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Interpolation Technique in Solving Second Rank Polarization Tensor with Error Analysis

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Abstract The Polarization Tensor (PT) serves as a fundamental property used to characterize the shape, size, and orientation of geometric shapes. Widely employed in engineering, particularly in electrical and magnetic domains for various objects with diverse metallic properties, PT has emerged as a valuable tool for researchers. Hence, this paper aims to provide a derivation of error analysis concerning high gradient problem of the first order PT. The derivation of error analysis is based on the numerical integration scheme which is Gaussian Quadrature. The numerical approximation of first order PT includes some errors due to computational approximations. Therefore, it is crucial to develop a method for estimating these errors to understand real-time inaccuracies, rather than just looking at errors after the computation is complete. The perturbation theory is used as a fundamental key for the derivation and representation of error in the PT problem. The derivation of the error analysis implemented the concept of Taylor series expansion of a function. Notably, our findings confirm that the computed PT error aligns with perturbation theory expectations. Our analysis prioritizes assessing the relative error in both data and computed solutions over norm-based evaluations, thus obviating dependency on problem-specific condition numbers. Future research directions include introducing variability in conductivity scenarios, which will further describe the robustness of algorithms developed in this paper.

Keywords: Perturbation theory, error analysis, polarization tensor, linear element integration.

Introduction

The term PT originated from work by Pólya (1947), where it was initially conceptualized as the virtual mass associated with solid motion within a fluid. Subsequently, Pólya and Szegö (1953) expanded on this concept, introducing the Generalized Polarization Tensor (GPT), which encompasses crucial details such as domain shape and material parameters (Ammari, Kang, & Kim, 2005). PT can be found in the application for example in characterizing the presence of conducting objects in electric and electromagnetic fields, described as perturbations (Ammari & Kang, 2007). This perturbation is expressed through PT as an asymptotic formula, with the primary term known as the first-order PT. Various methods, including the finite element method (FEM), numerical integration, semi-analytic methods, and the boundary element method (BEM), can be used to compute the first-order PT for different domains. Despite the diversity of methods employed, few studies have emphasized the significance of error analysis in their research. Error analysis becomes crucial when applying specific numerical methods to solve particular problems. Through error analysis, the computed errors resulting from numerical computations can be estimated and validated against established perturbation theory. In this paper, we aim to emphasize on the error analysis which is based on the perturbation theory where we compute the error based on the integration method using linear element integration of three points.

Many researchers have explored the methodologies of first order PT in numerical aspects. For instance, Capdeboscq *et al*. (2012) employed an approximate semi-algebraic method in two-dimensional

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Generalized Polarization Tensors (GPT). From their research, when computing objects with high conductivity, the accuracy of the numerical solution decreases as the number of points decreased. For validation purposes, the researchers compute GPTs for objects where the analytical solutions are known which are sphere and ellipsoid. Additionally, the researcher provides a MATLAB package which is user-friendly with a graphical interface (GUI), ideal for computing PT of two dimensional objects. However, this method's applicability is limited to two-dimensional object dimensions. In later years, PT computation has been expanded into three dimensions by the researchers such as Khairuddin & Lionheart (2013a) and Lu *et al*. (2015). The boundary integral equation of PT is evaluated using the Boundary Element Method (BEM) by both researchers. A novel technique which employed the recent developed software of BEM++ has been introduced by Khairuddin & Lionheart (2013a) to compute PT for geometries with known analytical solutions like discs and ellipses. BEM++ offered an advantage in handling problem domains with complex geometries, reducing the dimension of the problem to one. Magnetic Polarization Tensor (MPT) is one branch of PT where it can characterize the conductive permeable object. Lu *et al*. (2015) focused on computing the MPT for metallic objects in the shape of sphere and cylinder, deviating from general PT computation. However, a drawback of BEM is the production of a sparse matrix with mostly zero elements. In recent year, Wilson and Ledger (2021) compute MPT of a spectral signature by using proper orthogonal decomposition where they proposed an efficient reduced order model (ROM) in their computation.

In a parallel effort, a nonlinear system of equation has been developed by Khairuddin and Lionheart (2013b) to compute the first order PT using element integration of the Finite Element Method (FEM). The equation of the first-order PT is expressed as a nonlinear equation and the trapezoidal rule is used to compute the equation numerically. In terms of capability in handling complex geometries, FEM was chosen since FEM can handle the objects without altering the formulation or code. Furthermore, Khairuddin and Lionheart (2015) compared two methods which are a simple quadrature rule of numerical integration and BEM. The researchers utilized Matlab for computation with the quadrature method and combined functions from Python and C++ in BEM++ where they focused on computing the PT of ellipsoids. After the computation of the numerical solutions made, then the results are compared with the analytical solutions, where from the comparison, BEM shows better convergence for the specific conductivity of ellipsoids. From the utilization of the linear element integration, a symmetrical matrix for the singular integral operator is created, which is by subsequently solving it by multiplying with the area of each triangle (meshes) of the sphere geometry.

Additionally, Jin *et al*. (2021) utilized BEM to estimate magnetic polarization tensors for metallic samples, focusing on cylindrical metals with high conductivity. In a later year, Khairuddin *et al*. (2018) classified different conductivities of ellipsoid samples by using first order PT. The hp-FEM method was utilized to compute the polarization tensor of the object domain. In terms of the values of matrix of PT, Yunos *et al*. (2017) the resulting matrix of PT is positive definite, when the materials have conductivities greater than one, whereas for objects with conductivities ranging from zero to one, the resulting matrix of PT is negative definite. Amad *et al*. (2021) provided benchmark computations involving spheres and ellipsoids, utilizing the adaptive boundary element method to compute PT where they employed the boundary integral approach with the BEM++ library to solve the PT transmission problem. Recently, Bahuriddin *et al*. (2021) calculated the semi-principal axes of a spheroid from a given first-order PT, considering both oblate and prolate spheroids by introducing a new method that utilized the rotation of PT to estimate the semi-axes of the geometry, a continuation from the approach by Khairuddin *et al*. (2017). From the above literature, there is still lacking of the study for error analysis. Error analysis is crucial to ensuring the reliability as well as accuracy of the method used. Hence, this research aims to provide a comprehensive derivation of error analysis related to the first-order PT computed through linear element numerical integration of three points Gaussian quadrature. By addressing the computational errors and emphasizing the importance of real-time error estimation, our study is crucial in checking the validity of the numerical method employed which align with existing perturbation theory. The focus on relative error assessment over traditional norm-based evaluations marks a significant advancement in error analysis methodologies, ultimately enabling more accurate and reliable characterizations of geometric properties in engineering applications. Through this work, we contribute to a deeper understanding of the error analysis involved in PT computations.

The Derivation of Error Analysis of First Order PT

Mathematical Formulation of First Order PT

The formulation of mathematical representation of PT comes from a transmission problem where, by considering a Lipschitz domain, B in R^3 while the origin, O is in domain B. At this rate, domain B has a conductivity for which it is denoted as k and k must satisfy condition where $0 \le k \ne 1 \le +\infty$. The reason why the conductivity cannot be equal to 1 is, when the $k=1$, it has the same conductivity as the domain background B. The asymptotic expansion of generalized polarization tensor (GPT) which is widely used in the application of Electrical Imaging Tomography (EIT) (Holder, 2005; Adler *et al.*, 2015) and electrosensing fish (Khairuddin *et al.*, 2019) can be represented as

$$\begin{aligned} &\nabla(1+(k-1)\chi(B)\text{grad}(u))=0 \text{ in } R^3, \\ &u(x)-H(x)=O(1/|x|^2) \text{ as } |x|\to+\infty. \end{aligned} \quad (1)$$

χ is the characteristic equation of domain B with $u(x)$ as the solution of the equation and $H(x)$ is the harmonic function in equation. By referring to equation (1), equation of PT can be expressed as far field expansion of u

$$(u-H)(x)=\sum_{|i|,|j|=1}^{+\infty}\frac{(-1)^{|i|}}{i!\,j!}\partial_x^i\Gamma(x)M_{ij}(k,B)\partial^j H(0) \text{ as } |x|\to+\infty, \quad (2)$$

where i and j are the multi-indices. The fundamental solution of the Laplacian is denoted as $\Gamma(x)$ and the generalized PT denoted as M_{ij}. Ammari and Kang (2004) then represent PT as an integral equation over the boundary B, which denoted as ∂B which is given by

$$M_{ij}=\int_{\partial B}Y^j\phi_i(Y)d\sigma(Y). \quad (3)$$

From equation (3), Y_j is the element in ∂B while ϕ_i is the solution to the following linear system of equation which is expressed as

$$\phi_i(Y)=(\lambda I-K_B^*)^{-1}(V_X\cdot\nabla X^i)(Y). \quad (4)$$

In equation (4), λ is defined in terms of conductivity of an object B which is $\lambda=(k+1)/(2k-2)$, while K_B^* is an integral in the form of Cauchy Principal Integral (CPV) as in equation (5).

$$K_B^*\phi(X)=\frac{1}{4\pi}P.V.\int_{\partial B}\frac{\langle X-Y\rangle\cdot\langle V_x\rangle}{|X-Y|^3}\phi(Y)d\sigma(Y). \quad (5)$$

As written in equation (5), the notation for the distance between the element X and element Y is denoted as $X-Y$, while V_x denote the normal vector of element X. Therefore, in order to obtain the first order PT which is represented by M_{ij}, we need to mainly focus on equation (3), (4) and (5).

Error Analysis Using Taylor Series Expansion for the First Order Polarization Tensor

This section begins with the derivation of the stability analysis of a two-dimensional problem related to the first order Polarization Tensor (PT). The analysis aims to assess how perturbations influence the accuracy and reliability of the PT calculations. Specifically, we examine a perturbed Cauchy Principle Value Integral (CPV) integral, which plays a crucial role in the evaluation of the PT. The perturbation of CPV integral is represented as ΔK_B^*, allowing us to quantify its impact on the integral. With this procedure employed, it brings better reliability to the results and designation to the stability of the numerical techniques employed.

By employing Taylor series expansion, we can derive an expression that captures the behaviour of the CPV integral under perturbation. This approach facilitates a deeper understanding of how small changes in the parameters affect the overall stability and results of the integral. To start, the CPV integral is expressed as integral with a perturbation as follows

$$\begin{aligned} K_B^*\phi(x)&=\phi\left(\int_\sigma\varphi(y(\xi,\eta))Jac(\xi,\eta)d\xi d\eta+\int_\sigma R_2(\xi,\eta,\theta)d\xi d\eta\right), \\ &=K_B^*+\Delta K_B^*. \end{aligned} \quad (6)$$

where the perturbations is represented as integral containing remainder, R_2 as

$$\Delta K_B^*=\int_\sigma R_2(\xi,\eta,\theta)d\xi d\eta,$$

for which the remainder is expressed in term of Taylor series expansion containing function f in terms of ξ and η.

$$R_2(\xi,\eta,\theta) = \frac{1}{2!}\Big[f_{\xi\xi}\big(\alpha_\xi + \theta(\xi-\alpha_\xi), \alpha_\eta + \theta(\eta-\alpha_\eta)\big)(\xi-\alpha_\xi)^2 + 2f_{\xi\eta}\big(\alpha_\xi + \theta(\xi-\alpha_\xi), \alpha_\eta + \theta(\eta-\alpha_\eta)\big)(\xi-\alpha_\xi)(\eta-\alpha_\eta) + f_{\eta\eta}\big(\alpha_\xi + \theta(\xi-\alpha_\xi), \alpha_\eta + \theta(\eta-\alpha_\eta)\big)(\eta-\alpha_\eta)^2\Big]. \quad (7)$$

In order to identify the error for the CPV integral, for which the integration is focussing on the element Y (element X are treated as a constant element), therefore, function $f(\xi,\eta)$ can be expressed as the dot product of the distance between element X and element Y with normal vector, V_X.

$$f(Y(\xi,\eta)) = \frac{\langle X - Y(\xi,\eta), V_X\rangle}{|X - Y(\xi,\eta)|^3}. \quad (8)$$

Hence, to obtain the perturbation as mentioned in equation (6), partial derivatives of function in equation (8) must be obtained with respect to ξ and η. Then, the partial derivative of the function is substituted back into integration in equation (6). First, we define the equation in (8) in the form of ln function expressed as

$$\ln f(Y(\xi,\eta)) = \ln\langle X - Y(\xi,\eta), V_X\rangle - \frac{3}{2}\ln|X - Y(\xi,\eta)|^2. \quad (9)$$

First derivatives of f with regards to ξ is obtained by differentiating all components of equation (9), yield to

$$\frac{1}{f(\xi,\eta)}f_\xi(\xi,\eta) = \frac{1}{\langle X - Y(\xi,\eta), V_X\rangle}\langle X - Y_\xi(\xi,\eta), V_X\rangle - \frac{3}{2}\frac{1}{|X - Y(\xi,\eta)|^2}\big(2|X - Y(\xi,\eta)|\big(-Y_\xi(\xi,\eta)\big)\big),$$

$$= \frac{\langle X - Y_\xi(\xi,\eta), V_X\rangle}{\langle X - Y(\xi,\eta), V_X\rangle} - 3\frac{|X - Y(\xi,\eta)|\big(-Y_\xi(\xi,\eta)\big)}{|X - Y(\xi,\eta)|^2},$$

$$f_\xi(\xi,\eta) = \frac{f(\xi,\eta)\langle -Y_\xi(\xi,\eta), V_X\rangle}{\langle X - Y(\xi,\eta), V_X\rangle} - \frac{3f(\xi,\eta)|X - Y(\xi,\eta)|\big(-Y_\xi(\xi,\eta)\big)}{|X - Y(\xi,\eta)|^2}. \quad (10)$$

For second derivatives, $f_{\xi\xi}$ is obtained by differentiating the function in (10) once with respect to ξ. Therefore, firstly, equation (10) is simplified as

$$\begin{aligned} f_\xi(\xi,\eta) &= f(\xi,\eta)\cdot\frac{\langle -Y_\xi(\xi,\eta), V_X\rangle}{\langle X - Y(\xi,\eta), V_X\rangle} - 3f(\xi,\eta)\cdot\frac{|X - Y(\xi,\eta)|\big(-Y_\xi(\xi,\eta)\big)}{|X - Y(\xi,\eta)|^2}, \\ &= f(\xi,\eta)\cdot\frac{\langle -Y_\xi(\xi,\eta), V_X\rangle}{\langle X - Y(\xi,\eta), V_X\rangle} + 3f(\xi,\eta)\cdot\frac{|X - Y(\xi,\eta)|\big(Y_\xi(\xi,\eta)\big)}{|X - Y(\xi,\eta)|^2} \end{aligned} \quad (11)$$

Hence, by finding the partial derivatives of equation (11), we will get

$$\begin{aligned}
\frac{\partial}{\partial\xi}f_{\xi}(\xi,\eta) &= \frac{\partial}{\partial\xi}\left[f(\xi,\eta)\cdot\frac{\langle -Y_{\xi}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}+3f(\xi,\eta)\cdot\frac{|X-Y(\xi,\eta)|\left(Y_{\xi}(\xi,\eta)\right)}{|X-Y(\xi,\eta)|^2}\right],\\
&= \frac{\partial}{\partial\xi}\left[f(\xi,\eta)\cdot\frac{\langle -Y_{\xi}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}\right]+\frac{\partial}{\partial\xi}\left[3f(\xi,\eta)\cdot\frac{\left(Y_{\xi}(\xi,\eta)\right)}{|X-Y(\xi,\eta)|}\right],\\
&= f_{\xi}(\xi,\eta)\cdot\frac{\langle -Y_{\xi}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}+f(\xi,\eta)\frac{\partial}{\partial\xi}\left[\frac{\langle -Y_{\xi}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}\right]\\
&\quad +3f_{\xi}(\xi,\eta)\frac{\left(Y_{\xi}(\xi,\eta)\right)}{|X-Y(\xi,\eta)|}+3f(\xi,\eta)\frac{\partial}{\partial\xi}\left[\frac{\left(Y_{\xi}(\xi,\eta)\right)}{|X-Y(\xi,\eta)|}\right],\\
f_{\xi\xi}(\xi,\eta) &= f_{\xi}(\xi,\eta)\cdot\frac{\langle -Y_{\xi}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}\\
&\quad +f(\xi,\eta)\left[\frac{\langle -Y_{\xi\xi}(\xi,\eta),V_X\rangle\langle X-Y(\xi,\eta),V_X\rangle-\langle -Y_{\xi}(\xi,\eta),V_X\rangle\langle -Y_{\xi\xi}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle^2}\right],\\
&\quad +3f_{\xi}(\xi,\eta)\frac{\left(Y_{\xi}(\xi,\eta)\right)}{|X-Y(\xi,\eta)|}+3f(\xi,\eta)\left[\frac{Y_{\xi}(\xi,\eta)}{|X-Y(\xi,\eta)|}+\frac{Y(\xi,\eta)\langle X-Y(\xi,\eta),V_X\rangle}{|X-Y(\xi,\eta)|^3}\right].
\end{aligned} \tag{12}$$

Next, for $f_{\eta\eta}$, where we need to differentiate the function, f two times with respect to η. We begin the differentiation process by first differentiating the function f once with respect to η. Same process as to find f_{ξ} is implemented to obtain $f_{\eta\eta}$ for which firstly, we need to find f_{η}. After several steps, f_{η} is then expressed as

$$f_{\eta}(\xi,\eta)=\frac{f(\xi,\eta)\langle -Y_{\eta}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}-\frac{3f(\xi,\eta)|X-Y(\xi,\eta)|\left(-Y_{\eta}(\xi,\eta)\right)}{|X-Y(\xi,\eta)|^2}. \tag{13}$$

The steps that follow are taken by differentiating equation (13) second times with respect to η, yielding

$$\begin{aligned}
f_{\eta\eta}(\xi,\eta) &= f_{\eta}(\xi,\eta)\cdot\frac{\langle -Y_{\eta}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}\\
&\quad +f(\xi,\eta)\left[\frac{\langle -Y_{\eta\eta}(\xi,\eta),V_X\rangle\langle X-Y(\xi,\eta),V_X\rangle-\langle -Y_{\eta}(\xi,\eta),V_X\rangle\langle -Y_{\eta\eta}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle^2}\right],\\
&\quad +3f_{\eta}(\xi,\eta)\frac{\left(Y_{\eta}(\xi,\eta)\right)}{|X-Y(\xi,\eta)|}+3f(\xi,\eta)\left[\frac{Y_{\eta}(\xi,\eta)}{|X-Y(\xi,\eta)|}+\frac{Y(\xi,\eta)\langle X-Y(\xi,\eta),V_X\rangle}{|X-Y(\xi,\eta)|^3}\right].
\end{aligned} \tag{14}$$

Then, to get second derivatives for $f_{\xi\eta}$ we differentiate equation (14) with regard to η, which gives us

$$\begin{aligned}
f_{\xi\eta}(\xi,\eta) &= \frac{f_{\eta}(\xi,\eta)\langle -Y(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}+f(\xi,\eta)\left[\frac{\langle -Y_{\eta}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle}+\frac{\langle Y(\xi,\eta),V_X\rangle\langle Y_{\eta}(\xi,\eta),V_X\rangle}{\langle X-Y(\xi,\eta),V_X\rangle^2}\right]\\
&\quad +3f_{\eta}(\xi,\eta)\frac{Y(\xi,\eta)}{|X-Y(\xi,\eta)|}+3f(\xi,\eta)\frac{Y_{\eta}(\xi,\eta)}{|X-Y(\xi,\eta)|}+\frac{Y(\xi,\eta)\langle X-Y(\xi,\eta),V_X\rangle}{|X-Y(\xi,\eta)|^3}.
\end{aligned} \tag{15}$$

Finally, we substitute the equation (12), (14) and (15) into equation (7), which then yields to the addition of the perturbation to the linear system of the equation ϕ. Next, for the perturbation in the linear system, as in equation (4), since K_B^* contained perturbations denoted as ΔK_B^*, then automatically, the linear system of equation of ϕ will also contained perturbations. This perturbation in linear system can be expressed as

$$\phi_i + \Delta\phi_i = \left(\lambda I - \left(K_B^* + \Delta K_B^*\right)\right)^{-1}\left(V_X \cdot \nabla X^i\right)(Y),$$
$$\left(\lambda I - \left(K_B^* + \Delta K_B^*\right)\right)\left(\phi_i + \Delta\phi\right) = \left(V_X \cdot \nabla X^i\right)(Y). \tag{16}$$

Based on (16), there are two scenarios of perturbations are analyzed for a linear system of equations of PT. The first scenarios, which involve modifying the solution of a linear system of equation ϕ while maintaining the right-hand side unchanged. Second scenarios is where the singular integral operator, K_B^* is perturbed but the right-hand side remains unaffected. By letting $A + \Delta A = \lambda I - \left(K_B^* + \Delta K_B^*\right)$ where $\Delta A = \Delta K_B^*$ and $A = \lambda I - K_B^*$, the solution of the linear system, $x + \Delta x = (\phi_i + \Delta\phi)$ where $\Delta x = \Delta\phi$, while $x = \phi$ and $b = \left(V_X \cdot \nabla X^i\right)(Y)$, then we have, from equation (16), yield to

$$Ax = b,$$
$$(A + \Delta A)(x + \Delta x) = b. \tag{17}$$

The left perturbation theorem is used as in equation (18)

$$\frac{\|\Delta x\|}{\|x\|} \le \frac{\kappa(A)\dfrac{\|\Delta A\|}{\|A\|}}{1 - \kappa(A)\dfrac{\|\Delta A\|}{\|A\|}}. \tag{18}$$

for which $\kappa(A)$ is the condition of the matrix A defined as $\kappa(A) = \|A\|\|A^{-1}\|$. By using left perturbation theorem in equation (18), the perturbation based on linear systems can be expressed as

$$\frac{\|\Delta\phi\|}{\|\phi\|} \le \frac{\kappa(A)\dfrac{\|\Delta K_B^*\|}{\|A\|}}{1 - \kappa(A)\dfrac{\|\Delta K_B^*\|}{\|A\|}}. \tag{19}$$

The evaluation of error for the singular integral operator, K_B^* is outline and presented in Table 1 in the next subsection. Following that, based on equation (19), we have

$$\|\Delta\phi\| \le \frac{\kappa(A)\dfrac{\|\Delta K_B^*\|}{\|A\|}}{1 - \kappa(A)\dfrac{\|\Delta K_B^*\|}{\|A\|}}\|\phi\|. \tag{20}$$

In order to find the perturbation of integral M_{ij}, equation (20) is then being substituting it into equation (3), yield to the addition of the perturbation of the integral as in equation (21)

$$M_{ij} + \Delta M_{ij} = \int_{\partial B} y^j \left(\phi_i + \Delta\phi_i\right) d\sigma, \tag{21}$$

where

$$\Delta M_{ij} = \int_{\partial B} y^j \Delta\phi_i d\sigma. \tag{22}$$

By applying the norm to both sides of equation (22), we obtain equation (23), which will be the basis for our error analysis. The evaluation of the perturbation of integral M_{ij} is presented in Table 2, where, we also implemented linear element integration with three points in order to present the error analysis for the PT integral.

$$\begin{aligned}\|\Delta M_{ij}\| &= \left\|\int_{\partial B} y^j \Delta\phi_i d\sigma\right\|, \\ &\le \int_{\partial B} \|y^j\| \|\Delta\phi_i\| d\sigma, \\ &\le \int_{\partial B} \|y^j\| \left(\frac{\kappa(A)\frac{\|\Delta K_B^*\|}{\|A\|}}{1-\kappa(A)\frac{\|\Delta K_B^*\|}{\|A\|}} \|\phi\| \right) d\sigma, \\ &= \sum_{j=1}^{N} w_j \|y^j\| \left(\frac{\kappa(A)\frac{\|\Delta K_B^*\|}{\|A\|}}{1-\kappa(A)\frac{\|\Delta K_B^*\|}{\|A\|}} \|\phi\| \right) \sqrt{\det(J^T J)}.\end{aligned} \tag{23}$$

In the following section, the verification of the derived error analysis using Taylor series expansion for the method of linear element integration will be represented.

Verification of the Error Analysis

In seeking for the error analysis computation for the first order PT, the calculations of the PT for different conductivities is observed in the following Table 1 and Table 2. The residual error is an important part in numerical simulation when numerical methods is applied to a specific problem.

Based on the Table 1, as the number of surface elements increases, the relative error for ϕ is decreasing for all values of conductivities. This demonstrates that the higher the resolution of the elements, the higher the accuracy of the computed first order PT. This suggests that finer discretization of the elements would lead to more accurate results which is typical in numerical simulations. As depicted in Table 1, the ratio of left perturbations shows the behavior of the perturbations of matrix $(\lambda I - K_B)$. The ratio of the left perturbation error depends on the condition number, which, in this case, is $\kappa(\lambda I - K_B)$. As indicated in the table, as the number of surface elements increases, it leads to a smaller ratio of left perturbation error. Therefore, we can conclude that the numerical methods are stable under small changes. Hence, the numerical results as the linear element integration is implemented is meaningful and have a reliable result.

Table 1. Error analysis for the first order polarization tensor of a sphere using linear element integration based on equation (19)

Conductivity, k	Number of surface elements, N	Relative Error of ϕ $\frac{\|\Delta\phi\|}{\|\phi\|}$	Ratio of Left Perturbation Error $\frac{\kappa(\lambda I - K_B^*)\frac{\|\Delta K_B^*\|}{\|\lambda I - K_B^*\|}}{1-\kappa(\lambda I - K_B^*)\frac{\|\Delta K_B^*\|}{\|\lambda I - K_B^*\|}}$
0.1	44	0.005177	0.0158318
	72	0.003646	0.011116
	118	0.002759	0.007127
	230	0.001691	0.004260
1.5	44	0.001228	0.009025
	72	0.000925	0.006459
	118	0.000736	0.004233
	230	0.000453	0.002567
10	44	0.0060062	0.033549
	72	0.004894	0.024752
	118	0.003627	0.016497
	230	0.002273	0.003627

On the other hand, Table 2 presents error analysis for the first order PT of a sphere by using the same methods, which is linear element integration. The reason error analysis is conducted for whole computation of PT is because, we want to make sure is there any instability of the numerical methods occurs in the numerical computation. In Table 2, the behavior of the relative error for $\|\Delta M_{ij}\|$ is presented. A similar case when the error analysis for the linear system of equations is conducted, as number of surface elements increases, the error decreases, which shows the convergence of the numerical methods. We can conclude that the error with respect to N can be represented as $\|\Delta M^*\| \approx O(N^{-\vartheta})$ for which ϑ is the convergence rate dependent on the numerical methods used. Similar to ϕ, as N increases, the ratio of the left perturbation error decreases. This indicates that the system of equation is less sensitive to the perturbations as finer discretization is implemented. Hence, this suggests that the numerical stability of the method of linear element integration with condition number $\kappa(\lambda I - K_B)$ is maintained.

Table 2. Error analysis for the first order polarization tensor of a sphere using linear element integration based on equation (23)

Conductivity, k	Number of surface elements, N	Relative Error $\|\Delta M_{ij}\|$	Ratio of Left Perturbation Error $\dfrac{\kappa(\lambda I - K_B^*)\frac{\|\Delta K_B^*\|}{\|\lambda I - K_B^*\|}}{1-\kappa(\lambda I - K_B^*)\frac{\|\Delta K_B^*\|}{\|\lambda I - K_B^*\|}}$
0.1	44	5.30×10^{-7}	4.73×10^{-6}
	72	2.87×10^{-7}	3.21×10^{-6}
	118	1.29×10^{-7}	1.24×10^{-6}
	230	1.96×10^{-8}	5.00×10^{-7}
1.5	44	3.15×10^{-7}	7.03×10^{-7}
	72	2.08×10^{-7}	4.30×10^{-7}
	118	1.37×10^{-7}	2.01×10^{-7}
	230	7.65×10^{-8}	8.41×10^{-8}
10	44	2.34×10^{-6}	1.25×10^{-5}
	72	1.65×10^{-6}	8.10×10^{-6}
	118	1.17×10^{-6}	3.92×10^{-6}
	230	7.34×10^{-7}	1.72×10^{-6}

To conclude, based on the numerical solution for error analysis, as $N \to \infty$, the relative error decreases, showing the methods has converged. Meanwhile, in terms of the perturbations ratio, it decreases as N increases, which suggests that the method is stable under small perturbations. Additionally, for conductivity, higher values of conductivity, k increase the error in the solution. However, with sufficient resolution of the elements, the errors become comparable.

Conclusions

In this paper, we present a derivation based on perturbation theory for polarization tensor (PT). This method utilizes the perturbation theorem and involves the integration of the integral related to PT. Our findings indicate that this approach is both highly effective and stable for numerical computations involving PT. The results from our error analysis demonstrate that the method aligns well with theoretical predictions. Given that our study is restricted to cases with fixed conductivity parameters, we encourage future researchers to use our derived algorithms to perform error analyses. They should assess whether the computed error values conform to the perturbation theorem, thereby validating the robustness of our method.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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