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RESEARCH ARTICLE

Two-Point Diagonally Implicit Fractional Block Backward Differentiation Formula for Solving Fractional Differential Equations

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Abstract This paper presents the development of two-point diagonally implicit fractional block backward differentiation formula of order two with constant step size (2DIFBBDF(2)) for solving the fractional differential equations (FDEs). The method is derived based on the concept of fractional linear multistep method and the classical diagonally block backward differentiation formula (BBDF) method. Convergence and stability analyses of the method are also discussed. This method is proved to be A-stable for values of fractional order between 0.7 and 1.0. Next, numerical examples in the form of linear, non-linear and system of FDEs are presented to demonstrate the method's reliability and efficiency. The results obtained are compared with the existing methods whereas 2DIFBBDF(2) method outperforms the others in terms of accuracy indicating that it is an appropriate method for solving FDEs.

Keywords: Fractional differential equations, fractional linear multistep method, convergence, stability.

Introduction

A branch of mathematical analysis known as fractional calculus expands on the ideas of classical calculus by taking into account integrals and derivatives of arbitrary orders, which can include both real and complex values. In contrast to classical calculus, which deals with integrals and derivatives of integer order, fractional calculus provides an extended framework that enables differentiation and integration at fractional or non-integer orders. This expansion facilitates a wider array of applications and mathematical inquiries, when traditional calculus may be inadequate.

Fractional calculus is primarily used in the analysis of differential equations, especially those that include fractional derivatives of an unknown function. Traditional differential equations are generalized by the addition of fractional orders in these equations, which are known as FDEs. This kind of equations enable the modeling of complex phenomena that exhibit memory effects, as mentioned by Kilicman [1]. Several mathematical models that utilize FDEs include the Pharmacokinetics Model [2], the SIR Model [1], the Economic Growth Model [3], FitzHugh–Nagumo Model [4], the COVID-19 Model [5] and the Viscoelastic Model [6].

FDEs are a class of equations that incorporate derivatives of fractional order, typically expressed in the form $\frac{d^\alpha}{dx^\alpha}$, where $\alpha > 0$ and not necessarily an integer. These equations generalize ordinary differential equations (ODEs) by extending their order to non-integer values. An essential component of FDEs is the starting conditions, described by the fractional initial value problem (FIVP). The FIVP comprises of a FDE accompanied by an initial condition that delineates the value of the unknown function at a particular beginning point within a specified interval. Biala and Jator [7] assert that the FIVP is articulated in the subsequent manner:

$$ {}_{C}D_{t_0}^{\alpha} y(t) = f(t, y(t)), \qquad y(t_0) = y_0 \tag{1} $$

where ${}_{C}D_{t_0}^{\alpha}$ is the fractional Caputo's α-derivative operator, as defined in Equation (2) [8] and $0 < \alpha < 1$ is the fractional order:

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$$ {}_{C}D_{t_0}^{\alpha} = {}_{RL}D_{t_0}^{\alpha} y(t) - y(t_0), \quad (2) $$

where ${}_{RL}D_{t_0}^{\alpha} y(t)$ is defined as Riemann-Liouville differential operator:

$$ {}_{RL}D_{t_0}^{\alpha} y(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_{t_0}^{t} \frac{y(\tau)d\tau}{(t-\tau)^{\alpha-m+1}}, \quad (3) $$

with $m - 1 < \alpha < m \in \mathbb{Z}, \alpha > 0$ and $m = \lceil \alpha \rceil$.

A significant number of FDEs lack analytical solutions, indicating that in general, the equations do not always have exact solutions. There are conditions where numerical solutions to FDEs have shown to be more effective and convenient than analytical solutions. This is especially relevant when addressing substantial and complex problems. Consequently, numerical approaches have become more essential for solving FDEs. In literature, Biala and Jator [9] proposed a family of Implicit Adams Methods (IAMS) to solve FDEs. Ahmed [10] proposed the modified fractional Euler method (MFEM) to solve the systems of FDEs which are linear and nonlinear type. Gnitchogna and Atangana [11] discussed a method known as Laplace Adam-Bashforth method. A family of multistep explicit methods for solving FDEs was discussed by Bonab and Javidi [12] which are derived based on the third-order fractional backward differentiation formula. In 2020, Zabadi *et al.* [13] proposed an explicit multistep method which is fractional explicit Adams method of order 3 (FEAM3) to solve linear and non-linear FDEs. Furthermore, Zabidi *et al*. [14] continued to discuss an implicit method which utilised predict-correct numerical method of order two, known as fractional Adams method of explicit order 2, implicit order 2 (FAM22) to solve the FDEs. In 2022, Mechee and Aidi [15] discussed direct method for solving third-order fractional ODE which are one-stage explicit method and two-stages, three-stage explicit Runge-Kutta methods with the same step size.

Although numerous numerical methods have been created to solve FDEs, the majority of existing methods are non-block methods. These non-block methods generally resolve the equations sequentially, which can be highly computational and less efficient when addressing large systems. Lambert [16] found that the implicit method is more accurate than the explicit method for addressing stiff conditions. As a result, several researchers had proposed the numerical methods on solving the stiff problems, especially BBDF method to solve stiff ODEs. Zawawi *et al*. [17] has proposed diagonally implicit 2-point BBDF to solve stiff ODEs and the outcomes showed that diagonally BBDF method is efficient in solving the equations. Thus, the aim of this paper is to further modifying the diagonally BBDF method to solve the FDEs, which has been proven to be more efficient in solving stiff ODEs. The proposed method is able to compute two solution points simultaneously within a single step. Hence, this approach significantly reduces computation time and requires fewer steps compared to non-block methods, which calculate one solution point at a time. The BBDF method was introduced by Ibrahim *et al.* [26], where two variants which are the implicit r = 2-point BBDF method and the implicit r = 3-point BBDF method. In 2011, Nasir *et al.* [27] expands upon the concepts by Ibrahim *et al.* to solve first order stiff ODEs. They derived the new method by adding two back values as well as increase the order of the method to order 5 to form a new method to solve stiff ODEs. Furthermore. Zawawi *et al*. [17] modified the method proposed by Nasir *et al.* [27], known as diagonally implicit 2-point block backward differentiation formulas. Abasi *et al.* [29] derived a new method which involves the approximation of two solutions by incorporating two off-step points simultaneously at each iteration. Not only that, Ibrahim and Zawawi [28] proposed an order four block backward differentiation formula by incorporating independent parameter α. By adapting the diagonally BBDF method, we are inspired to investigate the potential of using the method to solve FDEs. This is because the BBDF method has gained significant popularity and has been shown to be effective at solving ODEs.

Derivation of the Method

This section describes the derivation of two-point diagonally implicit fractional block backward differentiation formula of order two with constant step size (2DIFBBDF(2)). Two back values which are t_{n-1} and t_n are used to approximate the values of y_{n+1} and y_{n+2} simultaneously with a constant step size, h. Two back values are chosen only because it can reduce the computational cost as more back values involved will result in a bigger system of equations where it will increase computational cost. The following are the notations that will be considered in the derivation part:

t_n	: Point at current time level
t_{n-1}	: Point at previous time level
y_n	: Approximate solution at time t_n
y_{n-1}	: Approximate solution at time t_{n-1}
y_{n+1}	: Future approximate solution at time t_{n+1} that will be computed by using proposed method
y_{n+2}	: Future approximate solution at time t_{n+2} that will be computed by using proposed method
h	: Step size

The method is formulated with the general formula of fractional linear multistep method (FLMM) as proposed by Galeone and Garrappa [18]:

$$\sum_{j=0}^{n} \gamma_j y_{n-j} = h^{\alpha} \sum_{i=0}^{n} \beta_i f(t_{n-j}, y_{n-j}) \tag{4}$$

where γ_j and β_i are the real parameters and the classical diagonally BBDF used by Ijam *et al.* [19] in the form of,

$$\sum_{j=0}^{i+1} \gamma_{j,i} y_{n+j-1} = h^{\alpha} \beta_i f_{n+i} \tag{5}$$

where $\gamma_{j,i}$ and β_i are the real parameters, h^{α} is the step length, $i = 1{,}2$ for y_{n+1} and y_{n+2} respectively. The derivation includes FLMM in Equation (4) linked to the linear difference operator L_h [18], defined as follows:

(i) For $i = 1$,

$$\begin{aligned} L_h[y(t), t, \alpha] &= \sum_{j=0}^{2} \gamma_{j,i} y_{n+j-1} - h^{\alpha} \beta_i f_{n+i} \\ &= \sum_{j=0}^{2} \gamma_{j,i} y_{n+j-1} - h^{\alpha} \beta_i \, {}^{C}D_{t_0}^{\alpha} y_{n+i} \\ &= \gamma_{0,1} y_{n-1} + \gamma_{1,1} y_n + \gamma_{2,1} y_{n+1} - h^{\alpha} \beta_1 \, {}^{C}D_{t_0}^{\alpha} y_{n+1} \\ &= 0. \end{aligned} \tag{6}$$

(ii) For $i = 2$,

$$\begin{aligned} L_h[y(t), t, \alpha] &= \sum_{j=0}^{3} \gamma_{j,i} y_{n+j-1} - h^{\alpha} \beta_i f_{n+i} \\ &= \sum_{j=0}^{3} \gamma_{j,i} y_{n+j-1} - h^{\alpha} \beta_i \, {}^{C}D_{t_0}^{\alpha} y_{n+i} \\ &= \gamma_{0,2} y_{n-1} + \gamma_{1,2} y_n + \gamma_{2,2} y_{n+1} + \gamma_{3,2} y_{n+2} - h^{\alpha} \beta_2 \, {}^{C}D_{t_0}^{\alpha} y_{n+2} \\ &= 0. \end{aligned} \tag{7}$$

Then, by using Taylor's series expansion [18], the terms y_{n-1}, y_n, y_{n+1}, y_{n+2}, ${}^{C}D_{t_0}^{\alpha} y_{n+1}$ and ${}^{C}D_{t_0}^{\alpha} y_{n+2}$ are expanded about t_n in Equation (6) and (7) as presented below:

$$\begin{aligned} y_{n-1} &= y_n + (-h) y_n' + \frac{(-h)^2}{2!} y_n'' + \frac{(-h)^3}{3!} y_n''' + \cdots \\ y_n &= y_n \\ y_{n+1} &= y_n + (h) y_n' + \frac{(h)^2}{2!} y_n'' + \frac{(h)^3}{3!} y_n''' + \cdots \\ y_{n+2} &= y_n + (2h) y_n' + \frac{(2h)^2}{2!} y_n'' + \frac{(2h)^3}{3!} y_n''' + \cdots \\ {}^{C}D_{t_0}^{\alpha} y_{n+1} &= \frac{(h)^{1-\alpha}}{\Gamma(2-\alpha)} y_n' + \frac{(h)^{2-\alpha}}{\Gamma(3-\alpha)} y_n'' + \frac{(h)^{3-\alpha}}{\Gamma(4-\alpha)} y_n''' + \cdots \end{aligned} \tag{8}$$

$$ {}^{C}D_{t_0}^{\alpha}y_{n+2} = \frac{(2h)^{1-\alpha}}{\Gamma(2-\alpha)}y_n' + \frac{(2h)^{2-\alpha}}{\Gamma(3-\alpha)}y_n'' + \frac{(2h)^{3-\alpha}}{\Gamma(4-\alpha)}y_n''' + \cdots $$

The formula for the first point, y_{n+1} can be computed by substituting Equation (8) into Equation (6), resulting in the following equation:

$$ \begin{aligned} &\gamma_{0,1}\left[y_n + (-h)y_n' + \frac{(-h)^2}{2!}y_n'' + \frac{(-h)^3}{3!}y_n''' + \cdots\right] \\ &+\gamma_{1,1}(y_n) + \gamma_{2,1}\left[y_n + (h)y_n' + \frac{(h)^2}{2!}y_n'' + \frac{(h)^3}{3!}y_n''' + \cdots\right] \\ &-h^{\alpha}\beta_1\left[\frac{(h)^{1-\alpha}}{\Gamma(2-\alpha)}y_n' + \frac{(h)^{2-\alpha}}{\Gamma(3-\alpha)}y_n'' + \frac{(h)^{3-\alpha}}{\Gamma(4-\alpha)}y_n''' + \cdots\right] = 0. \end{aligned} \tag{9} $$

Then, the value of value $\gamma_{2,1}$ is normalised to 1 in order to eliminate the arbitrary nature of the coefficients, yielding,

$$ \begin{aligned} &\gamma_{0,1}\left[y_n + (-h)y_n' + \frac{(-h)^2}{2!}y_n'' + \frac{(-h)^3}{3!}y_n''' + \cdots\right] \\ &+\gamma_{1,1}(y_n) + (1)\left[y_n + (h)y_n' + \frac{(h)^2}{2!}y_n'' + \frac{(h)^3}{3!}y_n''' + \cdots\right] \\ &-h^{\alpha}\beta_1\left[\frac{(h)^{1-\alpha}}{\Gamma(2-\alpha)}y_n' + \frac{(h)^{2-\alpha}}{\Gamma(3-\alpha)}y_n'' + \frac{(h)^{3-\alpha}}{\Gamma(4-\alpha)}y_n''' + \cdots\right] = 0. \end{aligned} \tag{10} $$

Subsequently, using factorizing, all coefficients of $y_n, y_n', y_n'', \ldots$ in Equation (10) are collected giving

$$ L_h[y(t), t, \alpha] = C_{0,1}\,y_n + \sum_{k=1}^{m} h^k C_{k,1} y_n^{(k)} + h^{m+1}R_{m+1}\ , k = 1,2,3, \ldots \tag{11} $$

where the remainder R_{m+1} is obtained using Taylor's expansions and the constants,

$$ \begin{aligned} C_{0,1} &= \gamma_{0,1} + \gamma_{1,1} + 1 \\ C_{1,1} &= -\gamma_{0,1} + 1 + \frac{(1)^{1-\alpha}}{\Gamma(2-\alpha)}\beta_1 \\ C_{2,1} &= \frac{(-1)^2}{2!}\gamma_{0,1} + \frac{1^2}{2!} - \frac{(1)^{2-\alpha}}{\Gamma(3-\alpha)}\beta_1 \end{aligned} \tag{12} $$

In order to determine the coefficient values of $\gamma_{0,1}, \gamma_{1,1}$ and β_1, the system in Equation (12) are solved simultaneously, resulting in

$$ \begin{aligned} \gamma_{0,1} &= -\frac{\alpha}{-4+\alpha}, \\ \gamma_{1,1} &= \frac{4}{-4+\alpha}, \\ \beta_1 &= -\frac{2\Gamma(3-\alpha)}{-4+\alpha}. \end{aligned} \tag{13} $$

The values in Equation (13) are then substituted into Equation (6) to obtain the following:

$$ -\frac{\alpha}{-4+\alpha}y_{n-1} + \frac{4}{-4+\alpha}y_n + y_{n+1} + \frac{2\Gamma(3-\alpha)}{-4+\alpha}h^{\alpha}\,{}^{C}D_{t_0}^{\alpha}y_{n+1} = 0 \tag{14} $$

Equation (14) is rearranged to obtain the approximation solution for the first point of the method, y_{n+1}.

$$ y_{n+1} = \frac{\alpha}{-4+\alpha}y_{n-1} - \frac{4}{-4+\alpha}y_n - \frac{2\Gamma(3-\alpha)}{-4+\alpha}h^{\alpha}\,{}^{C}D_{t_0}^{\alpha}y_{n+1} \tag{15} $$

The formula for the second point, y_{n+2} can be computed by substituting Equation (8) into Equation (7), resulting in the following:

$$\begin{aligned}
&\gamma_{0,2}\left[y_n+(-h)y_n'+\frac{(-h)^2}{2!}y_n''+\frac{(-h)^3}{3!}y_n'''+\cdots\right]\\
&+\gamma_{1,2}y_n+\gamma_{2,2}\left[y_n+(h)y_n'+\frac{(h)^2}{2!}y_n''+\frac{(h)^3}{3!}y_n'''+\cdots\right]\\
&+\gamma_{3,2}\left[y_n+(2h)y_n'+\frac{(2h)^2}{2!}y_n''+\frac{(2h)^3}{3!}y_n'''+\cdots\right]\\
&-h^\alpha\beta_2\left[\frac{(2h)^{1-\alpha}}{\Gamma(2-\alpha)}y_n'+\frac{(2h)^{2-\alpha}}{\Gamma(3-\alpha)}y_n''+\frac{(2h)^{3-\alpha}}{\Gamma(4-\alpha)}y_n'''+\cdots\right]=0.
\end{aligned} \tag{16}$$

Then, the value of value $\gamma_{3,2}$ is normalised to 1 in order to eliminate the arbitrary nature of the coefficients, resulting in,

$$\begin{aligned}
&\gamma_{0,2}\left[y_n+(-h)y_n'+\frac{(-h)^2}{2!}y_n''+\frac{(-h)^3}{3!}y_n'''+\cdots\right]\\
&+\gamma_{1,2}y_n+\gamma_{2,2}\left[y_n+(h)y_n'+\frac{(h)^2}{2!}y_n''+\frac{(h)^3}{3!}y_n'''+\cdots\right]\\
&+(1)\left[y_n+(2h)y_n'+\frac{(2h)^2}{2!}y_n''+\frac{(2h)^3}{3!}y_n'''+\cdots\right]\\
&-h^\alpha\beta_2\left[\frac{(2h)^{1-\alpha}}{\Gamma(2-\alpha)}y_n'+\frac{(2h)^{2-\alpha}}{\Gamma(3-\alpha)}y_n''+\frac{(2h)^{3-\alpha}}{\Gamma(4-\alpha)}y_n'''+\cdots\right]=0.
\end{aligned} \tag{17}$$

Subsequently, using factorizing, we collect all of the coefficients of $y_n, y_n', y_n'', \ldots$ in Equation (17) and collecting terms gives,

$$L_h[y(t),t,\alpha]=C_{0,1}\,y_n+\sum_{k=1}^{m}h^kC_{k,1}y_n^{(k)}+h^{m+1}R_{m+1}\ ,k=1,2,3,\ldots \tag{18}$$

where the remainder R_{m+1} is obtained using Taylor's expansions and the constant,

$$\begin{aligned}
C_{0,2}&=\gamma_{0,2}+\gamma_{1,2}+\gamma_{2,2}+1\\
C_{1,2}&=-\gamma_{0,2}+\gamma_{2,2}+2-\frac{(2)^{1-\alpha}}{\Gamma(2-\alpha)}\beta_2\\
C_{2,2}&=\frac{(-1)^2}{2!}\gamma_{0,2}+\frac{1^2}{2!}\gamma_{2,2}+\frac{2^2}{2!}-\frac{(2)^{2-\alpha}}{\Gamma(3-\alpha)}\beta_2\\
C_{3,2}&=\frac{(-1)^3}{3!}\gamma_{0,2}+\frac{1^3}{3!}\gamma_{2,2}+\frac{2^3}{3!}-\frac{(2)^{3-\alpha}}{\Gamma(4-\alpha)}\beta_2
\end{aligned} \tag{19}$$

In order to determine the coefficient values of $\gamma_{0,2}, \gamma_{1,2}, \gamma_{2,2}$ and β_2, the systems of (19) are solved simultaneously.

$$\begin{aligned}
\gamma_{0,2}&=\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18}\\
\gamma_{1,2}&=\frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18}\\
\gamma_{2,2}&=-\frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18}\\
\beta_2&=-\frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}
\end{aligned} \tag{20}$$

The values in Equation (20) are then substituted into Equation (7) to obtain the following,

$$\begin{aligned}
&\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18}y_{n-1}+\frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18}y_n-\frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18}y_{n+1}\\
&+y_{n+2}-\frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^\alpha\,{}^{C}D_{t_0}^{\alpha}y_{n+2}=0
\end{aligned} \tag{21}$$

The equation (21) is rearranged to obtain the approximation solution for the second point of the method, y_{n+2}.

$$\begin{aligned} y_{n+2} = &-\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18}y_{n-1} - \frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18}y_n \\ &+\frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18}y_{n+1} - \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^\alpha\, {}^C D_{t_0}^\alpha y_{n+2} \end{aligned} \tag{22}$$

Therefore, the general corrector formula of the 2DIFBBDF(2) method is as follows,

$$\begin{aligned} y_{n+1} &= \frac{\alpha}{-4+\alpha}y_{n-1} - \frac{4}{-4+\alpha}y_n - \frac{2\Gamma(3-\alpha)}{-4+\alpha}h^\alpha\, {}^C D_{t_0}^\alpha y_{n+1} \\ y_{n+2} &= -\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18}y_{n-1} - \frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18}y_n \\ &\quad +\frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18}y_{n+1} - \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^\alpha\, {}^C D_{t_0}^\alpha y_{n+2} \end{aligned} \tag{23}$$

Convergence and Stability of the Method

The convergence and stability of the 2DIFBBDF(2) method in Equation (23) are examined in this section where the fractional order, α, fluctuates between 0 and 1. The main focus of the values of α in this study is 0.7, 0.8, 0.9 and 1.0. Therefore, the method's analysis will take into account the subsequent definitions:

Theorem 1 [16]. If the method fulfils both consistency and zero stability, then the method is said to be convergent.

Definition 1 [16]. If the order of FLMM is larger than or equal to k, where $k > 1$, then the method is said to be consistent.

Definition 2 [16,18]. As $C_0 = C_1 = \cdots = C_k = 0$ and $C_{k+1} \neq 0$, the fractional linear multistep method is said to be of order k. The following is the formula for calculating the constant C_k:

$$\begin{aligned} C_0(n,\alpha) &= \sum_{j=0}^{n}\gamma_j \\ C_k(n,\alpha) &= \frac{1}{k!}\sum_{j=0}^{n}(j-2)^k\gamma_j - \frac{1}{\Gamma(k+1-\alpha)}\sum_{j=0}^{n}(j-2)^k\beta_j\,, k = 2,3,\ldots \end{aligned} \tag{24}$$

where k is the order of the proposed method, γ is the coefficient acquired from the proposed method, and β is the coefficient obtained from the proposed method. It is noted that the method's error constant is C_{k+1}.

Suppose that $n = 3$. The process of determining the order of the method as detailed in Equation (23) is as follows:

$$\begin{aligned} C_0(3,\alpha) &= \sum_{j=0}^{3}\gamma_j = \begin{bmatrix}0\\0\end{bmatrix}, \\ C_1(3,\alpha) &= \sum_{j=0}^{3}(j-2)\gamma_j - \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{3}(j-2)^{1-\alpha}\beta_j = \begin{bmatrix}0\\0\end{bmatrix}, \\ C_2(3,\alpha) &= \frac{1}{2!}\sum_{j=0}^{3}(j-2)^2\gamma_j - \frac{1}{\Gamma(3-\alpha)}\sum_{j=0}^{3}(j-2)^{2-\alpha}\beta_j = \begin{bmatrix}0\\0\end{bmatrix}, \\ C_3(3,\alpha) &= \frac{1}{3!}\sum_{j=0}^{3}(j-2)^3\gamma_j - \frac{1}{\Gamma(4-\alpha)}\sum_{j=0}^{3}(j-2)^{3-\alpha}\beta_j = \begin{bmatrix}e_1\\e_2\end{bmatrix}. \end{aligned} \tag{25}$$

Due to the presence of error constant at C_3, the method is of second order. According to Definition 1, the method is deemed to be consistent due to the existence of the order of the method. Table 1 presents the error constants corresponding to each value of α.

Table 1. Error constants of the method, e_1 and e_2 when $\alpha = 0.7, 0.8, 0.9$ and 1.0

α	e_1	e_2
0.7	$-\frac{301}{2277}$	0
0.8	$-\frac{7}{44}$	0
0.9	$-\frac{41}{217}$	0
1.0	$-\frac{2}{9}$	0

It has been noted that the error constants that are presented in Table 1 are always zero for second point of the corrector, regardless of the fractional order,α that is being considered. Consequently, this implies that the method obtains an enhanced accuracy than that predicted by the theoretical order alone. The consistency of zero error constants across multiple fractional orders further suggests that the suggested method preserves its accuracy consistently, no matter of the fractional derivative's order. This helps to the robustness and reliability of the approach when used to a wide variety of numerical examples with variable degrees of fractional behavior.

Definition 3 [16]. The FLMM is deemed zero stable if all roots of the characteristic polynomial associated with the initial conditions are within -1 and 1, and any root with a modulus equal to one is a simple root.

The stability properties of 2DIFBBDF(2) method are investigated. By considering the linear test problem from Zabidi *et al.* [13]:

$$\begin{gathered} {}^{C}D^{\alpha} y(t) = \lambda y(t), \quad \lambda \in \mathbb{C}, \\ y(t_0) = y_0, \end{gathered} \tag{26}$$

where λ is the eigenvalue, the exact solution is $y(t) = E_\alpha(\lambda(t-t_0)^\alpha)y_0$, and $E_\alpha(\cdot)$ represents the Mittag-Leffler function:

$$E_\alpha(t) = \sum_{k=0}^{\infty}\left(\frac{t^k}{\Gamma(\alpha k+1)}\right). \tag{27}$$

Substituting Equation (26) into Equation (23) forms

$$\begin{aligned} y_{n+1} &= \frac{\alpha}{-4+\alpha} y_{n-1} - \frac{4}{-4+\alpha} y_n - \frac{2\Gamma(3-\alpha)}{-4+\alpha} h^\alpha \lambda y_{n+1}, \\ y_{n+2} &= -\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18} y_{n-1} - \frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18} y_n \\ &\quad + \frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18} y_{n+1} - \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18} h^\alpha \lambda y_{n+2}. \end{aligned} \tag{28}$$

Then, substituting $h^\alpha \lambda = \bar{h}$ into Equation (28) leads to

$$\begin{aligned} y_{n+1} &= \frac{\alpha}{-4+\alpha} y_{n-1} - \frac{4}{-4+\alpha} y_n - \frac{2\Gamma(3-\alpha)}{-4+\alpha} \bar{h} y_{n+1}, \\ y_{n+2} &= -\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18} y_{n-1} - \frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18} y_n \\ &\quad + \frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18} y_{n+1} - \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18} \bar{h} y_{n+2}. \end{aligned} \tag{29}$$

Equation (29) is then arranged in the matrix form,

$$\begin{bmatrix} 1+\frac{2\Gamma(3-\alpha)}{-4+\alpha}\bar{h} & 0 \\ -\frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18} & 1+\frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}\bar{h} \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{-4+\alpha} & -\frac{4}{-4+\alpha} \\ -\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18} & -\frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18} \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}. \tag{30}$$

Equation (30) is equivalent to $AY_m = BY_{m-1}$, by which,

$$A = \begin{bmatrix} 1+\frac{2\Gamma(3-\alpha)}{-4+\alpha}\bar{h} & 0 \\ -\frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18} & 1+\frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}\bar{h} \end{bmatrix}, \quad Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix},$$
$$B = \begin{bmatrix} \frac{\alpha}{-4+\alpha} & -\frac{4}{-4+\alpha} \\ -\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18} & -\frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18} \end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}. \tag{31}$$

Then, by Equation (31), the stability polynomial of the method is calculated by using the formula:

$$\pi(r;\bar{h}) = de\,t(Ar-B), \tag{32}$$

where *r* is the root of stability polynomial, forming,

$$\begin{aligned}\pi(r;\bar{h}) &= det\,(Ar-B) \\ &= \det\left(\begin{bmatrix} 1+\frac{2\Gamma(3-\alpha)}{-4+\alpha}\bar{h} & 0 \\ -\frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18} & 1+\frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}\bar{h} \end{bmatrix} r - \begin{bmatrix} \frac{\alpha}{-4+\alpha} & -\frac{4}{-4+\alpha} \\ -\frac{2\alpha(\alpha+1)}{\alpha^2-5\alpha-18} & -\frac{3(\alpha^2-13\alpha+6)}{\alpha^2-5\alpha-18} \end{bmatrix}\right).\end{aligned} \tag{33}$$

Substituting $\bar{h} = 0$ into Equation (33), to determine the zero stability, forms the following polynomials and roots.

1. When $\alpha = 0.7$,

$$\pi(r;\bar{h}) = r^2 - \frac{65294}{69333}r - \frac{4039}{69333} \tag{34}$$

   and the roots, r_s are 1 and -0.05826.

2. When $\alpha = 0.8$,

$$\pi(r;\bar{h}) = r^2 - \frac{343}{356}r - \frac{13}{356} \tag{35}$$

   and the roots, r_s are 1 and -0.03652.

3. When $\alpha = 0.9$,

$$\pi(r;\bar{h}) = r^2 - \frac{7418}{7471}r - \frac{53}{7471} \tag{36}$$

   and the roots, r_s are 1 and -0.007094.

4. When $\alpha = 1.0$,

$$\pi(r;\bar{h}) = r^2 - \frac{34}{33}r + \frac{1}{33} \tag{37}$$

and the roots, r_s are 1 and 0.03030.

According to Definition 3, the method is proved to be zero stable as all the roots satisfy the condition $|r_s| \leq 1$. By referring to Theorem 1, the method is said to be convergent as it satisfies both consistency and zero stability.

The following definitions are used to further investigate the stability properties.

Definition 4 [16]. The method is considered to be absolutely stable in a region, R for a given $\bar{h}$ if all roots of the stability polynomial $\pi(r;\bar{h}) = 0$ satisfy the condition $|r_s| \leq 1$, where $s = 1,2,\ldots,k$.

Definition 5 [16]. The method is described as A-stable if its region of absolute stability includes the entire left-half plane where $R(\bar{h}) < 0$.

In order to plot the stability region, $r = e^{i\theta}$ is substituted into the stability polynomial in Equation (33) where $0 \leq \theta \leq 2\pi$. Then, the equation obtained is solved for $\bar{h}$. The stability region of the 2DIFBBDF(2) was plotted by using Maple software and is displayed in Figure 1.

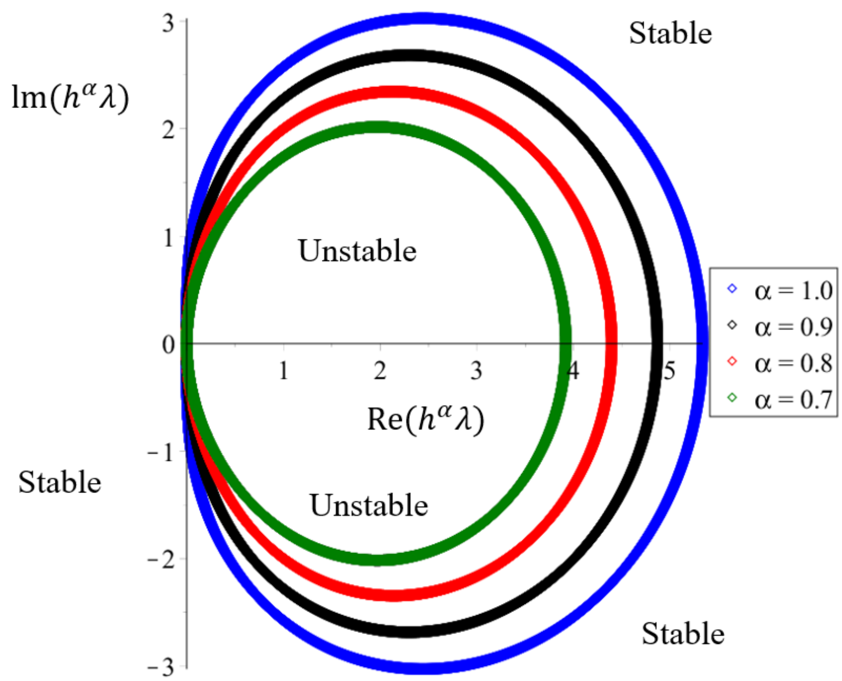

Figure 1. Stability region of 2DIFBBDF(2) method

From Figure 1, the stability regions of 2DIFBBDF(2) method when $\alpha = 0.7, \alpha = 0.8, \alpha = 0.9$ and $\alpha = 1.0$ are represented by the green, red, black and blue curves respectively. By referring to Definition 5, the proposed method is proved to be A-stable when $\alpha = 0.7, \alpha = 0.8, \alpha = 0.9$ and $\alpha = 1.0$. The region of absolute stability region seems to grow bigger as α decreases. This takes place because smaller fractional orders naturally make the system more damping. When the damping is made stronger, the approximate solutions will become more stable, and the method can deal with additional problems without becoming unstable. In real life, this behaviour is helpful because when bigger step sizes are used, the approximate solutions will still stable. Consequently, this will improve the efficiency of the computation for the proposed method. This is very important for stiff problems or systems with strong memory effects which are mostly in fractional orders. This condition is only valid for $\alpha \in (0,1]$.

Implementation of the Method

This section presents the implementation of 2DIFBBDF(2) method using Newton's iteration, expressed in the following form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F_j\left(y_{n+j}^{(i)}\right)\right)\left(F_j'\left(y_{n+j}^{(i)}\right)\right)^{-1}, j = 1,2,\ldots \tag{38}$$

Equation (30) is applied into the 2DIFBBDF(2) method where $y_{n+j}^{(i+1)}$ is the $(i+1)$th iteration values and $e_{n+j}^{(i+1)}$ is the difference between the iteration values of $(i+1)$th and (i)th, denoted:

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, j = 1,2,\ldots \tag{39}$$

Equation (39) is substituted into Equation (38) to obtain the following formula,

$$\begin{gathered} F_1 = y_{n+1} + \frac{2\Gamma(3-\alpha)}{-4+\alpha}h^\alpha f_{n+1} - \zeta_1, \\ F_2 = y_{n+2} - \frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18}y_{n+1} + \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^\alpha f_{n+2} - \zeta_{2.} \end{gathered} \tag{40}$$

where ζ_1 and ζ_2 are the backvalues. Thus,

$$\begin{gathered} e_{n+1}^{(i+1)} = -\frac{\left(y_{n+1} + \frac{2\Gamma(3-\alpha)}{-4+\alpha}h^\alpha f_{n+1} - \zeta_1\right)}{\left(1 + \frac{2\Gamma(3-\alpha)}{-4+\alpha}h^\alpha \frac{\partial f_{n+1}}{\partial y_{n+1}}\right)}, \\ e_{n+2}^{(i+1)} = -\frac{\left(y_{n+2} - \frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18}y_{n+1} + \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^\alpha f_{n+2} - \zeta_2\right)}{\left(1 + \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^\alpha \frac{\partial f_{n+2}}{\partial y_{n+2}}\right)}. \end{gathered} \tag{41}$$

Then, Equation (41) is arranged in the matrix form,

$$\begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \end{bmatrix} = -\begin{bmatrix} y_{n+1}^{(i)} + \frac{2\Gamma(3-\alpha)}{-4+\alpha}h^\alpha f_{n+1} - \zeta_1 & 0 \\ -\frac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18}y_{n+1}^{(i)} & y_{n+2}^{(i)} + \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^\alpha f_{n+2} - \zeta_2 \end{bmatrix} \begin{bmatrix} 1 + \frac{2\Gamma(3-\alpha)}{-4+\alpha}h^\alpha \frac{\partial f_{n+1}}{\partial y_{n+1}} & 0 \\ 0 & 1 + \frac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^\alpha \frac{\partial f_{n+2}}{\partial y_{n+2}} \end{bmatrix}^{-1}. \tag{42}$$

By using a similar 'CurveFitting' package and command Polynomial Interpolation in Maple, the predictor formula for the first point, y_{n+1} and second point, y_{n+2} are computed by using Lagrange polynomial and two back values points which are t_{n-1} and t_n as the interpolating points. The resulting predictor formula for first point, $y_{n+1}^{(p)}$ and second point, $y_{n+2}^{(p)}$ are:

$$\begin{gathered} y_{n+1}^{(p)} = -y_{n-1} + 2y_n \\ y_{n+2}^{(p)} = -2y_{n-1} + 3y_n \end{gathered} \tag{43}$$

In order to find the approximate solutions of y_{n+1} and y_{n+2}, predictor-corrector approach, the Predictor, Evaluator, Corrector, and Execution (PECE) mode is employed. PECE initiates the process by predicting a solution, which is subsequently evaluated and corrected. This approach minimises errors by eliminating the necessity of solving complicated equations at every step. This results in a better and more efficient implementation of the method, while maintaining good stability properties. PECE is more stable and reliable, particularly for difficult problems. The following are the calculations for the approximate solution of y_{n+1} and y_{n+2} in PECE mode:

Step 1 P (Predict) : Compute the predictor formula, $y_{n+j}^{(p)}$.

Step 2 E (Evaluate) : $D^{\alpha}y_{n+j}=f\left(x_{n+j},y_{n+j}^{(p)}\right)$.

Step 3 C (Correct) : Compute the corrector formula, $y_{n+j}^{(c)}$.

Step 4 E (Evaluate) : $D^{\alpha}y_{n+j}=f\left(x_{n+j},y_{n+j}^{(c)}\right)$.

Two-stage Newton's iteration is involved in the process of finding the approximate solution of y_{n+1} and y_{n+2}. The initial solution guess is derived from the predictor phase of the PECE strategy in the Newton iteration process. The iteration is initiated with this predicted value, which contributes to the enhancement of stability and an increase of convergence. By employing the predictor as the first guess, the number of Newton iterations required at each time step is decreased. The computational processes are:

Step 1 Compute $e_{n+1,n+2}^{(i+1)}=A^{-1}B$, where

$$A=\begin{bmatrix}1+\dfrac{2\Gamma(3-\alpha)}{-4+\alpha}h^{\alpha}\dfrac{\partial f_{n+1}}{\partial y_{n+1}} & 0\\ 0 & 1+\dfrac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^{\alpha}\dfrac{\partial f_{n+2}}{\partial y_{n+2}}\end{bmatrix},$$

$$B=\begin{bmatrix}-y_{n+1}^{(i)}-\dfrac{2\Gamma(3-\alpha)}{-4+\alpha}h^{\alpha}f_{n+1}+\zeta_1 & 0\\ \dfrac{6\alpha(\alpha-7)}{\alpha^2-5\alpha-18}y_{n+1}^{(i)} & -y_{n+2}^{(i)}-\dfrac{6(2^{-1+\alpha})\Gamma(4-\alpha)}{\alpha^2-5\alpha-18}h^{\alpha}f_{n+2}+\zeta_2\end{bmatrix}.$$

Step 2 Compute the corrected value of $y_{n+1,n+2}^{(i+1)}$ by using the value of $e_{n+1,n+2}^{(i+1)}$ obtained from Step 1.

Step 3 Solve $e_{n+1,n+2}^{(i+1)}=A^{-1}B$ for second stage iteration

Step 4 Obtain the updated values of $y_{n+1,n+2}^{(i+1)}$ from the second stage of iteration.

Numerical Results of the Method

In order to validate and illustrate the accuracy of the proposed method, several tested problems in the form linear, non-linear and systems of FDEs are solved using 2DIFBBDF(2) method. Numerical results were computed using C programming and the results obtained will be compared with the existing methods. The absolute error (AbsE) was computed by using the following formula:

$$error=\left|y_j(t_n)-y_j(t)\right| \tag{44}$$

where $y_j(t_n)$ is the approximate value and $y_j(t)$ is the exact value. The following is the notations that will be considered in this study:

h	:	Step size
AbsE	:	Absolute error
Method	:	Comparison method
2DIFBBDF(2)	:	Two-point Diagonally Implicit Fractional Block Backward Differentiation Formula of Order Two with Constant Step Size
FDE12	:	Fractional Differential Equation Code (FDE12.m) – Available in MATLAB
FAM22	:	Fractional Adams Method of Explicit Order 2, Implicit Order 2 [14]
FEAM3	:	Fractional Explicit Adams Method of Order 3 [13]
FRPS	:	Fractional Residual Power Series method [24]
RKHS	:	Reproducing Kernel Hilbert Space method [24]
2FBBDF(4)	:	Fourth-order 2-point Fractional Block Backward Differentiation Formula [25]

The following are five numerical examples of different types that have been chosen. Problem 1 is a basic linear FDE. Problem 2 is a nonlinear FDE with variable coefficients, while Problem 3 is an application problem, which is fractional Riccati differential equations. Problem 4 is simple linear FDE and Problem 5 is non-linear system of FDEs.

Problem 1. Consider the following basic linear FDE [20]:

$$D^{\alpha}y(t)=-y(t),\quad y(0)=1,\quad t\in[0,2],$$

with the exact solution given as:

$$y(t) = E_\alpha(-t^\alpha)$$

where $E_\alpha(z)$ is defined as Mittag-Leffler function:

$$E_\alpha(z) = \sum_0^\infty \frac{z^k}{\Gamma(k\alpha + 1)}.$$

Problem 2. Consider the following FIVP of FDE [12]:

$$D^\alpha y(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t^\alpha - \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + (t^{2\alpha} - t^2)^4 - y^4(t), \quad y(0) = 0, \qquad t \in [0,1],$$

with the exact solution given as:

$$y(t) = t^{2\alpha} - t^2.$$

Problem 3. Consider the following application problem of Riccati FDEs [21,22]:

$$D^\alpha y(t) = -y^2 + 1, \quad y(0) = 0, \quad t \in [0,1],$$

with the exact solution when $\alpha = 1$ given as:

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

Problem 4. Consider the following linear FDEs [23]:

$$D^\alpha y(t) = -10y + 10, \quad y(0) = 2, \quad t \in [0,1],$$

with the exact solution when $\alpha = 1$ given as:

$$y(t) = 1 + e^{-10t}.$$

Problem 5. Consider the following non-linear system of FDEs [24]:

$$D^{\alpha_1} y_1(t) = -1002 y_1 + 1000 y_2^2, \quad y_1(0) = 1,$$
$$D^{\alpha_2} y_2(t) = y_1 - y_2 - y_2^2, \quad y_2(0) = 1, \quad t \in [0,2],$$

with the exact solution of the system when $\alpha_1 = \alpha_2 = 1$ given as:

$$y_1 = e^{-2t}, \; y_2 = e^{-t}.$$

Table 2 presents the AbsE of 2DIFBBDF(2) method across different fractional order values (α) and step sizes (h) at the end point, $t = 2.0$. By referring to the table, the numerical results show that the AbsE remains within a tolerance range of 10^{-2} to 10^{-3}. Nonetheless, the inaccuracy persists within permissible boundaries as the AbsE is decreasing when the h is smaller. Not only that, the accuracy of 2DIFBBDF(2) method increases when α approaches to 1.0. Besides, Figure 2 shows the graph of approximate solutions for 2DIFBBDF(2) method and FDE12 when $\alpha = 0.8$ for which the range of t is from 0 to 2. Based on the graph, it can be observed that results from 2DIFBBDF(2) method are closer to the exact solution, which is 0, as compared to FDE12.

Table 2. Numerical results for Problem 1.

h	AbsE		
	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
1.00E-02	2.0087E-02	1.3677E-02	6.4684E-03
1.00E-04	1.7957E-02	1.2537E-02	6.3765E-03
1.00E-06	1.7889E-02	1.2521E-02	6.3764E-03

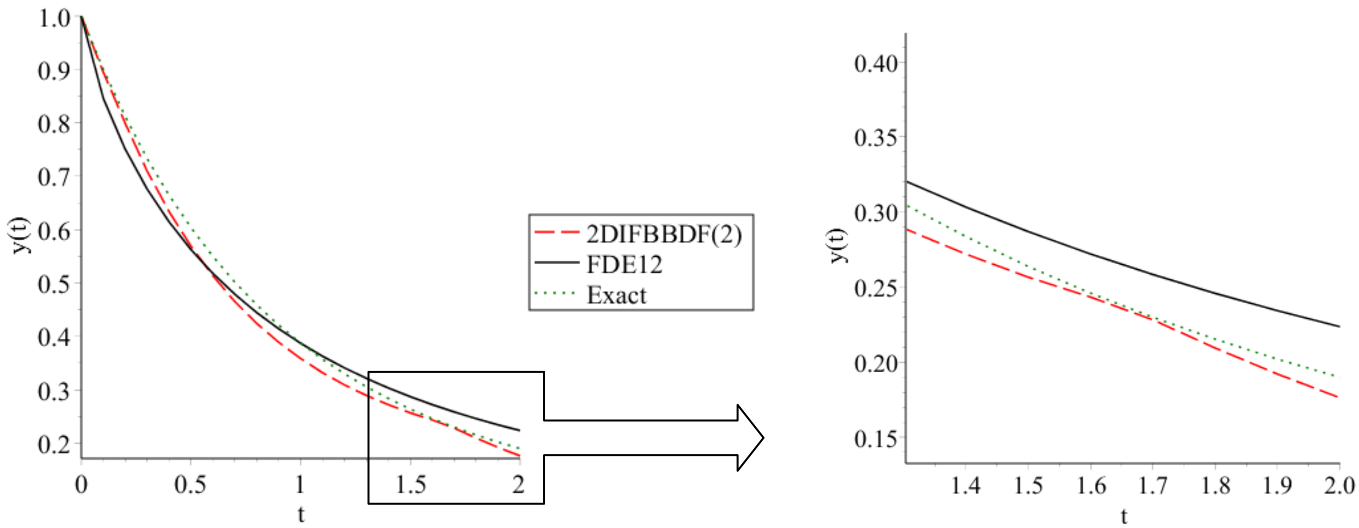

Figure 2. Graph of approximate solution for $y(t)$ on Problem 1 when $\alpha = 0.8$

The fractional-order values $\alpha = 0.7$, $\alpha = 0.8$ and $\alpha = 0.9$ were chosen to approximate the solution for Problem 2. Table 3 shows the numerical results for Problem 2 by using 2DIFBBDF(2) method at the end point, $t = 1.0$. According to the table, it underlines a distinct trend: the AbsE consistently decreases as h decreases, demonstrating that the approximation solution is getting more accurate. The behaviour of the solutions for α = 0.8 is plotted and examined in Figure 3 to further analyse the performance of the proposed method. This figure provides a graphical representation of the approximate solutions of 2DIFBBDF(2) method and FDE12 and compares them with the exact solution. The graph reveals that as the approximation solutions of 2DIFBBDF(2) method are getting closer to the exact solution as the value of t increases when compared with FDE12.

Table 3. Numerical results for Problem 2.

h	AbsE		
	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
1.00E-02	1.0588E-02	4.7437E-03	8.7759E-04
1.00E-04	6.5487E-04	1.1987E-04	6.2021E-06
1.00E-06	2.3754E-05	1.8327E-06	1.8348E-07

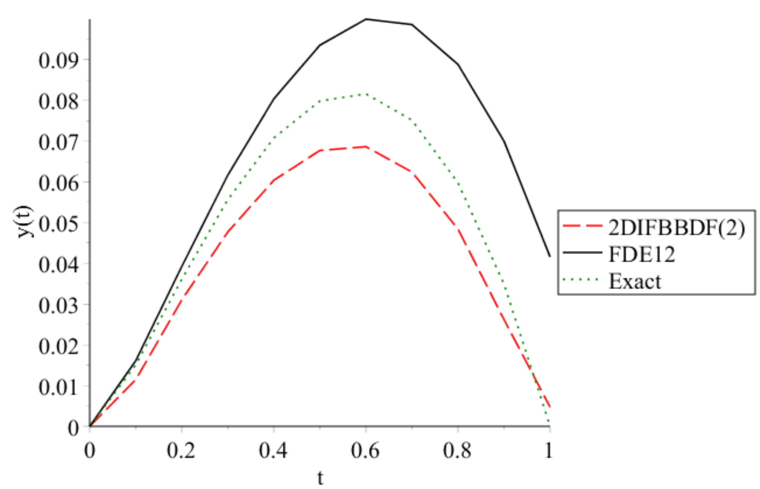

Figure 3. Graph of approximate solution for $y(t)$ on Problem 2 when $\alpha = 0.8$

In addition, this paper investigates the solution of the fractional Riccati differential equation as stated in Problem 3. The Riccati differential equation is renowned for its numerous applications, including the analysis of river flow dynamics, the modelling of linear systems with Markovian leaps and the application of the equation in econometric modelling [21]. This problem was solved by using 2DIFBBDF(2) method and the results are compared with those of FEAM3 and FAM22 in Table 4. From the table, 2DIFBBDF(2) method performs better in terms of AbsE as compared to FEAM3 and FAM22. Besides, Figure 4 demonstrates the graph of approximate solution for Problem 3 at various values of α. Based on the graph, the approximate solutions converge towards the exact solution when the α is getting closer to 1.0.

Table 4. Numerical results for Problem 3 when $\alpha = 1.0$

t	Method	AbsE
0.2	FAM22	9.8045E-07
	FEAM3	1.0653E-07
	2DIFBBDF(2)	9.0279E-08
0.4	FAM22	2.2901E-06
	FEAM3	2.0725E-06
	2DIFBBDF(2)	2.5344E-07
0.6	FAM22	4.5064E-06
	FEAM3	2.8479E-06
	2DIFBBDF(2)	4.8081E-07
0.8	FAM22	5.5022E-06
	FEAM3	3.3549E-06
	2DIFBBDF(2)	7.0771E-07
1.0	FAM22	6.3175E-06
	FEAM3	3.6079E-06
	2DIFBBDF(2)	8.6630E-07

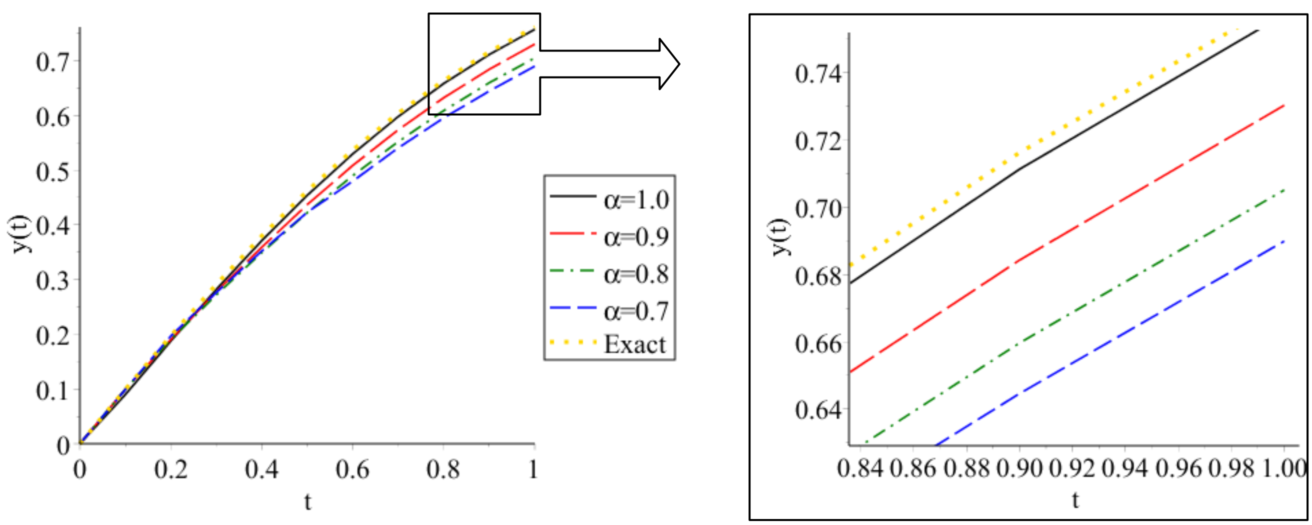

Figure 4. Graph of approximate solution for $y(t)$ on Problem 3 when $\alpha = 0.7, 0.8, 0.9$ and 1.0

Table 5 shows the AbsE for Problem 4 when solved by 2DIFBBDF(2) method and FDE12. According to the table, 2DIFBBDF(2) method generates more accurate results than FDE12. This indicates that the approximate solution approaches to the exact solution. Furthermore, Figure 5 displays the graph of approximate solutions at α = 0.7, 0.8, 0.9 and 1.0, indicating that the approximate solution is getting more accurate when the value of alpha is approaching to 1.0.

Table 5. Numerical results for Problem 4 when $\alpha = 1.0$

t	Method	AbsE
0.2	FDE12	1.3086E-04
	2DIFBBDF(2)	2.8855E-05
0.4	FDE12	1.7091E-05
	2DIFBBDF(2)	4.6787E-06
0.6	FDE12	2.2281E-06
	2DIFBBDF(2)	7.3788E-07
0.8	FDE12	2.9230E-07
	2DIFBBDF(2)	1.1403E-07
1.0	FDE12	4.2446E-08
	2DIFBBDF(2)	1.7349E-08

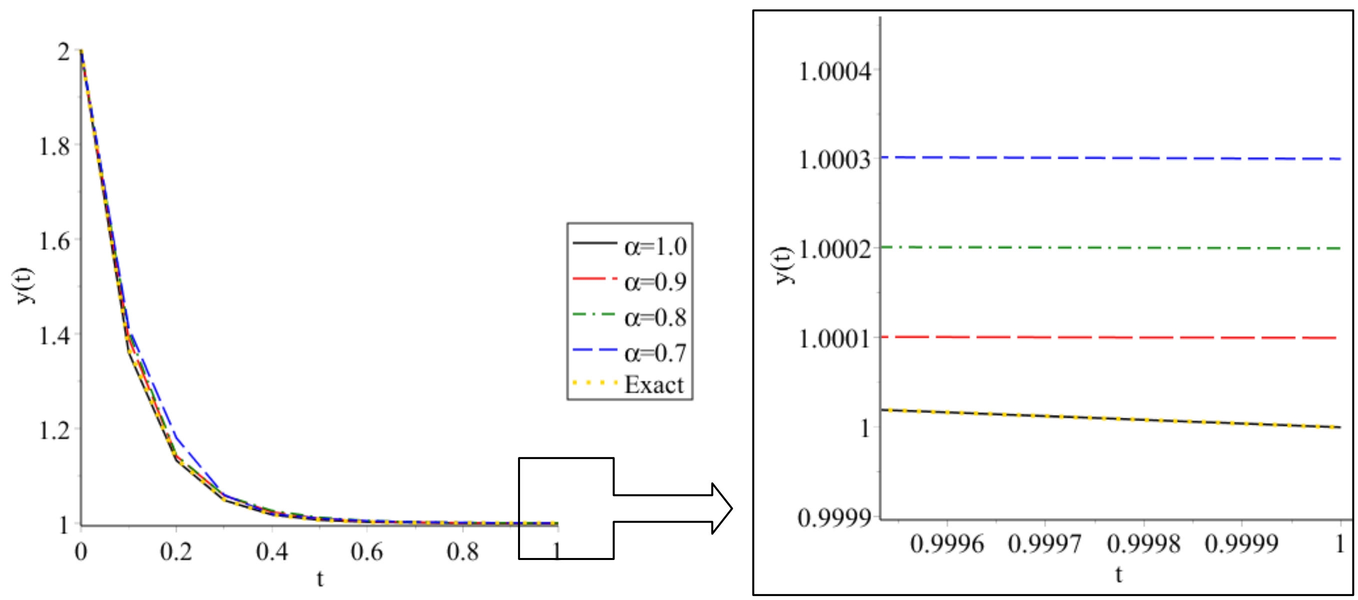

Figure 5. Graph of approximate solution for $y(t)$ on Problem 4 when $\alpha = 0.7, 0.8, 0.9$ and 1.0

Problem 5 is a non-linear system of FDEs. This problem is solved by 2DIFBBDF(2) method and the numerical results are tabulated in Table 6 together with the results of FRPS, RKHS and 2FBBDF(4). The findings demonstrate that the 2DIFBBDF(2) technique surpasses FRPS, RKHS and 2FBBDF(4), attaining smaller AbsE, hence yielding more accurate approximations of the exact solutions. Additionally, Figures 6 illustrates the approximate solutions for several values of alpha (0.7, 0.8, 0.9, and 1.0). The figures indicates that the approximation attains more accuracy as alpha nears 1.0, reflecting enhanced agreement in accordance with the exact solution.

Table 6. Numerical results for Problem 5 when $\alpha = 1.0$ in terms of AbsE

t	Method	$y_1(t)$	$y_2(t)$
0.0	FRPS	0.0000E+00	0.0000E+00
	RKHS	0.0000E+00	0.0000E+00
	2FBBDF(4)	0.0000E+00	0.0000E+00
	2DIFBBDF(2)	0.0000E+00	0.0000E+00
0.4	FRPS	5.5511E-17	0.0000E+00
	RKHS	1.2021E-06	1.2363E-06
	2FBBDF(4)	1.0983E-08	1.4116E-08
	2DIFBBDF(2)	6.9078E-11	1.1929E-10
0.8	FRPS	5.5511E-16	5.5511E-15
	RKHS	1.2840E-06	2.4759E-02
	2FBBDF(4)	3.9762E-10	9.4686E-09
	2DIFBBDF(2)	2.6758E-11	8.3589E-11
1.2	FRPS	1.7022E-12	0.0000E+00
	RKHS	9.1087E-07	1.9237E-01
	2FBBDF(4)	3.7542E-09	6.3514E-09
	2DIFBBDF(2)	5.1272E-11	5.8817E-11
1.6	FRPS	6.9266E-10	5.5511E-16
	RKHS	5.5924E-07	2.6255E-01
	2FBBDF(4)	4.0841E-09	4.0841E-09
	2DIFBBDF(2)	4.9602E-11	4.1581E-11
2.0	FRPS	7.2763E-08	3.7581E-09
	RKHS	3.1863E-07	4.3860E-01
	2FBBDF(4)	3.5532E-09	2.8578E-09
	2DIFBBDF(2)	4.0305E-11	2.9248E-11

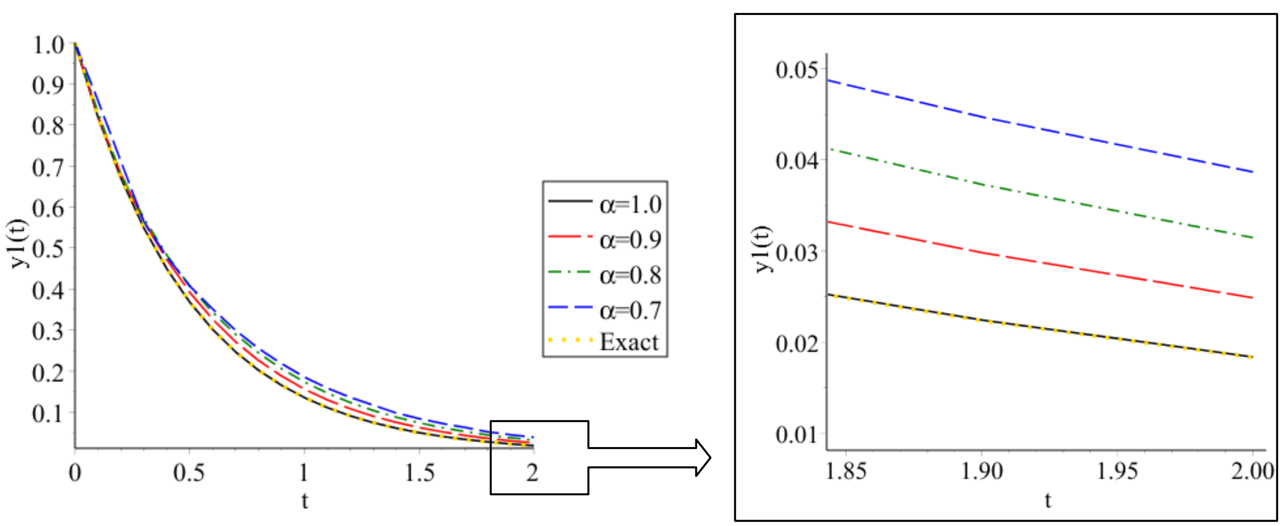

Figure 5. Graph of approximate solution for $y_1(t)$ on Problem 5 when $\alpha = 0.7, 0.8, 0.9$ and 1.0

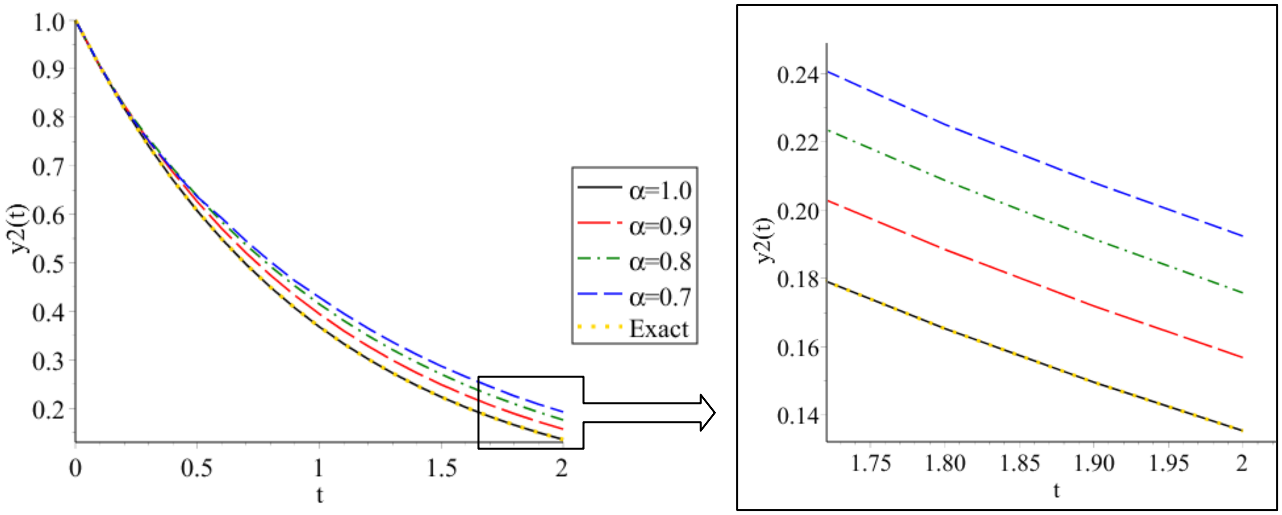

Figure 6. Graph of approximate solution for $y_2(t)$ on Problem 5 when $\alpha = 0.7, 0.8, 0.9$ and 1.0

Conclusions

In summary, this study presents a novel numerical method which is 2DIFBBDF(2) method. The stability analysis revealed that the proposed method is A-stable for fractional order, $\alpha = 0.7,\ 0.8,\ 0.9$ and 1.0. Besides, the numerical examples in the form of linear, non-linear and systems of FDEs solved by the proposed method yields better results in terms of AbsE than the existing methods in literatures. This is due to the zero stability of the method and the A-stability of the method which has been demonstrated as shown in the graph of stability region of the method. The outcomes show that better accuracy will be obtained when the value of alpha increases and nearer to 1.0. As the fractional order approaches one, the fractional derivative aligns with the ordinary first-order derivative. Thus, 2DIFBBDF(2) method is an alternative numerical method to solve different types of FDEs such as linear, non-linear and systems of FDEs. In addition, 2DIFBBDF(2) method is also an appropriate method to solve application problems of FDEs that involves memory effects, such as fractional Riccati differential equations.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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