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RESEARCH ARTICLE

# Stabilizability and Solvability of Fuzzy Reaction-Diffusion Equation using Modified Backstepping Control Method for Matrix Differential Equation

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**Abstract** In this article, an important type of fuzzy parabolic differential equations will be discussed, which is the one-dimensional fuzzy reaction-diffusion equation with fuzzy boundary conditions. This equation is one of the most widespread chemical fuzzy reaction-diffusion equations, as well as, studying the possibility of controlling and reducing the chemical pollution occurred in the chemical reactions. In order to reduce chemical contamination in the reaction medium, we observed that investigating this equation's stability is essential. In order to achieve stability, the fuzzy backstepping approach is proposed, which transforms the unstable system into a stable system after controlling the boundary conditions. Therefore, two different cases of Hukuhara derivatives must be considered, which are important in the study of fuzzy differential equations. Two cases are considered depending on the comparison between the lower and upper variable solution time derivative. Also, the proposed backstepping approach is applied based on the interval analysis of α-level sets. For this purpose, and in order to avoid the difficulty of separating the upper and lower solutions, the resulting non-fuzzy or crisp differential equations are converted into matrix differential equations, and then Consequently, we are able to remove the residual terms that are responsible for the instability of the open-loop. Moreover, this backstepping transformation is continuously invertible. Thus, the inverse transformation is used to obtain stabilizing state feedback for the original partial differential equation.

**Keywords:** Fuzzy differential equation, Hukuhara derivative, fuzzy backstepping method, matrix differential equation, fuzzy reaction-diffusion equation.

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## Introduction

In 1965, Zadeh and Chang introduced the idea of fuzzy numbers (FN) and investigated the derivative of fuzzy functions as well as arithmetic operations on these numbers [1-2]. In the decade that followed the fuzzy sets' debut, Zadeh carried on developing the theory's central ideas, including the research of fuzzy relations that meet the requirements of an equivalence relation. In 1975, Mamdani and Assilian created the first fuzzy logic controller [3].

The environment of fuzzy logic on differential equations have been developed after the description by Hukuhara in 1983, who proposed a theorem that guarantees the uniqueness of solutions to fuzzy initial-value problems [4]. Fuzzy ordinary differential equations (ODEs), which are then extended to fuzzy integer partial differential equations (PDEs), have increased manifold. It was applied in several fields including modeling robotics, quantum optics, engineering, medicine, gravity, artificial intelligence, and intelligence tests [5-7].

Lyapunov's Ph.D. dissertation served as the foundation for the current theory of stability in differential equations, "The General Problem of the Stability of Motion", in 1892. From that, the controllable systems appeared as an important when studying system's stability [8]. A technique for analyzing stability based on energy-like functions-now known as Lyapunov functions was presented by Lyapunov. George David Birkhoff created the stability hypothesis around the beginning of the 20th century [9]. The scientists Andronov and Pontryagin studied the branching theory and structural stability of dynamical systems in 1930 in Chapter 57 [10]. Control theory was developed on Lyapunov's second technique and stability ideas in 1950, especially for the creation of stable feedback systems [11]. Since 1970, nonlinear systems and numerical stability have been investigated. Advances in computer techniques have made it possible to use numerical analysis to explore stability in complex and nonlinear systems [12-14].

Researchers have also been interested in studying the stability of the fuzzy parabolic equations, because of its importance in various scientific, engineering and life fields, as it studies uncertainty in diffusion equations [15-17].

The control of systems described by fuzzy PDEs is an important subject that has attracted many researchers in fuzzy control engineering, as it helps describe various applications phenomena. An important control method that was proposed as a design for boundary control of fuzzy PDEs is the fuzzy backstepping approach [18-20].

The objective of fuzzy backstepping method is to Create one-to-one correspondence between the original solution fuzzy PDE and the solution of the stable target fuzzy PDE through certain transformation. This approach is not achieved only by applying the fuzzy backstepping transformation, since a residual term remains at the boundary for domain for definition for fuzz PDE. Hence, the boundary input is used to eliminate such residual term [21,22].

The main objective of this paper is to investigate the controllability of the one-dimensional fuzzy reaction-diffusion equation (RDE) with fuzzy boundary conditions, which models the most common chemical fuzzy RDEs, as well as the possibility of controlling and reducing chemical pollution during chemical reactions. To decrease chemical contamination in the reaction media, it has been determined that the stability of this equation must be investigated. To attain stability, the fuzzy backstepping strategy is given, which converts an unstable system into a stable one by adjusting the boundary conditions. The suggested backstepping technique uses interval analysis for $\alpha$-level sets. To accomplish this, and to avoid the difficulty of separating the upper and lower solutions, the resulting non-fuzzy or crisp differential equations are converted into matrix differential equations, which allow us to eliminate the residual terms that cause the open loop instability. Furthermore, this transformation (backstepping) is continuously invertible, therefore the inverse transformation yields the stabilizing state feedback for the original PDE.

# Preliminaries

In this section, some basic concepts which are necessary for this work are presented as preliminaries for completeness purpose.

**Definition 1, [23,24].** Triangular FN is a fuzzy set with the following function of membership:

$$\mu_{\tilde{A}}\,(x, a, m, s)= \begin{cases} m\left(1-\frac{|x-a|}{s}\right), a-s \le x \le a+s \\ 0, \text{otherwise} \end{cases} \tag{1}$$

where $a, m, s \in \mathbb{R}$ with $s \neq 0$ and $0 < m \le 1$.

**Definition 2, [25,26].** A triangular FN $\widetilde{M} = (a, m, b)$, where $a, m, b \in \mathbb{R}$, may be uniquely represented in terms of it's $\alpha$-level sets, as in the following closed interval of the real line:

$$M(\alpha) = [a-(a-m)\alpha, b-(b-m)\alpha], \alpha \in [0,1] \tag{2}$$

where $m$ the mean value of the FN $\widetilde{M}$, $a$ and $b$ are respectively the left and right spreads of the FN $\widetilde{M}$. Also, the triangular fuzzy number $\widetilde{M} = (a, m, b)$ may be written as an interval $\alpha$-level set as $M(\alpha) = [\underline{M}(\alpha), \overline{M}(\alpha)]$, where $\underline{M}(\alpha)$ and $\overline{M}(\alpha)$ refers respectively to the greatest and least lower bounds of $M(\alpha)$, and satisfies the following conditions:

1. $\underline{M}(\alpha)$ is a bounded left continuous monotonic increasing function, $\forall\ \alpha \in [0,1]$.
2. $\overline{M}(\alpha)$ is a bounded left continuous monotonic decreasing function, $\forall\ \alpha \in [0,1]$.
3. $\underline{M}(\alpha) \le \overline{M}(\alpha), 0 \le \alpha \le 1$.
4. $\underline{M}(\alpha) = \overline{M}(\alpha) = m$, if $\alpha = 1$, where $m$ is called the mean value of the FN $\widetilde{M}$.

**Remark 1, [27].** For two arbitrary FNs $\widetilde{M} = \left[\underline{M}(\alpha), \overline{M}(\alpha)\right]$ and $\widetilde{H} = \left[\underline{H}(\alpha), \overline{H}(\alpha)\right]$, the following algebraic operations may be fulfilled:

1. If k is any real number, then:
$$k\widetilde{W} = \begin{cases} \left[k\underline{M}(\alpha), k\overline{M}(\alpha)\right], if\ k \geq 0 \\ \left[k\overline{M}(\alpha), k\underline{M}(\alpha)\right], if\ k < 0 \end{cases}$$
2. $\widetilde{M} + \widetilde{H} = [\underline{M}(\alpha) + \underline{H}(\alpha), \overline{M}(\alpha) + \overline{H}(\alpha)]$.
3. $\widetilde{M} - \widetilde{H} = [\underline{M}(\alpha) - \overline{H}(\alpha), \overline{M}(\alpha) - \underline{H}(\alpha)]$.
4. $\widetilde{M}\widetilde{H} = $ [min s($\alpha$), max s($\alpha$)], where
$$s(\alpha) = \{\underline{M}(\alpha)\underline{H}(\alpha), \underline{M}(\alpha)\overline{H}(\alpha), \overline{M}(\alpha)\underline{H}(\alpha), \overline{M}(\alpha)\overline{H}(\alpha)\}$$

**Definition 3, [28,29].** Let $\tilde{A}, \tilde{B} \in \mathbb{R}_f$ and if there exist $\tilde{R} \in \mathbb{R}_f$, such that $\widetilde{\mathcal{B}} = \tilde{A} + \tilde{R}$, then $\tilde{R}$ is called the Hukuhara-difference of $\widetilde{\mathcal{B}}$ and $\tilde{A}$, which is denoted by $\widetilde{\mathcal{B}} \ominus \widetilde{M}$.

Hukuhara introduced the Hukuhara derivative in 1976, which became a foundation for studying fuzzy differential equations, potentially extending the nonfuzzy or crisp derivative, as it is presented in the next definition:

**Definition 4, [30].** The generalized derivative of a fuzzy-real valued function $\tilde{f}: (a, b) \longrightarrow \mathbb{R}_f$ at a point $x_0 \in \mathbb{R}$ is given by:

$$\tilde{f}'_{gH}(x_0) = \lim_{\mathcal{X} \to 0} \frac{\tilde{f}(x_0 + \hbar) \ominus_{gH} \tilde{f}(x_0)}{\hbar}$$

If $\tilde{f}'_{gH}(x_0) \in \mathbb{R}_f$, then we say that $\tilde{f}$ is generalized Hukuhara differentiable (*gH* -differentiable for short) at $x_0$, which is also written using interval $\alpha$-levels as:

$$\tilde{f}'_{gH}(x_0; \alpha) = \left[\underline{f}'_{gH}(x_0; \alpha), \overline{f}'_{gH}(x_0; \alpha)\right]$$

Furthermore, depending and related to property 1 of Remark 1, $\tilde{f}$ is said to be (i)-*gH*-differentiable at $x_0$ if:

$$\tilde{f}'_{i,gH}(x_0; \alpha) = \left[\underline{f}'(x_0; \alpha), \overline{f}'(x_0; \alpha)\right],\ 0 \leq \alpha \leq 1$$

and that $\tilde{f}$ is (ii)-*gH*-differentiable at $x_0$ if:

$$\tilde{f}'_{ii.gH}(x_0; \alpha) = \left[\overline{f}'(x_0; \alpha), \underline{f}'(x_0; \alpha)\right],\ 0 \leq \alpha \leq 1.$$

The fuzzy functions using Hukuhara differentiability are generalized for partial derivatives, as in crisp calculus as follows:

**Definition 5, [27,31].** Let $(x_0, y_0) \in D \subset \mathbb{R}^n$, then the first generalized Hukuhara partial derivative (abbreviated for simplicity as *p-gH*-derivative) of a fuzzy-valued function $\tilde{f}(x, y): D \longrightarrow \mathbb{R}_f$ at $(x_0, y_0)$ with respect to the variable $\mathcal{X}$ is denoted by $\partial_{x_{gH}}\tilde{f}(x_0, y_0)$ and is given by:

$$\partial_{x_{gH}}\tilde{f}(x_0, y_0) = \lim_{\hbar \to 0} \frac{\tilde{f}(x_0 + \hbar, y_0) \ominus_{gH} \tilde{f}(x_0, y_0)}{\hbar}.$$

while the partial derivative with respect to $y$ at $(x_0, y_0)$ is denoted by $\partial_{y_{gH}}\tilde{f}(x_0, y_0)$, and

$$\partial_{y_{gH}}\tilde{f}(x_0, y_0) = \lim_{k \to 0} \frac{\tilde{f}(x_0, y_0 + k) \ominus_{gH} \tilde{f}(x_0, y_0)}{k}$$

provided that $\partial_{x_{gH}}\tilde{f}(x_0, y)$ and $\partial_{y_{gH}}\tilde{f}(x_0, y_0)$ belongs to $\mathbb{R}_f$.

Similarly, as in Definition 4, $\tilde{f}$ is said to be (i)-p-*gH*-differentiable at $(x_0, y_0)$ if:

$$\tilde{f}'_{ip,gH}(x_0, y_0; \alpha) = \left[\underline{f}'(x_0, y_0; \alpha), \overline{f}'(x_0, y_0; \alpha)\right],\ 0 \leq \alpha \leq 1$$

and that $f$ is (ii)-p-*gH*-differentiable at $(x_0, y_0)$ if:

$$\tilde{f}'_{iip.gH}(x_0, y_0; \alpha) = \left[\overline{f}'(x_0, y_0; \alpha), \underline{f}'(x_0, y_0; \alpha)\right],\ 0 \leq \alpha \leq 1.$$

Also, the closed interval $\alpha$-level sets of the *p-gH*-derivatives are defined as in ordinary derivatives.

**Definition 6, [32].** Suppose that $\tilde{f}: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is the fuzzy-valued function of two variables are rule that assigns to each order pair of real numbers, $(x,t)$ in a set $D,$ a unique FN denoted by $\tilde{f}(x, t)$. The set $D$ is the domain of $\tilde{f}(x, t)$ and $f\ (x, t; \alpha) = \left[\underline{f}(x, t; \alpha), \overline{f}\ (x, t; \alpha)\right]$, $\forall\ \alpha \in [0,1]$. If there exist partial crisp derivatives of $\underline{f}\,(x;\alpha)$ and $\overline{f}\,(x;\alpha)$ with respect to $x \in \mathrm{D}$ and the interval $\left[\frac{\partial \underline{f}\,(x;\alpha)}{\partial x}, \frac{\partial \overline{f}\,(x;\alpha)}{\partial x}\right]$, for $(x,t) \in D$, $\alpha \in [0,1]$ defines the $\alpha$-level set of a FN, then the differentiation of $\tilde{f}(x,\mathrm{t})$ can be written using $\alpha$-level sets as:

$$\partial \tilde{f}\,(\mathcal{X},t;\alpha) = \left[\frac{\partial \underline{f}\,(x,t;\alpha)}{\partial x}, \frac{\partial \overline{f}\,(x,t;\alpha)}{\partial x}\right].$$

## Fuzzy Reaction-Diffusion Equation

Fuzzy RDE is often utilized in various fields of engineering and another science, such as fuzzy space experiments, robots, and nuclear reactors, and in various fields where uncertainty exists [17,33].

The fuzzy parabolic diffusion equation consider in this work is given by the following [10]:

$$\tilde{u}_t(x,t) = \gamma \tilde{u}_{xx}(x,t) + \tilde{\lambda}\tilde{u}(x,t) \quad (3)$$
$$\tilde{u}(0,t) = \tilde{0}, \quad (4)$$
$$\tilde{u}(1,t) = \tilde{U}(t) \quad (5)$$
$$\tilde{u}(x,0) = \tilde{g}(x) \quad (6)$$

where $\tilde{u}_t(x,t)$ and $\tilde{u}_{xx}(x,t)$ are defined using fuzzy partial derivatives in the Hukuhara differentiability, $\tilde{U}(t)$ is a control input, and the $\gamma$ is a constant greater than zero, $\tilde{\lambda}$ is a fuzzy triangular number and $\tilde{g}(x)$ is fuzzy function.

## Analysis of the Fuzzy Reaction-Diffusion Equation

Our focus in this section is on analysing fuzzy RDE with p-$gH$-differentiability using $\alpha$-level sets, which will be carried based on assuming [27,32], $u(x,t;\alpha) = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$, then substituting in the governing PDE given by Eq. (3), we get:

$$\left.\begin{aligned} \min\left\{\gamma \frac{\partial^2}{\partial x^2}\underline{u}(x,t;\alpha) + \underline{\lambda}\,\underline{u}(x,t;\alpha), \gamma \frac{\partial^2}{\partial x^2}\overline{u}(x,t;\alpha) + \overline{\lambda}\overline{u}(x,t;\alpha)\right\} = \frac{\partial}{\partial t}\underline{u}(x,t;\alpha) \\ \max\left\{\gamma \frac{\partial^2}{\partial x^2}\underline{u}(x,t;\alpha) + \underline{\lambda}\,\underline{u}(x,t;\alpha), \gamma \frac{\partial^2}{\partial x^2}\overline{u}(x,t;\alpha) + \overline{\lambda}\overline{u}(x,t;\alpha)\right\} = \frac{\partial}{\partial t}\overline{u}(x,t;\alpha) \end{aligned}\right\} \quad (7)$$

**Case (i):** If $\underline{u}_t(x,t;\alpha) \le \overline{u}_t(x,t;\alpha)$, then the possible PDEs resulted from system (7) are:

$$\underline{u}_t(x,t;\alpha) = \gamma \underline{u}_{xx}(x,t;\alpha) + \underline{\lambda u}(x,t;\alpha) \quad (8)$$
$$\overline{u}_t(x,t;\alpha) = \gamma \overline{u}_{xx}(x,t;\alpha) + \overline{\lambda}\overline{u}(x,t;\alpha) \quad (9)$$

It is clear that Eqs. (8) and (9) are unstable when compared with the crisp RDE, with initial and boundary conditions for lower and upper cases, respectively [34]:

$$\underline{u}(0,t;\alpha) = \underline{0} \quad (10)$$
$$\underline{u}(1,t;\alpha) = \underline{U}(t;\alpha) \quad (11)$$
$$\underline{u}(x,0;\alpha) = \underline{g}(x;\alpha)$$

and

$$\overline{u}(0,t;\alpha) = \overline{0} \quad (12)$$
$$\overline{u}(1,t;\alpha) = \overline{U}(t;\alpha) \quad (13)$$
$$\overline{u}(x,0) = \overline{g}(x;\alpha)$$

**Case (ii):** If $\underline{u}_t(x,t;\alpha) > \overline{u}_t(x,t;\alpha)$, then the possible PDEs resulted from system (7) are:

$$\left.\begin{aligned} \underline{u}_t(x,t;\alpha) = \gamma \overline{u}_{xx} + \overline{\lambda}\overline{u}(x,t;\alpha) \\ \overline{u}_t(x,t;\alpha) = \gamma \underline{u}_{xx} + \underline{\lambda u}(x,t;\alpha) \end{aligned}\right\} \quad (14)$$

with initial and boundary conditions given also as in case (i) as fuzzy numbers, which are parameterized using interval $\alpha$-level sets.

## Fuzzy Backstepping Control Method for Reaction-Diffusion Equation

In this section, we shall study the stabilizability and solvability of the unstable fuzzy RDE by using fuzzy backstepping approach [35].

It is remarkable that the open-loop system (3-5) will be unstable because the appearance of the source of instability term $\tilde{\lambda}\tilde{u}$.

Using the principal of coordinate transformation of the backstepping method defined using fuzzy Volterra integral equation (fuzzy VIE) of the form [34,36]:

$$\widetilde{W}(x,t) = \tilde{u}(x,t) - \int_0^x k(x,\xi)\tilde{u}(\xi,t)d\xi \quad (15)$$

where $k(x,\xi)$ is a gain kernel function, in addition to feedback control function defined as:

$$\tilde{u}(1,t) = \int_0^1 k(1,\xi)\tilde{u}(\xi,t)d\xi \quad (16)$$

The main purpose of the transformation is to establish a one-to-one correspondence between Eqs. (3)-(5), which has (unstable) solution and the stable target system solution [37]. This target system has the form:

$$\widetilde{W}_t(x,t) = \gamma\widetilde{W}_{xx}(x,t) \quad (17)$$

$$\widetilde{W}(0,t) = 0 \quad (18)$$

$$\widetilde{W}(1,t) = 0 \quad (19)$$

Now, analyzing Eqs. (15)-(19) using the pgh-derivational with the α-level set concepts approach, as follows:

Starting the analyses of stabilizing the fuzzy target system given by Eqs. (17)-(19) via applying the fuzzy backstepping transformation method based on interval $\alpha$-level sets and theory of interval fuzzy numbers. Therefore, we must consider the following two cases [27,32]:

**Case(i):** If $\underline{u}_t(x,t;\alpha) \leq \overline{u}_t(x,t;\alpha)$ then:

$$\underline{W}_t(x,t;\alpha) = \gamma\underline{W}_{xx}(x,t;\alpha) \quad (20)$$

$$\overline{W}_t(x,t;\alpha) = \gamma\overline{W}_{xx}(x,t;\alpha) \quad (21)$$

With boundary conditions

$$\underline{W}(0,t;\alpha) = 0 \quad (22)$$

$$\underline{W}(1,t;\alpha) = 0 \quad (23)$$

and

$$\overline{W}(0,t;\alpha) = 0 \quad (24)$$

$$\overline{W}(1,t;\alpha) = 0 \quad (25)$$

with lower and upper backstepping transformations to be defined as:

$$\underline{W}(x,t;\alpha) = \underline{u}(x,t;\alpha) - \int_0^x k(x,\xi)\overline{u}(\xi,t;\alpha)d\xi \quad (26)$$

$$\overline{W}(x,t;\alpha) = \overline{u}(x,t;\alpha) - \int_0^x k(x,\xi)\,\underline{u}(\xi,t;\alpha)d\xi \quad (27)$$

and lower and upper feedback control functions given by:

$$\underline{u}(1,t;\alpha) = \int_0^1 k(1,\xi)\overline{u}(\xi,t;\alpha)d\xi \quad (28)$$

$$\overline{u}(1,t;\alpha) = \int_0^1 k(1,\xi)\,\underline{u}(\xi,t;\alpha)d\xi \quad (29)$$

where it is notable that there is some difficulty when separating the crisp lower and upper transformations given by Eqs. (26) and (27), therefore rewriting the system given by Eqs. (8)-(28) in matrix differential equations see [38], by defining:

$\mathcal{V}(x,t;\alpha) = \begin{bmatrix}\underline{u}\\ \overline{u}\end{bmatrix}, \mathcal{V}_t(x,t;\alpha) = \begin{bmatrix}\underline{u}_t\\ \overline{u}_t\end{bmatrix}, \mathcal{V}_{xx}(x,t;\alpha) = \begin{bmatrix}\underline{u}_{xx}\\ \overline{u}_{xx}\end{bmatrix}$ and $I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

The above assumptions may be considered as the main modification of this work, since we will concern with lower and upper solutions together based on the theory of matrix differential equations.

Hence, the linear control system will be obtained which is defined in matrix form by the following matrix differential equation:

$$\mathcal{V}_t(x,t;\alpha) = \gamma I\mathcal{V}_{xx}(x,t;\alpha) + \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\mathcal{V}(x,t;\alpha) \quad (30)$$

with initial and boundary conditions:

$$\mathcal{V}(0,t;\alpha) = \begin{bmatrix}\underline{u}(0,t;\alpha)\\ \overline{u}(0,t;\alpha)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \quad (31)$$

$$\mathcal{V}(1,t;\alpha) = \begin{bmatrix}\underline{u}(1,t;\alpha)\\ \overline{u}(1,t;\alpha)\end{bmatrix}$$

Now, since $\begin{bmatrix}\underline{u}(1,t;\alpha)\\ \overline{u}(1,t;\alpha)\end{bmatrix} = \begin{bmatrix}\underline{U}(t;\alpha)\\ \overline{U}(t;\alpha)\end{bmatrix}$, then:

$$\mathcal{V}(1,t;\alpha) = U(t;\alpha)$$

$$U(t;\alpha) = \begin{bmatrix}\underline{U}(t;\alpha)\\ \overline{U}(t;\alpha)\end{bmatrix} \quad (32)$$

By letting $\mathcal{V}(x,0;\alpha) = \mathfrak{S}(x;\alpha)$, where, $\mathfrak{S}(x;\alpha) = \begin{bmatrix} \underline{g}(x) \\ \overline{g}(x) \end{bmatrix}$.

Then rewrite Eqs. (26)-(29) in matrix form will implies the next matrix differential equation:

$$\mathcal{W}(x,t;\alpha) = I\mathcal{V}(x,t;\alpha) - \int_0^x k(x,\xi) I' \, \mathcal{V}(x,t;\alpha) d\xi \quad (33)$$

Where $\mathcal{W}(x,t;\alpha) = \begin{bmatrix} \underline{W}(x,t;\alpha) \\ \overline{W}(x,t;\alpha) \end{bmatrix}$, $k(x,\xi)$ is the gain kernel and $I' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, with feedback control:

$$\mathcal{V}(1,t;\alpha) = \int_0^1 k(1,\xi) \, I' \mathcal{V}(x,t;\alpha) d\xi \quad (34)$$

and hence the target system is:

$$\mathcal{W}_t(x,t;\alpha) = \gamma I \mathcal{W}_{xx}(x,t;\alpha) \quad (35)$$

with target system boundary conditions:

$$\mathcal{W}(0,t;\alpha) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (36)$$

$$\mathcal{W}(1,t;\alpha) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (37)$$

where $\gamma > 0$.

## Evaluation of the Gain Kernel of Fuzzy Reaction Diffusion Equation

In this section, the objective is to find the function gain kernel $k(x,\xi)$ that makes system (30)-(32) with the feedback control (34) behave as the target system (35)-(37).

First differentiate Eq. (33) with respect to *t*, getting:

$$\mathcal{W}_t(x,t;\alpha) = I\mathcal{V}_t(x,t;\alpha) - \int_0^x k(x,\xi) I' \mathcal{V}_t\,(\xi,t;\alpha) d\xi \quad (38)$$

and when substituting Eq. (30) into Eq. (38), implies:

$$\mathcal{W}_t(x,t;\alpha)\mathcal{V}(x,t;\alpha) = \gamma I \mathcal{V}_{xx}(x,t;\alpha) + \begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \mathcal{V}(x,t;\alpha) \; - \int_0^x k(x,\xi) \left( \gamma I' \mathcal{V}_{\xi\xi}(x,t;\alpha) + \begin{bmatrix} 0 & \overline{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix} \mathcal{V}(x,t;\alpha) \right) d\xi$$

$$= \gamma I \mathcal{V}_{xx}(x,t;\alpha) + \begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \mathcal{V}(x,t;\alpha) - \gamma I' k(x,x) \mathcal{V}_x + \gamma I' k(x,0) \mathcal{V}_x(0,t) + \gamma I' k_\xi \mathcal{V}(x) - \gamma I' k_\xi \mathcal{V}(0) - \int_0^x k_{\xi\xi}(x,\xi) \; \gamma I' \mathcal{V}(\xi,t;\alpha) d\xi - \int_0^x k(x,\xi) \begin{bmatrix} 0 & \overline{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix} \mathcal{V}(x,t;\alpha) d\xi \quad (39)$$

Similarly differentiating Eq. (33) twice with respect to $x$, we get*:*

$$\mathcal{W}_{xx}(x,t;\alpha) = I\mathcal{V}_{xx}(x,t;\alpha) - \frac{d}{dx} k(x,x) I' \mathcal{V}(x,t;\alpha) - I' k(x,x) \mathcal{V}_x - I' k_x \mathcal{V}(x,t;\alpha) - \int_0^x k_{xx} \, I' \mathcal{V}(\xi,t;\alpha) d\xi \quad (40)$$

Subtract Eq. (39) from Eq. (40) yielding to:

$$\mathcal{W}_t(x,t;\alpha) - \gamma I \mathcal{W}_{xx}(x,t;\alpha) = \gamma I \mathcal{V}_{xx}(x,t;\alpha) + \begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \mathcal{V}(x,t;\alpha) \; - \gamma I' k(x,x) \mathcal{V}_x + \gamma I' k(x,0) \mathcal{V}_x(0,t) + \gamma I' k_\xi \mathcal{V}(x) - \gamma I' k_\xi \mathcal{V}(0) - \int_0^x k_{\xi\xi}(x,\xi) \; \gamma I' \mathcal{V}(\xi,t;\alpha) d\xi - \int_0^x k(x,\xi) \begin{bmatrix} 0 & \overline{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix} \mathcal{V}(\xi,t;\alpha) d\xi + I\mathcal{V}_{xx}(x,t;\alpha) + \; 2\gamma \frac{d}{dx} k(x,x) I' \mathcal{V}(x,t;\alpha) + \gamma I' \mathcal{V}_x k(x,x) + \gamma I' k_x \mathcal{V}(x,t;\alpha) + \gamma \int_0^x k_{xx} \, I' \mathcal{V}(\xi,t;\alpha) d\xi$$

For the right-hand side to be zero for all $v(x,t;\alpha)$, the following have to be satisfied:

$$\gamma \int_0^x k_{\xi\xi}(x,\xi) \, I' d\xi - \gamma \int_0^x k_{\xi\xi}(x,\xi) \, I' d\xi = \int_0^x k(x,\xi) \begin{bmatrix} 0 & \overline{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix} d\xi$$

which is equivalent to:

$$\gamma k_{xx}(x,x) I' - \gamma k_{\xi\xi} k(x,x) I' = k(x,\xi) \begin{bmatrix} 0 & \overline{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}$$

and hence:

$$2\gamma \frac{d}{dx} k(x,x) I' = \begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \quad (41)$$

Integrating Eq. (41) with respect to $x$, implies to:

$$\left.\begin{aligned} k_{xx}(x,x) - k_{\xi\xi}(x,x) &= \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{\gamma} k(x,\xi) \\ k(x,x) &= -\frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{2\gamma} x \\ k(x,0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}\right\} \tag{42}$$

and hence, we have three conditions to be satisfied on $k(x,t)$.

Now, to find a solution of Eq. (42), consider the gain kernel as follows when starting with the new variable transformation $\Upsilon = x + \xi, \eta = x - \xi$:

$$\left.\begin{aligned} k(x,\xi) &= Q(\Upsilon,\eta) \\ k_{xx}(x,\xi) &= Q_{\Upsilon\Upsilon}(\Upsilon,\eta) + 2Q_{\Upsilon\eta}(\Upsilon,\eta) + Q_{\eta\eta}(\Upsilon,\eta) \\ k_{\xi\xi}(x,\xi) &= Q_{\Upsilon\Upsilon}(\Upsilon,\eta) - 2Q_{\Upsilon\eta}(\Upsilon,\eta) + Q_{\eta\eta}(\Upsilon,\eta) \end{aligned}\right\} \tag{43}$$

$$Q(\Upsilon,0) = -\frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{4\gamma} \tag{44}$$

$$Q(\Upsilon,\eta) = \Upsilon \tag{45}$$

Therefore, after substituting Eqs. (43) back into Eq. (42), the gain kernel as a PDE become:

$$Q_{\Upsilon\eta}(\Upsilon,\eta) = \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma} Q(\Upsilon,\eta) \tag{46}$$

Integrating Eq. (46) with respect to $\eta$ from 0 to $\eta$, implies to:

$$Q_{\Upsilon}(\Upsilon,\eta) = -\frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{4\gamma} + \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma} \int_0^{\eta} Q(\Upsilon,\varsigma)d\varsigma$$

while integrating the last result with respect to $\Upsilon$, from $\eta$ to $\Upsilon$, obtaining:

$$Q(\Upsilon,\eta) = -\frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{4\gamma}(\Upsilon - \eta) + \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma} \int_{\eta}^{\Upsilon}\int_0^{\eta} Q(\tau,\varsigma)d\varsigma d\tau \tag{47}$$

As a result, the solution of Eqs. (42) is equivalent to Eq. (47).

For this the purpose, we begin by suggesting an initial guess solution for Eq. (47), then recursively substituting the resulting solution back into Eq. (47) and then continue in this manner until we obtain the solution of the Eq. (47). The initial guess solution may be chosen to be as:

$$Q^0(\Upsilon,\eta) = 0 \tag{48}$$

and setting up the recursive formula (47) as follows:

$$Q^{n+1}(\Upsilon,\eta) = -\frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{4\gamma}(\Upsilon - \eta) + \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma} \int_{\eta}^{\Upsilon}\int_0^{\eta} Q^n(\tau,\varsigma)d\varsigma d\tau, \text{ for all } n = 0,1,\ldots \tag{49}$$

If the last integral equation converges, we can write the solution $Q(\Upsilon,\eta)$ as:

$$Q(\Upsilon,\eta) = \lim_{n\to\infty} Q^n(\Upsilon,\eta)$$

It is notable that, the difference between the two consecutive solutions is defined by:

$$\Delta Q^n(\Upsilon,\eta) = Q^{n+1}(\Upsilon,\eta) - Q^n(\Upsilon,\eta) \tag{50}$$

and hence:

$$\begin{aligned} \Delta Q^n(\Upsilon,\eta) &= -\frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{4\gamma}(\Upsilon - \eta) + \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma} \int_{\eta}^{\Upsilon}\int_0^{\eta} Q^n(\tau,\varsigma)d\varsigma d\tau + \frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{4\gamma}(\Upsilon - \eta) - \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma} \int_{\eta}^{\Upsilon}\int_0^{\eta} Q^{n-1}(\tau,\varsigma)d\varsigma d\tau \\ &= \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma}\left(\int_{\eta}^{\Upsilon}\int_0^{\eta} Q^n(\tau,\varsigma)d\varsigma d\tau - \int_{\eta}^{\Upsilon}\int_0^{\eta} Q^{n-1}(\tau,\varsigma)d\varsigma d\tau\right) \\ &= \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma}\left(\int_{\eta}^{\Upsilon}\int_0^{\eta}(Q^n(\tau,\varsigma) - Q^{n-1}(\tau,\varsigma))d\varsigma d\tau\right) \\ &= \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma} \int_{\eta}^{\Upsilon}\int_0^{\eta} \Delta Q^{n-1}(\tau,\varsigma)\, d\varsigma d\tau \end{aligned} \tag{51}$$

There is another way to write Eq. (50) as the series:

$$Q(\Upsilon,\eta) = \sum_{n=0}^{\infty} \Delta Q^n(\Upsilon,\eta)$$

Finding $\Delta Q^n$ from Eq. (48) when beginning with

$$Q^1(\Upsilon,\eta) = -\frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{2\gamma}(\Upsilon - \eta) + \frac{\begin{bmatrix} 0 & \bar{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix}}{4\gamma} \int_{\eta}^{\Upsilon}\int_0^{\eta} Q^0(\tau,\varsigma)d\varsigma d\tau$$

and from Eq. (46) one can obtain:

$$Q^1(\Upsilon,\eta) = -\frac{\begin{bmatrix} \underline{\lambda} & 0 \\ 0 & \bar{\lambda} \end{bmatrix}}{4\gamma}(\Upsilon - \eta)$$

and so:

$$\Delta Q^n(\Upsilon,\eta) = -\frac{\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}}{4\gamma}\frac{(\Upsilon-\eta)\Upsilon^n\eta^n}{n!(n+1)!}\left(\frac{\begin{bmatrix}0 & \bar{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{4\gamma}\right)^n \tag{52}$$

the resolution of the integral equation obtain the following solution:

$$Q(\Upsilon,\eta) = -\sum_{n=0}^{\infty}\frac{\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}}{4\gamma}\frac{(\Upsilon-\eta)\Upsilon^n\eta^n}{n!(n+1)!}\left(\frac{\begin{bmatrix}0 & \bar{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{4\gamma}\right)^n \tag{53}$$

Furthermore, comparing the series (53) with the first order modified Bessel function as the solutions of the following ODE [13]:

$x^2y'' + xy' - (x^2+m^2)y = 0$

which is a Bessel ODE of order $m$ that could be solved by assuming the solution to be $y(x) = I_m(x)$, then:

$I_m(x) = \sum_{n=0}^{\infty}\frac{(x/2)^{m+2n}}{n!(n+m)!}$

and then for the first order Bessel ODE, i.e., if $m = 1$, the modified first order Bessel function is:

$$I_1(x) = \sum_{n=0}^{\infty}\frac{(x/2)^{1+2n}}{n!(n+1)!} \tag{54}$$

Comparing Eq. (53) with Eq. (54), we obtain:

$$\begin{aligned}Q(\Upsilon,\eta) &= -\frac{\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}}{2\gamma}(\Upsilon-\eta)\sum_{n=0}^{\infty}\frac{\left(\Upsilon\eta\frac{\begin{bmatrix}0 & \bar{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{4\gamma}\right)^{n+1}}{n!(n+)!\Upsilon\eta}\\ &= -\frac{\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}}{2\gamma}(\Upsilon-\eta)\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & \bar{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}\Upsilon\eta}}{\sqrt{\frac{\begin{bmatrix}0 & \bar{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}\Upsilon\eta}}\end{aligned} \tag{55}$$

or returning with respect to the original $x, \xi$, variables:

$$k(x,\xi) = -\frac{\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}}{2\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & \bar{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & \bar{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(x^2-\xi^2)}} \tag{56}$$

## Inverse Transformation for the Gain Kernel

The inverse transformation of the VIE given by Eq. (33) is:

$$\mathcal{V}(x,t) = \mathcal{W}(x,t) + \int_0^x L(x,\xi)\mathcal{W}(\xi,t)d\xi \tag{57}$$

where $L(x,\xi)$ is the transformation kernel.

Differentiating Eq. (57) with respect to time $t$ implies to:

$$\begin{aligned}\mathcal{V}_t(x,t) &= \mathcal{W}_t(x,t) + \int_0^x L(x,\xi)\mathcal{W}_t(\xi,t)\,d\xi\\ &= \gamma I\mathcal{W}_{xx}(x,t) + \int_0^x \gamma IL(x,\xi)\mathcal{W}_{xx}(\xi,t)d\xi\\ &= \gamma\big(IW_{xx}(x,t) + IL(x,x)W_x(x,t) - IL(x,0)W_x(x,t) - IL_\xi(x,x)W(x,t) + \int_0^x IL_{\xi\xi}(x,\xi)W(\xi,t)d\xi\big)\end{aligned} \tag{58}$$

and differentiating twice with respect to $x$ give:

$$\mathcal{V}_{xx}(x,t) = IW_{xx}(x,t) + IL_x(x,x)W(x,t) + IW(x,t)\frac{d}{dx}L(x,x) + IL(x,x)W_x(x,x) + \int_0^x IL_{xx}(x,\xi)W(\xi,t)d\xi \tag{59}$$

subtract then Eq. (58) from (59) one can obtain:

$$\begin{aligned}I\mathcal{V}_t(x,t) - \gamma\mathcal{V}_{xx}(x,t) = {}&\gamma IW_{xx}(x,t) + \gamma IL(x,x)W_x(x,t) - \gamma IL_\xi(x,x)W(x,t) + \gamma IL_x(x,x)W(x,x) -\\ &IW_{xx}(x,x) - 2\gamma IW(x,x)\frac{d}{dx}L(x,x) - \gamma IL(x,)W_x(x,x) + \gamma\int_0^x IL_{\xi\xi}(x,\xi)W(\xi,t) -\\ &\gamma\int_0^x IL_{xx}(x,\xi)W(\xi,t)\,d\xi\end{aligned}$$

from Eq. (30) getting:

$$\begin{aligned}&\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}W(x,t) + \begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}\int_0^x L(x,\xi)W(\xi,t)d\xi = \gamma I\left(-2W(x,x)\frac{d}{dx}L(x,x)\right) - \gamma IL(x,0)W(0,x) +\\ &\gamma I\int_0^x(L_{\xi\xi}(x,\xi) - L_{xx}(x,\xi))W(\xi,t)\,d\xi\end{aligned} \tag{60}$$

Equation (60) gives the condition on $L(x,x)$ as in the following:

$$\left.\begin{aligned} &\gamma\left(L_{xx}(x,x)-L_{\xi\xi}(x,x)\right)=-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}L(x,\xi)\\ &L(x,0)=0\\ &L(x,x)+L(x,x)=-\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{2\gamma}x\end{aligned}\right\} \quad (61)$$

Comparing system (61) with system (42), one can notice that:

$$L(x,\xi)=-k(x,\xi;\underline{\lambda})$$
$$L(x,0)=\begin{bmatrix}0\\0\end{bmatrix}$$
$$L(x,x)=-\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{4\gamma}x$$

Now, to find a solution of Eq. (61), consider the gain kernel when starting with the new variable transformation $\Upsilon=x+\xi,\ \eta=x-\xi$. Then, the inverse kernel $L(x,\xi)=H(\gamma,\eta)$. Differentiate $L(x,\xi)$ twice with respect to $x$, will give:

$$L_{xx}(x,\xi)=H_{\gamma\gamma}(\gamma,\eta)+2H_{\gamma\eta}(\gamma,\eta)+H_{\eta\eta}(\gamma,\eta)$$

while differentiating $L(x,\xi)$ with respect to $\xi$ implies to:

$$L_{\xi\xi}(x,\xi)=H_{\gamma\gamma}(\gamma,\eta)-2H_{\gamma\eta}(\gamma,\eta)+H_{\eta\eta}(\gamma,\eta)$$

Substituting $L_{xx}(x,\xi)$ and $L_{\xi\xi}(x,\xi)$ into Eq. (61), getting:

$$H_{\gamma\eta}(\gamma,\eta)=-\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{4\gamma} \quad (62)$$

$$H(\gamma,\gamma)=0 \quad (63)$$

$$H(\gamma,0)=-\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{4\gamma}\xi \quad (64)$$

Integrating Eq. (62) with respect to $\eta$ from 0 to $\eta$, and then integrating the result with respect to $\gamma$, from $\eta$ to $\gamma$ obtaining the integral equation:

$$H(\Upsilon,\eta)=\frac{-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{4\gamma}(\Upsilon-\eta)-\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{4\gamma}\int_{\eta}^{\Upsilon}\int_{0}^{\eta}H(\tau,\varsigma)d\varsigma d\tau$$

Comparing the last integral equation with Eq. (49), yielding to:

$$H(\Upsilon,\eta)=(-1)^n\sum_{n=0}^{\infty}\frac{(\Upsilon-\eta)\Upsilon^n\eta^n}{n!(n+1)!}\left(\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{4\gamma}\right)^{n+1} \quad (65)$$

Comparing Eq. (65) with the first order modified Bessel function given by Eq. (54) as the solution of the ODE and obtaining:

$$H(\Upsilon,\eta)=-\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{\gamma}(\Upsilon-\eta)\frac{I_1\sqrt{\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{\gamma}\Upsilon\eta}}{\sqrt{\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{\gamma}\Upsilon\eta}} \quad (66)$$

or returning to the original $x,\xi$ variables, we have:

$$L(x,\xi)=-\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{\gamma}\xi\frac{J_1\sqrt{\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{\gamma}(x^2-\xi^2)}} \quad (67)$$

Then, the resulted control design functions related to Eqs. (30)-(34) are given by:

$$\mathcal{V}_t(x,t;\alpha)=\gamma\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathcal{V}_{xx}(x,t;\alpha)+\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\mathcal{V}(x,t;\alpha)$$
$$\mathcal{V}(0,t;\alpha)=\begin{bmatrix}0\\0\end{bmatrix}$$
$$\mathcal{V}(1,t;\alpha)=U(t;\alpha)$$
$$\mathcal{V}(1;\alpha)=-\int_0^1\frac{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}{2\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & \lambda\\ \lambda & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & \lambda\\ \lambda & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}\mathcal{V}(\xi,t;\alpha)d\xi \quad (68)$$

with the following transformation of the VIE and inverse transformation of VIE

$$\mathcal{W}(x,t;\alpha) = \mathcal{V}(x,t;\alpha) + \int_0^x \frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{2\gamma}\xi \frac{I_1\sqrt{\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(x^2-\xi^2)}}\mathcal{V}(\xi,t;\alpha)d\xi \tag{69}$$

$$\mathcal{V}(x,t;\alpha) = \mathcal{W}(x,t;\alpha) - \int_0^x \frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{2\gamma}\xi \frac{J_1\sqrt{\frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{\gamma}(x^2-\xi^2)}}\mathcal{W}(\xi,t;\alpha)d\xi. \tag{70}$$

while the target system is:

$$\mathcal{W}_t(x,t;\alpha) = \gamma\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathcal{W}_{xx}(x,t;\alpha) \tag{71}$$

$$\mathcal{W}(0,t;\alpha) = \begin{bmatrix}0\\0\end{bmatrix},\ \mathcal{W}(1,t;\alpha) = \begin{bmatrix}0\\0\end{bmatrix} \tag{72}$$

Now, applying case (ii) fuzzy backstepping method for the fuzzy RDE, which is evaluated previously in Section 4, and will be mentioned again as in case (i), as follows:

**Case (ii):** If $\underline{u}_t(x,t;\alpha) > \overline{u}_t(x,t;\alpha)$, then the possible RDE resulted from system (3)-(6) are:

$\underline{u}_t(x,t;\alpha) = \gamma\overline{u}_{xx}(x,t;\alpha) + \overline{\lambda}\overline{u}(x,t;\alpha)$

$\overline{u}_t(x,t;\alpha) = \gamma\underline{u}_{xx}(x,t;\alpha) + \underline{\lambda}\underline{u}(x,t;\alpha)$

with the same initial and boundary conditions given in case (i) as fuzzy numbers, which are parameterized using $\alpha$-level sets.

Taking the fuzzy backstepping transformation method for implies the following lower and upper cases integral equations:

$$\underline{W}(x,t;\alpha) = \overline{u}(x,t;\alpha) - \int_0^x k(x,\xi)\underline{u}(\xi,t;\alpha)d\xi \tag{73}$$

$$\overline{W}(x,t;\alpha) = \underline{u}(x,t;\alpha) - \int_0^x k(x,\xi)\overline{u}(\xi,t;\alpha)d\xi \tag{74}$$

with lower and upper feedback control followings given by:

$$\underline{u}(1,t;\alpha) = \int_0^1 k(1,\xi)\underline{u}(\xi,t;\alpha)d\xi \tag{75}$$

$$\overline{u}(1,t;\alpha) = \int_0^1 k(1,\xi)\overline{u}(\xi,t;\alpha)d\xi \tag{76}$$

As in case (i), there is some difficulties when separating the crisp upper and lower transformations given by Eqs. (14), and therefore rewriting Eqs. (14) and Eqs. (72)-(76) using matrix differential equation by letting:

$$\mathcal{H}(x,t;\alpha) = \begin{bmatrix}\underline{u}(x,t;\alpha)\\ \overline{u}(x,t;\alpha)\end{bmatrix}$$

and

$$\mathcal{F}(x,t;\alpha) = \begin{bmatrix}\underline{W}(x,t;\alpha)\\ \overline{W}(x,t;\alpha)\end{bmatrix}.$$

Then substituting in Eq. (10) and Eqs. (64)-(67), the transformed matrix differential equation will have form:

$$\mathcal{H}_t(x,t;\alpha) = \gamma\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathcal{H}_{xx}(x,t;\alpha) + \begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}\mathcal{H}(x,t;\alpha) \tag{77}$$

and the target system will be as:

$$\mathcal{F}_t(x,t;\alpha) = \gamma\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathcal{F}_{xx}(x,t;\alpha) \tag{78}$$

$$\mathcal{F}(x,t;\alpha) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathcal{H}(x,t;\alpha) - \int_0^x k(x,\xi)I\mathcal{H}(\xi,t;\alpha)d\xi \tag{79}$$

$$\left.\begin{aligned} k_{xx}(x,\xi) - k_{\xi\xi}(x,\xi) &= \frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}k(x,\xi)\\ \frac{d}{dx}k(x,x) &= -\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{2}x\\ k(x,0) &= \begin{bmatrix}0\\0\end{bmatrix}\end{aligned}\right\} \tag{80}$$

Following the same steps as from Eq. (38) to Eq. (42) getting:

$$k(x,\xi) = -\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(x^2-\xi^2)}}. \tag{81}$$

Also, from Eqs. (46)-(59), one can get:

$$L(x,\xi) = -\frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{\gamma}\xi\frac{J_1\sqrt{\frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{\gamma}(x^2-\xi^2)}} \tag{82}$$

Therefore, the control design results related to Eq. (77) are:

$$\mathcal{H}_t(x,t;\alpha) = \gamma I\mathcal{H}_{xx}(x,t;\alpha) + \begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}\mathcal{H}(x,t;\alpha)$$

$$\mathcal{H}(0,t;\alpha) = \begin{bmatrix}0\\0\end{bmatrix}, \mathcal{H}(1,t;\alpha) = U(t;\alpha)$$

$$\mathcal{H}(1,t;\alpha) = -\int_0^1\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}\mathcal{H}(\xi,t;\alpha)d\xi \tag{83}$$

with the following transformation of the Volterra integral and inverse of Volterra integral,

$$\mathcal{F}(x,t;\alpha) = \mathcal{H}(x,t;\alpha) + \int_0^x\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & \overline{\lambda}\\ \underline{\lambda} & 0\end{bmatrix}}{\gamma}(x^2-\xi^2)}}\mathcal{H}(\xi,t;\alpha)d\xi \tag{84}$$

$$\mathcal{H}(x,t;\alpha) = \mathcal{F}(x,t;\alpha) - \int_0^x\frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{\gamma}\xi\frac{J_1\sqrt{\frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}}{\gamma}(x^2-\xi^2)}}\mathcal{F}(\xi,t;\alpha)d\xi \tag{85}$$

while the target system is:

$$\mathcal{F}_t(x,t;\alpha) = \gamma\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathcal{F}_{xx}(x,t;\alpha)$$

$$\mathcal{F}(0,t;\alpha) = \begin{bmatrix}0\\0\end{bmatrix}, \mathcal{F}(1,t;\alpha) = \begin{bmatrix}0\\0\end{bmatrix}$$

## Results and Discussion

Fuzzy RDEs have many real-life applications, especially in chemistry, where the mathematical model corresponding to several physical phenomena's, in which most of them are the space and time concentration changes of one or more chemicals substances, such as diffusion, which causes the compounds to disperse over a surface in space, and local chemical reactions, which change the substances into one another, which causes the substances to spread out over a surface in space. This can describe non-chemical fuzzy processes in biology, geology, physics, and ecology. They are mathematically represented as semi-linear fuzzy parabolic PDEs and can be rewrite in general form as:

$$\tilde{u}_t(x,t) = \gamma\tilde{u}_{xx}(x,t) + \tilde{\lambda}\tilde{u}(x,t)$$

with initial and boundary conditions:

$$\tilde{u}(x,0) = \tilde{g}(x)$$

$$\tilde{u}(0,t) = \tilde{0},\ \tilde{u}(1,t) = \tilde{U}(t)$$

In this section, we will examine the effectiveness of the above theoretical results that we obtained, taking different values for triangular fuzzy number $\tilde{\lambda}$. In addition, observe their effect on the kernel given by Eq. (56) and also observe their effect on the stability of Eqs. (30) and (77), with $\tilde{\lambda}$ is considered to be triangular FNs, namely $\widetilde{10}, \widetilde{15}, \widetilde{20}$ and $\widetilde{25}$.

Then from Eq (30), the related of $\alpha$-level system is given by:

$$\mathcal{V}_t(x,t;\alpha) = \gamma I\mathcal{V}_{xx}(x,t;\alpha) + \begin{bmatrix}\underline{\lambda} & 0\\ 0 & \overline{\lambda}\end{bmatrix}\mathcal{V}(x,t;\alpha)$$

Then for $\tilde{\lambda} = \widetilde{10}$, and stated in the Definition 2.2, then interval $\alpha$-level set is given by:

$$\widetilde{10} = [9+\alpha, 11-\alpha],\ \alpha \in [0,1]$$

and hence, the gain kernel is:

$$k(\xi) = -\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}} \tag{86}$$

In the backstepping approach, obtaining a negative kernel is crucial for ensuring the stability of the controlled system. This concept is especially important when dealing with boundary control problems in PDEs where the goal is to stabilize the system by appropriately influencing its boundary behavior.

The control function given by:

$$\tilde{u}(1,t) = -\int_0^1 \frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma} \xi \frac{I_1\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}} \tilde{u}(\xi,t)d\xi \quad (87)$$

The transformed Volterra integral equation is:

$$\widetilde{W}(x,t;\alpha) = \tilde{u}(x,t;\alpha) + \int_0^x \frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{4\gamma} \xi \frac{I_1\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{4\gamma}(x^2-\xi^2)}}{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{4\gamma}\sqrt{(x^2-\xi^2)}} \tilde{u}(\xi,t;\alpha)d\xi \quad (88)$$

the inverse kernel gives as:

$$L(x,\xi;\alpha) = -\frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma} \xi \frac{J_1\sqrt{\frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma}(x^2-\xi^2)}}. \quad (89)$$

We notice from the above equations that we have obtained a negative nucleus, which makes the system stable. This idea is particularly crucial when working with PDE boundary control issues, where the objective is to stabilize the system by suitably affecting its boundary behavior.

Also, the inverse transformation Volterra function is:

$$\tilde{u}(x,t;\alpha) = \widetilde{W}(x,t;\alpha) - \int_0^x \frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma} \xi \frac{J_1\sqrt{\frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma}(x^2-\xi^2)}} \widetilde{W}(\xi,t;\alpha)d\xi$$

Similarly, as in case (i) with $\tilde{\lambda} = \widetilde{10}$, we get:

$$\mathcal{H}_t(x,t;\alpha) = \gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathcal{H}_{xx}(x,t;\alpha) + \begin{bmatrix} 0 & \overline{\lambda} \\ \underline{\lambda} & 0 \end{bmatrix} \mathcal{H}(x,t;\alpha)$$

$$k(\xi) = -\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma} \xi \frac{I_1\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}} \quad (90)$$

and the control function is:

$$U(1,t;\alpha) = -\int_0^1 \frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma} \xi \frac{I_1\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}} \mathcal{H}(\xi,t;\alpha)d\xi$$

with transformation interval equation is:

$$L(x,\xi) = \frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma} \xi \frac{I_1\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}}$$

and

$$\mathcal{F}(x,t;\alpha) = \mathcal{H}(x,t;\alpha) + \int_0^1 \frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma} \xi \frac{I_1\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix} 0 & 11-\alpha \\ 9+\alpha & 0 \end{bmatrix}}{\gamma}(1-\xi^2)}} \mathcal{H}(\xi,t;\alpha)d\xi$$

$$= \mathcal{F}(x,t;\alpha) - \int_0^x \frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma} \xi \frac{J_1\sqrt{\frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix} 9-\alpha & 0 \\ 0 & 11+\alpha \end{bmatrix}}{4\gamma}(x^2-\xi^2)}} \mathcal{F}(\xi,t;\alpha)d\xi$$

We notice that in both cases of Hukuhara we get a negative kernel, which means that the methodology of backstepping approach achieves stability for the fuzzy systems of differential equations.

As in the above, similar calculations may be carried out for the other values of $\tilde{\lambda}$ for both cases (i) and (ii) and the results of the required functions are given in Table 1:

**Table 1.** Results of the kernel functions of Eq. (30), inverse of the kernel (57) and the control function of Eq. (30) related to different values of $\tilde{\lambda}$

| $\tilde{\lambda}$ | Kernel $k(x,\xi)$ | Inverse of kernel | Control function |
|---|---|---|---|
| $\widetilde{10}$ | $k(\xi) = -\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}$ | $L(x,\xi) = -\frac{\begin{bmatrix}9-\alpha & 0\\ 0 & 11+\alpha\end{bmatrix}}{4\gamma}\xi\frac{J_1\sqrt{\frac{\begin{bmatrix}9-\alpha & 0\\ 0 & 11+\alpha\end{bmatrix}}{4\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}9-\alpha & 0\\ 0 & 11+\alpha\end{bmatrix}}{4\gamma}(x^2-\xi^2)}}$ | $U(1) = -\int_0^1\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 11-\alpha\\ 9+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}\mathcal{V}(\xi)d\xi$ |
| $\widetilde{15}$ | $k(\xi) = -\frac{\begin{bmatrix}0 & 16-\alpha\\ 14+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 16-\alpha\\ 14+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 16-\alpha\\ 14+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}$ | $L(x,\xi) = -\frac{\begin{bmatrix}14-\alpha & 0\\ 0 & 16+\alpha\end{bmatrix}}{4\gamma}\xi\frac{J_1\sqrt{\frac{\begin{bmatrix}14-\alpha & 0\\ 0 & 16+\alpha\end{bmatrix}}{4\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}14-\alpha & 0\\ 0 & 16+\alpha\end{bmatrix}}{4\gamma}(x^2-\xi^2)}}$ | $U(1) = -\int_0^1\frac{\begin{bmatrix}0 & 16-\alpha\\ 14+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 16-\alpha\\ 14+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 16-\alpha\\ 14+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}\mathcal{V}(\xi)d\xi$ |
| $\widetilde{20}$ | $k(\xi) = -\frac{\begin{bmatrix}0 & 21-\alpha\\ 19+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 21-\alpha\\ 19+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 21-\alpha\\ 19+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}$ | $L(x,\xi) = -\frac{\begin{bmatrix}19-\alpha & 0\\ 0 & 21+\alpha\end{bmatrix}}{4\gamma}\xi\frac{J_1\sqrt{\frac{\begin{bmatrix}19-\alpha & 0\\ 0 & 21+\alpha\end{bmatrix}}{4\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}19-\alpha & 0\\ 0 & 21+\alpha\end{bmatrix}}{4\gamma}(x^2-\xi^2)}}$ | $U(1) = -\int_0^1\frac{\begin{bmatrix}0 & 21-\alpha\\ 19+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 21-\alpha\\ 19+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 21-\alpha\\ 19+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}\mathcal{V}(\xi)d\xi$ |
| $\widetilde{25}$ | $k(\xi) = -\frac{\begin{bmatrix}0 & 26-\alpha\\ 24+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 26-\alpha\\ 24+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 26-\alpha\\ 24+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}$ | $L(x,\xi) = -\frac{\begin{bmatrix}24-\alpha & 0\\ 0 & 26+\alpha\end{bmatrix}}{4\gamma}\xi\frac{J_1\sqrt{\frac{\begin{bmatrix}24-\alpha & 0\\ 0 & 26+\alpha\end{bmatrix}}{4\gamma}(x^2-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}24-\alpha & 0\\ 0 & 26+\alpha\end{bmatrix}}{4\gamma}(x^2-\xi^2)}}$ | $U(1) = -\int_0^1\frac{\begin{bmatrix}0 & 26-\alpha\\ 24+\alpha & 0\end{bmatrix}}{\gamma}\xi\frac{I_1\sqrt{\frac{\begin{bmatrix}0 & 26-\alpha\\ 24+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}{\sqrt{\frac{\begin{bmatrix}0 & 26-\alpha\\ 24+\alpha & 0\end{bmatrix}}{\gamma}(1-\xi^2)}}\mathcal{V}(\xi)d\xi$ |

From the above results, we can conclude the following:

1. The lower value of $\lambda$, the more we can control the diffusion of reactants in chemical reactions**.**
2. Also, in fuzzy reactions, the amount of pollution depends on $\alpha \in [0,1]$, factor as well, as pollution increases as it approaches zero, and decreases as it approaches 1.

# Conclusions

In this work, we are able to stabilize the fuzzy RDE using the backstepping approach, which is achieved by transforming those equations resulting from pgh-derivational with the α-level set concepts. approach including. The same equation with both upper and lower functions. Also, we are able to get rid of this problem by transforming the equations into a system of differential matrices. This work is of great importance in reducing pollution resulting from chemical reactions, so that we can narrow the area affected by the reaction of chemical materials.

It is also notable that the two cases considered in this work give the same results, or the results of the first case are very close to the results of the second case, which means that the stability of the fuzzy RDE is not affected by changing the state of the equation, whether increasing or decreasing.

Moreover, for future studies, we can apply the method of transforming the equations after removing the fuzziness into a system of differential matrices in the case of the appearance of the upper and lower functions in the same equation and apply any mathematical method for the purpose of finding solutions to fuzzy PDEs.

# Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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