edges $b_i b_{i+k}$ are colored with color 1 for all $0 \leq i \leq n-1$. Note that with φ, for some constant $0 \leq c \leq \lambda - 1$, the spokes $a_{j_0\lambda+c} b_{j_0\lambda+c}$ and $a_{j_1\lambda+c} b_{j_1\lambda+c}$, $0 \leq j_0, j_1 \leq m-1$ are the same color if and only if j_0 and j_1 are both even or both odd (same indexed elements of T_{j_0} and T_{j_1}). Note that since $\frac{n}{\lambda}$ is even, $\frac{k}{\lambda}$ is necessarily odd as $\gcd(\frac{n}{\lambda},\frac{k}{\lambda})=1$.

Then,

$$\varphi(a_{j_0\lambda+c}b_{j_0\lambda+c}\in T_{j_0})\neq\varphi\left(a_{(j_0\lambda+c)+k}b_{(j_0\lambda+c)+k}\in T_{j_0+\frac{k}{\lambda}}\right)$$

since

$$(j_0\lambda+c)+k=\left(j_0+\frac{k}{\lambda}\right)\lambda+c\,.$$

For ease of explanation, let $a_ib_i\in T_j$ where $i=j\lambda+l$ for some $0\le j\le m-1$ and some $0\le l\le\lambda-1$ for the rest of this proof.

In φ, $\deg_1(b_i)=3$ if and only if b_i is adjacent to a_i in G_1. It follows that $\deg_1(b_i)\neq\deg_1(b_{i+k})$. If λ is odd, all the edges of the outer rim are of color 2. Since $\deg_1(a_i)=1\neq3=\deg_1(b_i)$ and each a_i is not adjacent to another vertex in the outer rim, the subgraph G_1 is locally irregular. If λ is even, $\deg_1(a_i)\in\{1,2\}\neq3=\deg_1(b_i)$. Now we consider adjacent vertices in the outer rim for even λ. By φ, the edge $a_{j\lambda+l-1}a_{j\lambda+l}\in E(G_1)$ if and only if j is odd and l is odd for $0\le j\le m-1$ and $0\le l\le\lambda-1$. Notice that the vertices $a_{j\lambda+l-1}$ and $a_{j\lambda+l}$ are adjacent to each other and no other vertex in the outer rim in G_1. Here, $\deg_1(a_{j\lambda+l-1})=1\neq2=\deg_1(a_{j\lambda+l})$. So, the subgraph G_1 is locally irregular.

Consider the subgraph induced by edges color 2, G_2. Here, we will consider separate cases where λ is odd and λ is even.

Case 1: If λ is odd, the coloring of the spokes with φ is equivalent to coloring a_ib_i with color 1 if i is even and color 2 otherwise. All the edges of the outer cycle are assigned with color 2 and since $\varphi(a_ib_i)\neq\varphi(a_{i+1}b_{i+1})$, it follows that $\deg_2(a_i)\neq\deg_2(a_{i+1})$ for all $0\le i\le n-1$. Furthermore, a_i and b_i are adjacent in G_2 if and only if $\deg_2(a_i)=3$. Since $\deg_2(b_i)=1\neq3=\deg_2(a_i)$, and a vertex b_{i_0} is not adjacent to another vertex b_{i_1} in G_2 for $0\le i_0,i_1,\le n-1$, $i_0\neq i_1$, the subgraph G_2 is locally irregular.

Case 2: For λ is even, we will consider the vertex $a_{j\lambda+l}$, where $0\le j\le m-1$ and $0\le l\le\lambda-1$. Clearly, the following holds:

(1) If j is even and l is even, $\deg_2(a_{j\lambda+l})=2$.

(2) If j is even and l is odd, then $\deg_2(a_{j\lambda+l})=3$.

(3) If j is odd and l is even, then $\deg_2(a_{j\lambda+l})=2$.

(4) If j is odd and l is odd, then $\deg_2(a_{j\lambda+l})=1$.

Therefore, for the same j, adjacent vertices $a_{j\lambda+l}$ and $a_{j\lambda+l+1}$ have distinct color 2 degrees for $0\le l\le\lambda-2$. Now we consider the case where $l=\lambda-1$. The vertex $a_{j\lambda+\lambda-1}$ is adjacent to the vertex $a_{(j+1)\lambda}$. Note that since λ is even, $\lambda-1$ is odd. If j is even, $\deg_2(a_{j\lambda+\lambda-1})=3\neq2=\deg_2(a_{(j+1)\lambda})$. If j is odd, $\deg_2(a_{j\lambda+\lambda-1})=1\neq2=\deg_2(a_{(j+1)\lambda})$. Therefore, adjacent vertices of the outer rim have distinct color 2 degrees. Now we shall consider vertices of the inner rim that are in G_2. Note that the spokes $a_{j\lambda+l}b_{j\lambda+l}$ are color 2 if j is even and l is odd or if j is odd and l is even. In both cases, $\deg_2(b_{j\lambda+l})=1$. Since $\deg_2(a_{j\lambda+l})\in\{2,3\}$, $\deg_2(a_{j\lambda+l})\neq\deg_2(b_{j\lambda+l})$. Again, a vertex b_{i_0} is not adjacent to another vertex b_{i_1} in G_2 for $0\le i_0,i_1,\le n-1$, $i_0\neq i_1$. So, the subgraph G_2 is locally irregular. □

Corollary 2.6 is a direct consequence of Theorem 2.5.

Corollary 2.6. *If* $gcd(k,n)=1$ *and* n *is even, then* $\chi'_{\text{irr}}(P(n,k))=2$.

Now, we investigate the case where $\gcd(k,n)=1$ and both k,n are odd.

Theorem 2.7. *If* $\gcd(k,n)=1$, *both* $k\geq 3$ *and* $n\geq 3k+2$ *are odd, then* $\chi'_{\text{irr}}(P(n,k))=2$.

Proof. We color the edges with two colors accordingly:

- all the edges on the path $a_{k-1}a_k\ldots a_{n-1}a_0$ are colored with 1;
- for $k+1\leq i\leq n-1$ and i is even, all the edges on the path $b_{i-k}b_ia_i$ are colored with 1;
- for $k+1\leq i\leq n-1$ and i is odd, all the edges on the path $b_{i-k}b_ia_i$ are colored with 2;
- all the edges on the path $b_{n-k}b_0b_ka_k$ are colored with 2;
- the edge $b_{n-k+1}b_1$ is colored with 2;
- for $2\leq i\leq k-1$ and i is even, the edge $b_{n-k+i}b_i$ is colored with 2;
- for $2\leq i\leq k-1$ and i is odd, the edge $b_{n-k+i}b_i$ is colored with 1;
- the spokes a_0b_0, a_1b_1 and a_2b_2 are colored with 2;
- all the edges on the path $a_0a_1a_2$ are colored with 2;
- for $3\leq i\leq k-1$ and i is even, all the edges on the path $a_{i-1}a_ib_i$ is colored with 2;
- for $3\leq i\leq k-1$ and i is odd, all the edges on the path $a_{i-1}a_ib_i$ is colored with 1.

Let $P(n,k)=G_1\cup G_2$ be the edge decomposition of $P(n,k)$ into edge disjoint subgraphs according to this coloring where G_1 contains all edges of color 1 and G_2 contains all edges of color 2. The degree of a vertex u in the subgraph G_1 will be denoted by $\deg_1(u)$ whereas the degree of a vertex v in the subgraph G_2 will be denoted by $\deg_2(v)$.

Consider the subgraph G_1. The vertex a_0 is adjacent to a_{n-1} and $\deg_1(a_0)=1$ whereas $\deg_1(a_{n-1})=3$. For even $i\in\{2,3,\ldots,k-1\}$, a_i is adjacent to a_{i+1} and $\deg_1(a_i)=1$ whereas $\deg_1(a_{i+1})=2$ .For odd $i\in\{2,3,\ldots,k-1\}$, a_i is adjacent to all the vertices in $\{a_{i-1},b_i\}$ and $\deg_1(a_i)=2$, $\deg_1(a_{i-1})=1$ and $\deg_1(b_i)=3$. Next, a_k is adjacent to all the vertices in $\{a_{k-1},a_{k+1}\}$ and $\deg_1(a_k)=2$, $\deg_1(a_{k-1})=1$ and $\deg_1(a_{k+1})=3$. For even $i\in\{k+1,k+2,\ldots,n-2\}$, a_i is adjacent to all the vertices in $\{b_i,a_{i-1},a_{i+1}\}$ and $\deg_1(a_i)=3$, $\deg_1(b_i)=2$, and $\deg_1(a_{i-1})=2=\deg_1(a_{i+1})$. Next, a_{n-1} is adjacent to all the vertices in $\{b_{n-1},a_{n-2},a_0\}$ and $\deg_1(a_{n-1})=3$, $\deg_1(b_{n-1})=2$, $\deg_1(a_{n-2})=2$ and $\deg_1(a_0)=1$. For odd $i\in\{k+1,k+2,\ldots,n-2\}$, a_i is adjacent to all the vertices in $\{a_{i-1},a_{i+1}\}$ and $\deg_1(a_i)=2$, and $\deg_1(a_{i-1})=3=\deg_1(a_{i+1})$. The vertex b_1 is adjacent to b_{k+1} and $\deg_1(b_1)=1$ whereas $\deg_1(b_{k+1})=2$. For odd $i\in\{3,4,\ldots,k-2\}$, b_i is adjacent to all the vertices in $\{b_{n-k+i},a_i,b_{i+k}\}$ and $\deg_1(b_i)=3$, $\deg_1(b_{n-k+i})=1$, $\deg_1(a_i)=2$ and $\deg_1(b_{i+k})=2$ .For odd $i\in\{k,k+1,\ldots,n-k-1\}$, b_i is adjacent to b_{i+k} and $\deg_1(b_i)=1$ whereas $\deg_1(b_{i+k})=2$. For odd $i\in\{n-k+3,n-k+4,\ldots,n-1\}$, b_i is adjacent to b_{i+k} and $\deg_1(b_i)=1$ whereas $\deg_1(b_{i+k})=3$ .Next, b_{k+1} is adjacent to all the vertices in $\{b_1,a_{k+1}\}$ and $\deg_1(b_{k+1})=2$, $\deg_1(b_1)=1$, and $\deg_1(a_{k+1})=3$. For even $i\in\{k+3,k+4,\ldots,2k-2\}$, b_i is adjacent to all the vertices in $\{b_{i-k},a_i\}$ and $\deg_1(b_i)=2$, $\deg_1(b_{i-k})=3$, and $\deg_1(a_i)=3$. For even $i\in\{2k,2k+1,\ldots,n-1\}$ b_i is adjacent to all the vertices in $\{b_{i-k},a_i\}$ and $\deg_1(b_i)=2$, $\deg_1(b_{i-k})=1$, and $\deg_1(a_i)=3$. Therefore, G_1 is locally regular.

Consider the subgraph G_2. The vertex a_0 is adjacent to all the vertices in $\{b_0, a_1\}$ and $\deg_2(a_0) = 2$, $\deg_2(b_0) = 3$ and $\deg_2(a_1) = 3$. The vertex a_1 is adjacent to all the vertices in $\{a_0, b_1, a_2\}$ and $\deg_2(a_1) = 3$, $\deg_2(a_0) = 2 = \deg_2(a_2) = \deg_2(b_1)$. For even $i \in \{2,3,\ldots,k-1\}$, a_i is adjacent to all the vertices in $\{a_{i-1}, b_i\}$ and $\deg_2(a_i) = 2$, $\deg_2(a_{i-1}) = 1$ and $\deg_2(b_i) = 3$. For odd $i \in \{2,3,\ldots,k-1\}$, a_i is adjacent to a_{i+1} and $\deg_2(a_i) = 1$ whereas $\deg_2(a_{i+1}) = 2$. For odd $i \in \{k, k+1, \ldots, n-k-1\}$, a_i is adjacent to b_i and $\deg_2(a_i) = 1$ whereas $\deg_2(b_i) = 2$. Next, a_{n-k+1} is adjacent to b_{n-k+1} and $\deg_2(a_{n-k+1}) = 1$ whereas $\deg_2(b_{n-k+1}) = 3$. For odd $i \in \{n-k+3, n-k+4, \ldots, n-2\}$, a_i is adjacent to b_i and $\deg_2(a_i) = 1$ whereas $\deg_2(b_i) = 2$. Now, b_0 is adjacent to all the vertices in $\{b_{n-k}, a_0, b_k\}$ and $\deg_2(b_0) = 3$, $\deg_2(b_{n-k}) = 1$, $\deg_2(b_k) = 2$ and $\deg_2(a_0) = 2$. The vertex b_1 is adjacent to all the vertices in $\{b_{n-k+1}, a_1\}$ and $\deg_2(b_1) = 2$, $\deg_2(b_{n-k+1}) = 3$, $\deg_2(a_1) = 3$. For even $i \in \{2,3,\ldots,k-1\}$, b_i is adjacent to all the vertices in $\{b_{n-k+i}, a_i, b_{i+k}\}$ and $\deg_2(b_i) = 3$, $\deg_2(b_{n-k+i}) = 1$, $\deg_2(b_{i+k}) = 2$ and $\deg_2(a_i) = 2$. For odd $i \in \{k, k+1, \ldots, 2k-1\}$, b_i is adjacent to all the vertices in $\{b_{i-k}, a_i\}$ and $\deg_2(b_i) = 2$, $\deg_2(b_{i-k}) = 3$, and $\deg_2(a_i) = 1$. For odd $i \in \{2k+1, 2k+2, \ldots, n-k-1\}$, b_i is adjacent to all the vertices in $\{b_{i-k}, a_i\}$ and $\deg_2(b_i) = 2$, $\deg_2(b_{i-k}) = 1$, and $\deg_2(a_i) = 1$. The vertex b_{n-k+1} is adjacent to all the vertices in $\{b_{n-2k+1}, a_{n-k+1}, b_1\}$ and $\deg_2(b_{n-k+1}) = 3$, $\deg_2(b_{n-2k+1}) = 1$, $\deg_2(b_1) = 2$, and $\deg_2(a_{n-k+1}) = 1$. For odd $i \in \{n-k+3, n-k+4, \ldots, n-2\}$, b_i is adjacent to all the vertices in $\{b_{i-k}, a_i\}$ and $\deg_2(b_i) = 2$, $\deg_2(b_{i-k}) = 1$, and $\deg_2(a_i) = 1$. For even $i \in \{k+1, k+2, \ldots, n-k-2\}$, b_i is adjacent to b_{i+k} and $\deg_2(b_i) = 1$ whereas $\deg_2(b_{i+k}) = 2$. For even $i \in \{n-k, n-k+1, \ldots, n-1\}$, b_i is adjacent to b_{i+k} and $\deg_2(b_i) = 1$ whereas $\deg_2(b_{i+k}) = 3$. So, G_2 is locally irregular.

□

P(n,k) for *k*=1

Lemma 2.8. *For every* $P(n,1)$*, where* $n \notin \{3,5\}$, $\chi'_{\mathrm{irr}}(P(n,1)) = 2$.

Proof. It is sufficient to show that there is a 2-liec φ of $P(n,1)$. The case n is even follows from Theorem 2.5. Suppose $n \geq 7$ is odd. Here, we define a 2-liec φ of $P(n,1)$ as follows. We color the following edges with color 1:

(i) the 2-length path $b_1 b_2 b_3$;
(ii) the edge $b_{n-2} b_{n-1}$.
(iii) the edge $a_2 b_2$;
(iv) the edge $a_i b_i$ for all odd $3 \leq i \leq n-2$;
(v) the edge $a_i a_{i+1}$ for all $i = 2,3,\ldots,n-1$;

Then, we assign color 2 for the rest of the edges. Figure 3 depicts the 2-liec φ for $P(n,1)$ where color 1 is red and color 2 is blue. By using the coloring φ above, it is easily observed that the spokes are of alternating colored 1 and 2, and each subgraph induced by color 1 and 2 is locally irregular.

□

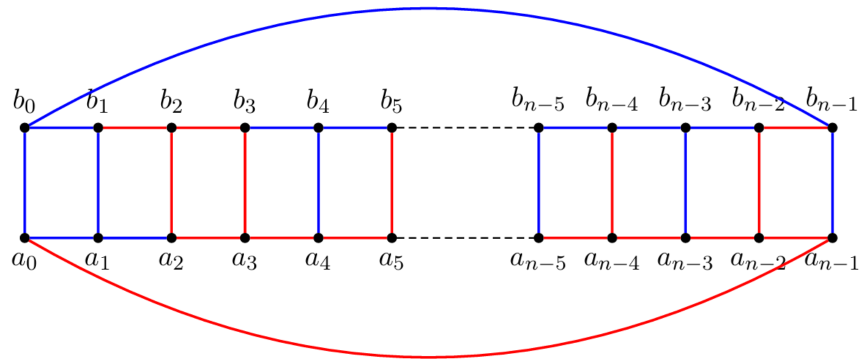


Figure 3. A 2-liec φ of $P(n,1)$ for odd $n \geq 7$

Lemma 2.9. *For the generalised Petersen graph* $P(3,1)$, $\chi'_{\text{irr}}(P(3,1)) = 3$.

Proof. There exists a locally irregular edge coloring $\varphi : P(3,1) \to \{1,2,3\}$ as follows. We let $\varphi(a_0a_1) = \varphi(a_1a_2) = \varphi(a_1b_1) = 1$, $\varphi(a_0a_2) = \varphi(a_0b_0) = \varphi(b_0b_1) = \varphi(b_0b_2) = 2$ and $\varphi(b_1b_2) = \varphi(a_2b_2) = 3$. Therefore, $\chi_{irr'}(P(3,1)) \leq 3$.

To show that the equality holds, we shall show that any edge coloring with 2 colors in $P(3,1)$ will induce a monochromatic subgraph (i.e. subgraph induced by the same color edges) having two adjacent vertices with the same color degree. Suppose there exists an edge coloring φ of $P(3,1)$, using two colors, induces two locally irregular subgraphs with the colors 1 and 2. If all the edges on the triangle $a_0a_1a_2$ are of the same color, then any possible coloring of the edge set $\{a_0b_0, a_1b_1, a_2b_2\}$ will result in which two of the vertices of the triangle $a_0a_1a_2$ having the same color degree. So, without loss of generality, we may assume that the triangle $a_0a_1a_2$ contains two edges of color 1 and one edge of color 2. Since $a_0a_1a_2$ is vertex transitive, we may further assume that $\varphi(a_0a_2) = 2$ and $\varphi(a_0a_1) = \varphi(a_1a_2) = 1$.

If both the edges a_0b_0 and a_2b_2 are of the same color, then a_0 and a_2 would be of the same color degree, a contradiction. Thus, the edges a_0b_0 and a_2b_2 cannot be of the same color.

By symmetry, we may assume that $\varphi(a_0b_0) = 2$ and $\varphi(a_2b_2) = 1$. Since a_1 and a_2 are adjacent, $\deg_1(a_2) = 2$ and $\varphi(a_0a_1) = \varphi(a_1a_2) = 1$, we must have $\varphi(a_1b_1) = 1$ (see Figure 4(a)).

Now, if both the edges b_0b_1 and b_0b_2 are not of the same color, then $\deg_2(a_0) = 2 = \deg_2(b_0)$, a contradiction. Therefore, both the edges b_0b_1 and b_0b_2 are of the same color. Similarly, both the edges b_2b_0 and b_2b_1 are of the same color, for otherwise $\deg_1(a_2) = 2 = \deg_1(b_2)$. Hence, all the edges on the triangle $b_0b_1b_2$ must be of the same color, but this is not possible (see Figures 4(b) and (c)).

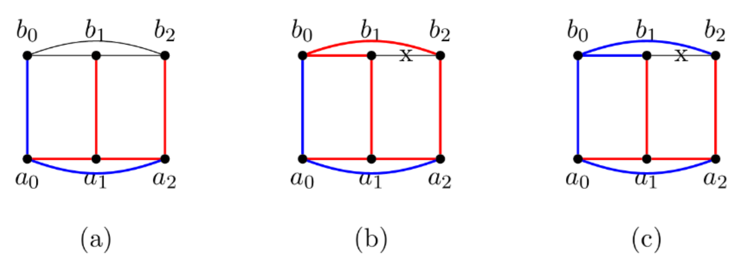


Figure 4. Attempted 2-edge coloring φ of $P(3,1)$

Lemma 2.10. *For the generalized Petersen graph* $P(5,1)$, $\chi'_{\text{irr}}(P(5,1)) = 3$.

Proof. There exists a 3-locally irregular edge coloring φ of $P(5,1)$ as follows. Let $\varphi(a_0a_1) = \varphi(a_0a_4) = \varphi(a_3a_4) = \varphi(b_2b_3) = \varphi(b_3b_4) = \varphi(a_3b_3) = \varphi(a_4b_4) = 1$, $\varphi(a_0b_0) = \varphi(b_0b_1) = \varphi(a_1b_1) = \varphi(b_0b_4) = 2$ and $\varphi(b_1b_2) = \varphi(a_2b_2) = \varphi(a_1a_2) = \varphi(a_2a_3) = 3$. Therefore, $\chi_{irr'}(P(5,1)) \le 3$.

To show that equality holds, we shall show that any edge coloring with two colors of $P(5,1)$ will induce a monochromatic subgraph having two adjacent vertices with the same degree. Suppose there exists an edge coloring φ of $P(5,1)$ with colors 1 and 2 which induces two locally irregular subgraphs.

If all edges of the outer rim (or the inner rim) of $P(5,1)$ are colored with the same color, then it is easy to see that all possible edge colorings of the 5 spokes will result having at least two adjacent vertices on the outer rim (or inner rim) with same color degree. So, we may assume that all the edges on the outer rim and inner rim are not of the same color.

Case 1. The outer rim has four edges of color 1 and one edge of color 2. Since the 5-cycle outer rim is vertex transitive, we may assume that the edges on the path $a_0a_1a_2a_3a_4$ are of color 1 and the edge a_0a_4 is of color 2. Note that both a_0b_0 and a_4b_4 cannot be of the same color, for otherwise, a_0 and a_4 are of the same color degree. Without loss of generality, we may assume that a_0b_0 is of color 2, and a_4b_4 is of color 1.

To avoid a_3 and a_4 having the same color, a_3b_3 must be of color 1. Next, a_2b_2 must be of color 2, for otherwise $\deg_1(a_2) = 3 = \deg_1(a_3)$, a contradiction. Similarly, to avoid a_1 and a_2 having same color degree, a_1b_1 must be of color 1. Figure 5(a) depicts such coloring.

If the edges b_0b_1 and b_0b_4 are of different color, then $\deg_2(b_0) = 2 = \deg_2(a_0)$, a contradiction. Similarly, the edges b_0b_4 and b_3b_4 cannot be of different color, for otherwise,

Therefore, all the edges on the path $b_1b_0b_4b_3$ are of the same color. Suppose the edges on the path $b_1b_0b_4b_3$ are of color 1. Note that $\deg_1(a_3) = 3$ and $\deg_1(b_0) = 2$. If the edge b_1b_2 is of color 2, then $\deg_1(b_0) = 2 = \deg_1(b_1)$, a contradiction. If the edge b_1b_2 is of color 1, then $\deg_1(b_1) = 3 = \deg_1(a_1)$, again a contradiction (see Figure 5(b)). So, we may assume that the edges on the path $b_1b_0b_4b_3$ are of color 2. Now, the edge b_2b_3 must be of color 1, for otherwise, $\deg_2(b_3) = 2 = \deg_2(b_4)$. Since the edge a_2b_2 is of color 2, it follows that the edge b_1b_2 must be of color 2. Then $\deg_2(b_1) = 2 = \deg_2(b_2)$, a contradiction (see Figure 5(c)).

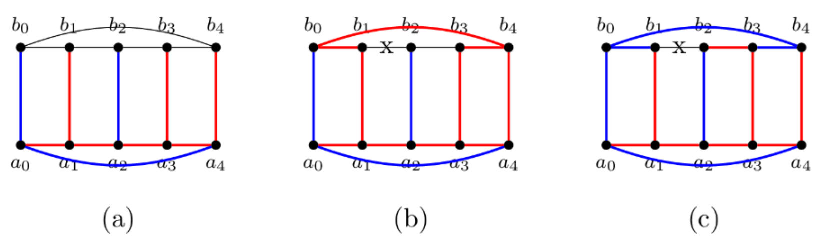


Figure 5. Possible coloring for Case 1 on $P(5,1)$.

The case the inner rim has four edges of color 2 and one edge of color 1 is similar. So, we may assume that the outer or inner rims do not have four edges of the same color and one edge of different color.

Case 2. The outer rim has three edges of color 1 and two edges of color 2 such that the edges of color 1 form a path of length 3. Without loss of generality, we may assume the edges a_0a_1, a_1a_2, and a_2a_3 are of color 1 and the edges a_0a_4 and a_3a_4 are of color 2. Note that the spokes a_1b_1 and a_2b_2 cannot be of the same color, for otherwise, a_1 and a_2 have the same color degree. By symmetry, we may assume that the edge a_1b_1 is of color 1 and a_2b_2 is of color 2. If the edge a_3b_3 is of color 1, then $\deg_1(a_3) = 2 = \deg_1(a_2)$, a contradiction. Thus, the edge a_3b_3 is of color 2. This implies that the edge a_4b_4 is of color 2, for otherwise, the vertices a_4 and a_3 are of the same color degree (see Figure 6(a)).

Now, if the edges b_3b_4 and b_3b_2 are of different color, then $\deg_2(a_3) = 2 = \deg_2(b_3)$, a contradiction. Therefore, the edges b_3b_4 and b_3b_2 are of the same color. Suppose the edges b_3b_4 and b_3b_2 are of color 2. This implies that the edges b_1b_2 and b_0b_4 are of color 1. Next, the edge b_0b_1 is of color 2, for otherwise $\deg_1(a_1) = 3 = \deg_1(b_1)$. Now, if the edge a_0b_0 is of color 1, then $\deg_1(a_0) = 2 = \deg_1(b_0)$, a contradiction. If the edge a_0b_0 is of color 2, then $\deg_2(a_0) = 2 = \deg_2(b_0)$, again a contradiction (see Figure 6(b)). The case both the edges b_3b_4 and b_3b_2 are of color 1 is depicted in Figure 6(c) and will result in a contradiction as well.

The case the inner rim has three edges of color 2 and two edges of color 1 such that the edges of color 2 form a path of length 3 is similar. So, we are left with the following case.

Case 3. The outer rim has three edges of color 1 and two edges of color 2 such that the edges with color 1 are two disjoint paths of length 2 and length 1 respectively. Without loss of generality, we may assume the edges a_0a_1, a_1a_2, and a_3a_4 are of color 1 and the edges a_0a_4 and a_2a_3 are of color 2.

Note that both the spokes a_4b_4 and a_3b_3 cannot be of the same color, for otherwise $\deg_1(a_3) = \deg_1(a_4)$. By symmetry, we may assume that the spokes a_3b_3 and a_4b_4 are of color 2 and 1, respectively. If the edges b_0b_4 and b_3b_4 are of different color, then $\deg_1(b_4) = 2 = \deg_1(a_4)$, a contradiction. So, the edges b_0b_4 and b_3b_4 are of the same color. Similarly, the edges b_3b_4 and b_3b_2 are of the same color.

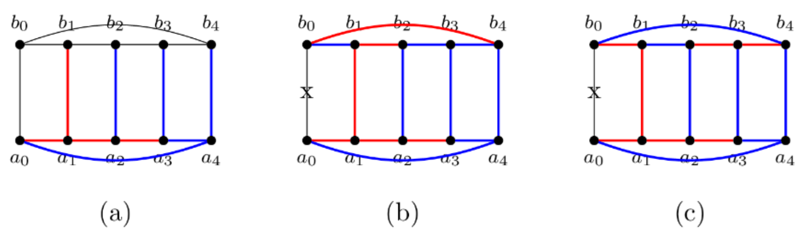


Figure 6. Possible coloring, Case 2 of $P(5,1)$.

Suppose all the edges on the path $b_0b_4b_3b_2$ are of color 1. If all the edges on the path $b_0b_1b_2$ are of color 2, then the inner rim has three edges of color 1 and two edges of color 2 such that the edges of color 1 form a path of length 3. We return to Case 2. If only one of the edges on the path $b_0b_1b_2$ is of color 2, then the inner rim has four edges of color 1 and an edge of color 2, in which we return to Case 1. If all the edges on the path $b_0b_1b_2$ are of color 1, then the inner rim is a 5-cycle with all edges of color 1, which is not possible as we have excluded this case from the very beginning. The case all the edges on the path $b_0b_4b_3b_2$ are of color 2 is similar.

The case the inner rim has three edges of color 2 and two edges of color 1 such that the edges with color 2 are two disjoint paths of length 2 and length 1 respectively is similar.

□

By Lemmas 2.8, 2.9 and 2.10, we have the following.

Theorem 2.11. *Suppose* $n \geq 3$. *For every* . $P(n,1)$,

$$\chi'_{\text{irr}}(P(n,1)) = \begin{cases} 3, & \text{if } n = 3,5; \\ 2, & \text{otherwise.} \end{cases}$$

P(n,k) for *k*=2

For the ease of explanation, we used a similar isomorphic graph of $P(n,2)$, say $H(n,2)$, as defined in [23], as in Figures 7 and 8.

For ease of illustration, in the rest of the section, we show the isomorphic graph $H(n,2)$ horizontally.

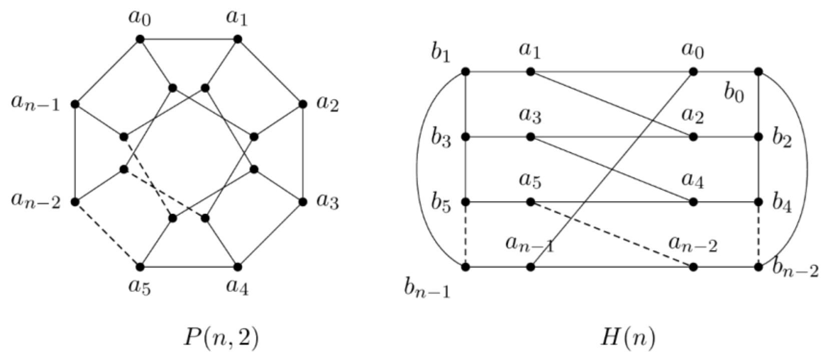


Figure 8. [23] *H(n)* is isomorphic to *P(n,2)* where n is even.

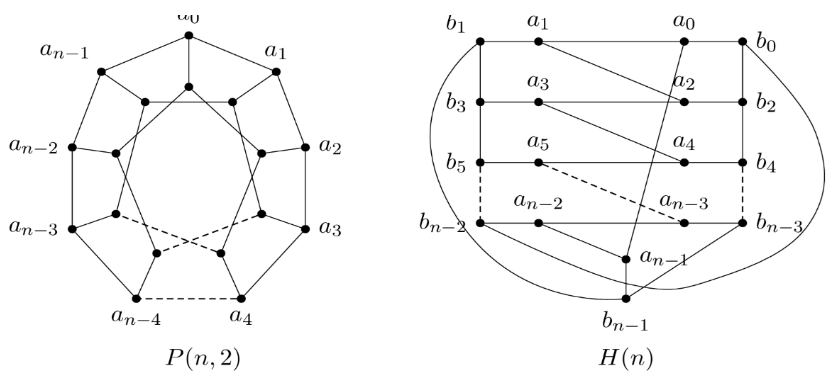


Figure 7. [23] *H(n)* is isomorphic to *P(n,2)* where n is odd.

Lemma 2.12. $\chi'_{\text{irr}}(P(n,2)) = 2$ *if* $n \geq 8$ *is even.*

Proof. It is sufficient to show a 2-liec φ of $P(n,2)$. Note that the operations on the subscripts are in modulo n.

Case 1. Suppose $n \equiv 0 \pmod 4$. Then $\gcd(n,k) = 2$ and $\dfrac{n}{\gcd(n,k)}$ is even. Therefore, it follows from Theorem 2.5 that $\chi'_{\text{irr}}(G) = 2$.

Case 2. Suppose $n \equiv 2 \pmod 4$. First, we assign color 1 to the following edges:

(i) the paths of length 3, $a_{i-2}a_{i-1}a_ia_{i+1}$, for . $i = 0,5$ ..

(ii) for $n \geq 10$, the paths of length 3, $a_{i-2}a_{i-1}a_ia_{i+1}$ for $i = 10,14,18,\ldots,n$.

(iii) the spokes a_ib_i and $a_{i+1}b_{i+1}$ for $i = 0,5$.

(iv) the edge $b_i b_{i+2}$ for $i = 0, 5$.
(v) the spokes $a_i b_i$ and $a_{i+1} b_{i+1}$ for $i = 10, 14, 18, \ldots, n$.

Then we assign color 2 to the rest of the edges. For $0 \le i \le 8$, it is easily observed from Figure 9 that adjacent vertices have different degrees.

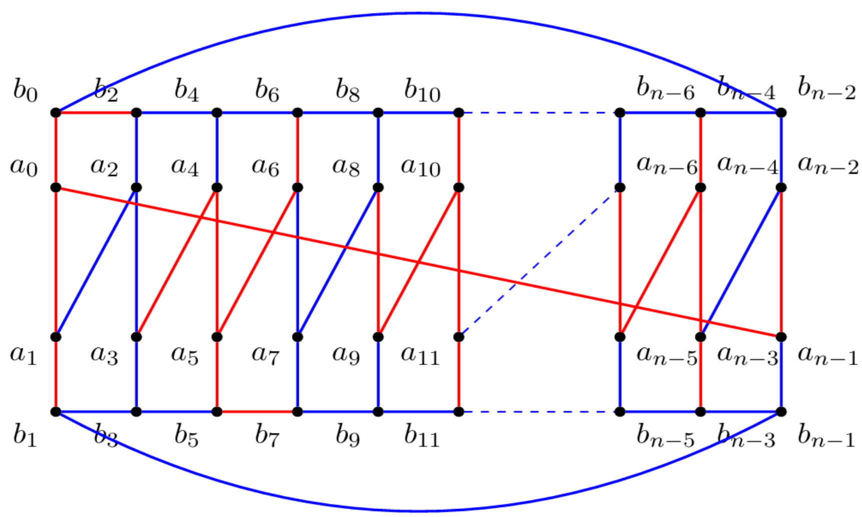
For $9 \le i \le n-1$, the above coloring will result in the following color degrees for vertices in the outer rim:

$$\deg_1(a_i) = \begin{cases} 3, & \text{if } i \equiv 2 \pmod 4 \\ 2, & \text{if } i \equiv 1,3 \pmod 4 \\ 1, & \text{if } i \equiv 0 \pmod 4 \end{cases} \quad \text{and, } \deg_2(a_i) = \begin{cases} 0, & \text{if } i \equiv 2 \pmod 4 \\ 1, & \text{if } i \equiv 1,3 \pmod 4 \\ 2, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

For $9 \le i \le n-1$, the above coloring will result in the following color degrees for the vertices in the inner rim:

$$\deg_1(b_i) = \begin{cases} 1, & \text{if } i \equiv 2,3 \pmod 4 \\ 0, & \text{if } i \equiv 0,1 \pmod 4 \end{cases} \quad \text{and, } \deg_2(b_i) = \begin{cases} 2, & \text{if } i \equiv 2,3 \pmod 4 \\ 3, & \text{if } i \equiv 0,1 \pmod 4 \end{cases}$$

Therefore, we can see that the resulting graph can be decomposed into two locally irregular graphs under φ.



□

Figure 9. The two edge coloring of $P(n,2)$, where $n \equiv 2 \pmod 4$

Lemma 2.13. *For* $P(6,2)$, $\chi'_{irr}(P(6,2)) = 3$.

Proof. Clearly, $\chi'_{irr}(P(6,2)) \ge 2$. It has been shown in [20] that $\chi'_{irr}(P(6,2)) \le 3$. Here, we shall show that any possible edge coloring of $P(6,2)$ with two colors, say color 1 and 2, cannot induce two locally irregular subgraphs. This implies $\chi'_{irr}(P(6,2)) = 3$.

Note that all the edges on the triangle $b_1 b_3 b_5 b_1$ cannot be of the same color. In fact, if they are of the same color, then any possible coloring of the spokes $a_1 b_1$, $a_3 b_3$ and $a_5 b_5$ will result to two of the vertices in the triangle $b_1 b_3 b_5 b_1$ having the same color degree. The same applies to the triangle $b_0 b_2 b_4 b_0$, i.e., all the edges cannot be of the same color. Since each of the vertices on the triangle $b_1 b_3 b_5 b_1$ is vertex transitive, we may assume that the edges $b_1 b_3$ and $b_3 b_5$ are of color 1 and the edge $b_1 b_5$ is of color 2. Now, the both the edges $a_1 b_1$ and $a_5 b_5$ cannot be of the same color, for otherwise,

$\deg_2(b_1) = \deg_2(b_5)$. By symmetry, we may assume that a_1b_1 is of color 2 and a_5b_5 is of color 1. This implies that a_3b_3 is of color 1, for otherwise, $\deg_1(b_3) = 2 = \deg_1(b_5)$.

We denote such coloring, as depicted in Figure 10 as the base case B of a 2-liec of $P(6,2)$.

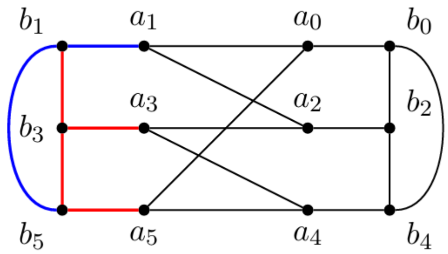


Figure 10. The base case B of a 2-liec of P(6,2)

Now, from B, if the edges a_0a_1 and a_1a_2 are of different color, then $\deg_2(b_1) = 2 = \deg_2(a_1)$. Thus, both the edges a_0a_1 and a_1a_2 must be of the same color. Suppose the edges a_0a_1 and a_1a_2 are of color 1 (see Figure 11(a)). If the edges a_0b_0 and a_0a_5 are of different color, then $\deg_1(a_1) = 2 = \deg_1(a_0)$. So, the edges a_0b_0 and a_0a_5 must be of the same color. Similarly, the edges a_5a_0 and a_5a_4 must be of the same color. If the edges a_0b_0, a_5a_0 and a_5a_4 are of color 1, then $\deg_1(a_0) = 3 = \deg_1(a_5)$. If the edges a_0b_0, a_5a_0 and a_5a_4 are of color 2, then $\deg_2(a_0) = 2 = \deg_2(a_5)$. Hence, we may assume that the edges a_0a_1 and a_1a_2 are of color 2 (see Figure 11(b)). As before, the edges a_5a_0 and a_5a_4 are of the same color. If the edges a_5a_0 and a_5a_4 are color 2, then $\deg_2(a_0) = 3 = \deg_2(a_1)$ provided that the edge a_0b_0 is of color 2, whereas $\deg_2(a_0) = 2 = \deg_2(a_5)$ provided that the edge a_0b_0 is of color 1. So, we may assume that the edges a_5a_0 and a_5a_4 are color 1. Now, both the edges a_3a_2 and a_3a_4 cannot be of color 1, for otherwise, $\deg_1(a_3) = 3 = \deg_1(b_3)$. If both the edges a_3a_2 and a_3a_4 is of color 2, then either $\deg_2(a_3) = 2 = \deg_2(a_2)$ or $\deg_2(a_1) = 3 = \deg_2(a_2)$. If the edge a_3a_2 is of color 2 and the edge a_3a_4 is of color 1, then $\deg_1(a_3) = 2 = \deg_1(a_4)$ or $\deg_1(a_5) = 3 = \deg_1(a_4)$. So, we are left with the case where the edge a_3a_2 is of color 1 and the edge a_3a_4 is of color 2. This means the edges a_4b_4 and a_2b_2 are of color 2. Now, the edges b_2b_0 and b_2b_4 must be of the same color, for otherwise, $\deg_2(a_2) = 2 = \deg_2(b_2)$. Similarly, the edges b_4b_2 and b_4b_0 must be of the same color. This implies that all the edges on the triangle $b_0b_2b_4b_0$ are of the same color, contradicting our early assumption.

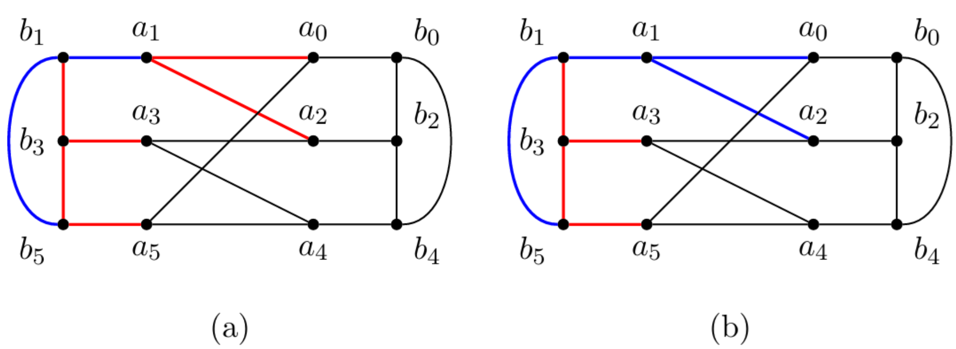


Figure 11. The edges a_0a_1 and a_1a_2 are of the same color.

□

Lemma 2.14. *Suppose* $n \geq 5$. *Then* $\chi'_{irr}(P(n,2)) = 2$ *if* $n \equiv 1(\text{mod } 4)$.

Proof. It is sufficient to show a 2-liec φ of $P(n,2)$. First, we assign color 1 to the following edges:

(i) the paths of length 3, $a_{i-2}a_{i-1}a_ia_{i+1}$ for $i = 0$.
(ii) the spokes a_ib_i and $a_{i+1}b_{i+1}$ for $i = 0$.
(iii) the edge b_ib_{i+2} for $i = 0$.
(iv) the paths of length 3, $a_{i-2}a_{i-1}a_ia_{i+1}$ for $i = 5, 9, 13, \ldots, n$.
(v) the spokes a_ib_i and $a_{i+1}b_{i+1}$ for $i = 5, 9, 13, \ldots, n$.

Then we assign color 2 to the rest of the edges. For $0 \leq i \leq 3$, it is easily observed from Figure 12 that adjacent vertices have distinct degrees.

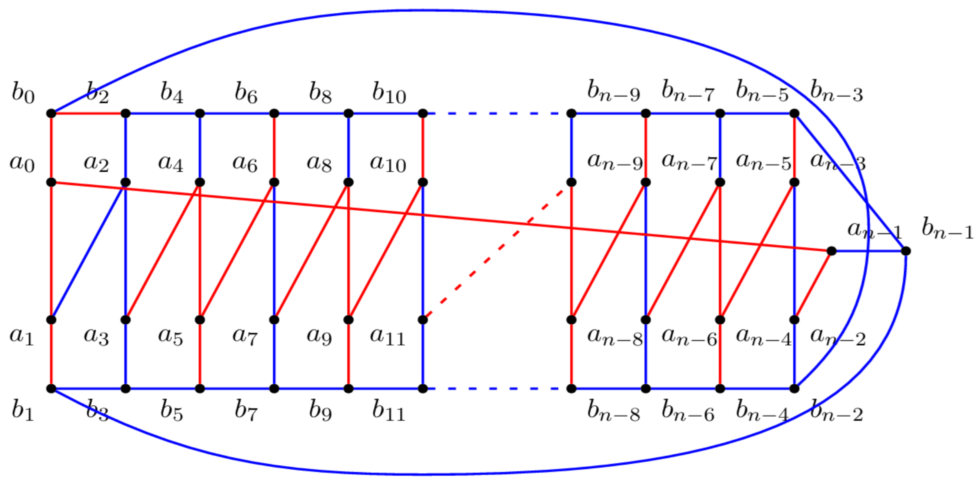


Figure 12. The 2-liec of $P(n,2)$ where $n \equiv 1(\text{mod } 4)$

For $4 \leq i \leq n-1$, the above coloring will result in the following color 1 degrees for the outer rim:

$$\deg_1(a_i) = \begin{cases} 3, & \text{if } i \equiv 1(\text{mod } 4) \\ 2, & \text{if } i \equiv 0,2(\text{mod } 4) \\ 1, & \text{if } i \equiv 3(\text{mod } 4) \end{cases} \text{ and, } \deg_2(a_i) = \begin{cases} 0, & \text{if } i \equiv 1(\text{mod } 4) \\ 1, & \text{if } i \equiv 0,2(\text{mod } 4) \\ 2, & \text{if } i \equiv 3(\text{mod } 4) \end{cases}$$

For $4 \leq i \leq n-1$, the above coloring will result in the following color degree for vertices in the inner rim:

$$\deg_1(b_i) = \begin{cases} 1, & \text{if } i \equiv 1,2(\text{mod } 4) \\ 0, & \text{if } i \equiv 0,3(\text{mod } 4) \end{cases} \text{ and, } \deg_2(b_i) = \begin{cases} 2, & \text{if } i \equiv 1,2(\text{mod } 4) \\ 3, & \text{if } i \equiv 0,3(\text{mod } 4) \end{cases}$$

□

It has been shown that in [20] that for $n = 7, 11$, $\chi'_{irr}(P(n,2)) = 3$. So, we are left with the case $n \equiv 3(\text{mod } 4)$ and $n \neq 7, 11$.

Lemma 2.15. *Suppose* $n \equiv 3(\text{mod } 4)$ *and* $n \neq 7, 11$. *Then* $\chi'_{irr}(P(n,2)) = 2$.

Proof. It is sufficient to show a 2-liec φ of $P(n,2)$. First, we assign color 1 to the following edges:

(i) the paths of length 3, $a_{i-2}a_{i-1}a_ia_{i+1}$ for $i = 0, 5, 10$.
(ii) the spokes . a_ib_i . and $a_{i+1}b_{i+1}$ for $i = 0, 5, 10$.
(iii) the edge b_ib_{i+2} for $i = 0, 5, 10$.
(iv) the paths of length 3, $a_{i-2}a_{i-1}a_ia_{i+1}$ for $i = 15, 19, 23, \ldots, n$.
(v) the spokes a_ib_i and $a_{i+1}b_{i+1}$ for $i = 15, 19, 23, \ldots, n$.

Then we assign color 2 to the rest of the edges. For $0 \le i \le 13$, it is easily observed from Figure 13 that adjacent vertices have distinct degrees.

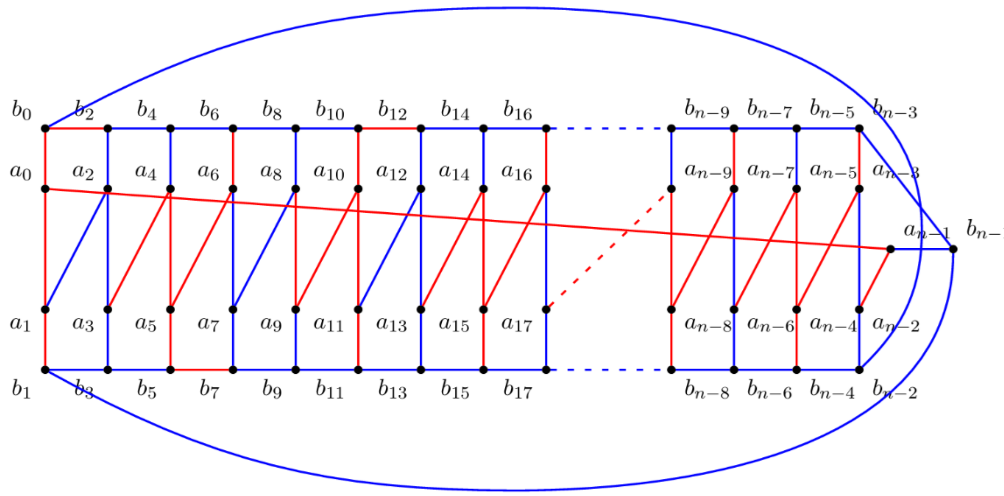


Figure 13. The 2-liec of $P(n,2)$ where $n \equiv 3 \pmod 4$

For $14 \le i \le n-1$, the above coloring will result in the following color degrees for the vertices in the outer rim:

$$\deg_1(a_i) = \begin{cases} 3, & \text{if } i \equiv 3 \pmod 4 \\ 2, & \text{if } i \equiv 0,2 \pmod 4 \\ 1, & \text{if } i \equiv 1 \pmod 4 \end{cases} \quad \text{and, } \deg_2(a_i) = \begin{cases} 0, & \text{if } i \equiv 3 \pmod 4 \\ 1, & \text{if } i \equiv 0,2 \pmod 4 \\ 2, & \text{if } i \equiv 1 \pmod 4 \end{cases}$$

For $14 \le i \le n-1$, the above coloring will result in the following color degrees for the vertices in the inner rim:

$$\deg_1(b_i) = \begin{cases} 1, & \text{if } i \equiv 0,3 \pmod 4 \\ 0, & \text{if } i \equiv 1,2 \pmod 4 \end{cases} \quad \text{and, } \deg_2(b_i) = \begin{cases} 2, & \text{if } i \equiv 0,3 \pmod 4 \\ 3, & \text{if } i \equiv 1,2 \pmod 4 \end{cases}$$

□

Since B.Lužar et. al. [20] have shown that $\chi'_{\text{irr}}(P(7,2) = \chi'_{\text{irr}}(P(11,2) = 3$, the following theorem follows from Lemmas 2.12, 2.13, 2.14, 2.15.

Theorem 2.16. Suppose $n \ge 5$. Then

$$\chi'_{\text{irr}}(P(n,2)) = \begin{cases} 3, & \text{if } n = 6,7,11; \\ 2, & \text{otherwise.} \end{cases}$$

Concluding Remarks

In this paper, we determine the locally irregular chromatic index $\chi'_{\text{irr}}(G)$ for G is the complete multipartite graph, see Theorem 2.4. Besides that, we showed the exact value of the $\chi'_{\text{irr}}(P(n,k))$ for $k = 1,2$ and proposed a 2-liec or 3-liec for these graphs. We also show that $\chi'_{\text{irr}}(P(n,k)) = 2$ when $gcd(k,n) = 1$ when n is even and when both k,n are odd, see Theorems 2.5, 2.7 and Corollary 2.6. We believe that Conjecture 1.5 is true is general.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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