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RESEARCH ARTICLE

Enhancing the Ability of the EWMA Control Chart to Detect Changes in the Mean of a Time-Series Model

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Abstract We improved the ability of the exponentially weighted moving average (EWMA) control chart to detect small shifts in the mean of a long-memory fractionally integrated autoregressive process with an exogenous variable under exponential white noise. We first designed the structure of the control chart and then evaluated its performance in terms of the average run length (ARL) via a simulation study. We first derived an analytical ARL using explicit formulas by solving integral equations and an approximated ARL derived by utilizing the numerical integral equation approach. Banach's fixed-point theorem proved that the analytical ARL exists and is unique. We then compared the out-of-control ARL values using both methods via a simulation study; the out-of-control ARL results for the analytical and approximated ARLs were similar. Moreover, the methods provided comparable accuracy in terms of the percentage difference in expected ARL and standard deviation of the run length. However, the explicit formula approach proved to be more advantageous in terms of faster computational speed and is thus recommended in this situation. An illustrative example using real data is also provided to demonstrate the practicability of the analytical ARL method.

Keywords: Average run length (ARL), expected ARL (EARL), approximate ARL, analytical ARL, Banach's fixed-point theorem, numerical integral equation, exponential white noise.

Introduction

Statistical process control (SPC) involves using tools to monitor changes in process parameters. Control charts are used to monitor study variables and identify situations when the process becomes out-of-control. They have been developed for and are most widely applied in fields such as medicine, ecology, and technology. A process is considered to be in-control or out-of-control if variations in the process parameter of interest remain within or deviate outside of the lower and upper control limits, respectively. Detecting unusual changes in a process can enable remedial action to restore it to the in-control state. The process mean can be monitored throughout production using an economical sampling scheme [1].

The concept of the control chart used for monitoring the process mean was first developed by Shewhart [2]. While this control chart is effective at detecting significant shifts in the process mean, it becomes less efficient for shifts of small magnitude. While the Shewhart control chart relies only on information from the most recent observation, the cumulative sum (CUSUM) [3] and exponentially weighted moving average (EWMA) control charts [4] are memory-type control charts more suited to detect small-to-moderate shifts in the process mean. The focus of the present study is enhancing the EWMA statistic for a one-sided EWMA control chart to detect small-to-moderate shifts in the process mean of a stationary time-series model. In numerous disciplines, including environmental sciences (high tide, concentration of particulate matter, ozone level, etc.), economics, engineering (loading capacity, tear strength, etc.), and finance (financial risk, and interest and unemployment rates), the one-sided problem is considered to be of critical importance.

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Assuming the observations under consideration for the desired quality characteristic are independent and identically distributed constitutes the primary limitation of memory-type control. On the other hand, observations from some manufacturing processes can be autocorrelated. [5]. Autocorrelation in data has been considered using various time series models, such as autoregressive (AR), moving average (MA), autoregressive moving average (ARMA), and autoregressive integrated moving average (ARIMA). These models often exhibit short-memory processes in the underlying data. However, some exhibit long-memory processes, which often appear in environmental, health, manufacturing, and financial data [6-8]. In the current work, we examined the EWMA control chart used for monitoring long-memory processes while considering the presence of autocorrelation in the observations.

The AR fractionally integrated MA order (*p*, *d*, *q*), ARFIMA(*p*, *d*, *q*) model was proposed by Granger and Joyeux [9] and Hosking [10] to handle long-memory time-series processes. It has been demonstrated that the ARFIMA model outperforms the ARIMA model for long-memory processes [11]. ARFIMA model implementations on EWMA and CUSUM control charts incorporate elements of autocorrelation and long-memory [11], [12], [13]. The econometric model-variable correlation influences economic forecasting. Exogenous variables such as government investment policies, exchange rates, interest rates, and inflation rates, among others, influence a forecasting econometric model. Incorporating an exogenous variable into a forecasting process enhances the predictive capacity of the model. A special case of the ARFIMA model with an exogenous variable (ARFIMAX(*p*, *d*, *q*, *r*)) simplified by assuming $q = 0$ (i.e., an ARFIX(*p*, *d*, *r*)) process) was investigated in the present study. Model errors (the discrepancy between actual and predicted values), depicted as white noise in a time-series model with autocorrelated observations, should be as low as feasible. White noise can be distributed normally or non-normally such as exponentially [14], [15], [16]. In the present study, we investigated the ARFIX model with exponential white noise.

The performance of comparison of the control chart can be measured based on the average run length (ARL), which is the average number of plotted samples before a control chart produces an out-of-control signal. Choosing an appropriate in-control ARL (ARL_0) is essential when selecting a control chart. In addition, the control chart's performance improves as the out-of-control ARL (ARL_1) decreases. Multiple methods such as Monte Carlo simulation, the Markov chain approach, the numerical integral equation (NIE) method, and explicit formulas can be used to compute the ARL, for example, Crowder [17] presented a numerical method employing an integral equation of the second kind alongside a computer program for evaluating ARL in a normally distributed process utilizing a two-sided EWMA control chart. Recently, Areepong and Peerajit [18] employed integral equation solutions to compute the ARL of a generalized seasonal ARFIMAX$(P, D, Q, r)_s$ process running on a CUSUM control chart. Finally, Peerajit [19] derived explicit formulas for the ARL of a one-sided CUSUM control to detect a mean shift in a FIMAX model with underlying exponential white noise. Moreover, Banach's fixed-point theorem can demonstrate the existence and uniqueness of the ARL, obtained through explicit formulas [20], [21]. The objective of the present work was to devise computation methods for the ARL by using techniques for solving integral equations.

Materials and Methods

This section provides an overview of the EWMA scheme, the long-memory ARFIX model with exponential white noise, and derivations of the ARL using explicit formulas and the NIE method.

The EWMA Control Chart

The EWMA control chart incorporates historical and current data for monitoring a change in the mean of a process. The most recent observations are given more importance by being assigned higher weights while the less recent observations are given lower weights. The EWMA control chart [4] is based on the statistic

$$E_t = (1-\pi)E_{t-1} + \pi Y_t, \quad Y_t;\ t = 1, 2, \ldots, \tag{1}$$

where process parameter $Y_t;\ t = 1, 2, \ldots,$ is taken from a long-memory ARFIX process with exponential white noise, E_{t-1} is the previous value of the statistic, E_0 is the initial value equal to φ, and $\pi \in (0, 1]$ is a smoothing parameter. The performance of the EWMA scheme is sensitive to minor process parameter shifts when using small values for π. The mean and variance of the EWMA statistic are

$$\mathrm{E}(E_t) = \mu,\ \mathrm{V}(E_t) = \sigma^2 \frac{\pi}{2-\pi}\left(1-(1-\pi)^{2t}\right), \text{ respectively,}$$

where μ and σ^2 are the mean and variance of the in-control process, respectively. The respective upper control limit (UCL), central line (CL), and lower control limit (LCL) of the EWMA control chart are

$$\begin{aligned} \mathrm{UCL} &= \mathrm{E}(E_t) + L\sqrt{\mathrm{V}(E_t)}, \\ \mathrm{CL} &= \mathrm{E}(E_t), \\ \mathrm{LCL} &= \mathrm{E}(E_t) - L\sqrt{\mathrm{V}(E_t)}, \end{aligned} \tag{2}$$

where L is the signaling coefficient of the EWMA statistic, the value of which is determined by the selection of the smoothing constant π and the desired in-control ARL (ARL_0). An out-of-control signal occurs whenever E_t exceeds the UCL or falls below the LCL.

The Characteristics of the Proposed Time-Series Model

Y_t in Equation (1) can be used to model a time series with long memory. Let $Y_t,\ t = 1, 2, \ldots$, represent the sequence of a fractionally integrated (FI) AR process with an exogenous variable order $(p,\ d,\ r)$, where non-integer values are used for the fractional differencing parameter (d) and $d \in (0, 0.5)$ indicates the presence of a long-memory process. Thus, the long memory $\mathrm{ARFIX}(p,\ d,\ r)$ process can be defined as follows:

$$\left(1 - \sum_{i=1}^{p} \phi_i B^i\right)(1-B)^d Y_t = \varepsilon_t + \sum_{j=1}^{r} \omega_j X_{jt}. \tag{3}$$

where $\phi_i,\ i = 1, 2, \ldots, p$ are AR coefficients and $X_{jt},\ j = 1, 2, \ldots, r$ holds the values for r exogenous time-varying predictors at time *t* with exogenous coefficients denoted as ω_j, and ε_t is the white noise. Note that ε_t follow exponential distribution $\varepsilon_t \sim Exp(\lambda)$.

The fractional difference operator $(1-B)^d$ is determined by employing a formal binomial series expansion:

$$(1-B)^d = 1 - dB - \frac{d(1-d)}{2}B^2 - \ldots, \tag{4}$$

where B is the backward shift operator, d is the degree of differencing parameter, when $0 < d < 0.5$ is a long-memory (or long-range dependent) process.

By substituting Equation (4) into Equation (3), the general long-memory $\mathrm{ARFIX}(p,\ d,\ r)$ process with exponential white noise for the EWMA scheme becomes

$$Y_t = \varepsilon_t + \sum_{j=1}^{r} \omega_j X_{jt} + \sum_{i=1}^{p} \phi_i Y_{t-i} + d\left(Y_{t-1} - \sum_{i=1}^{p} \phi_i Y_{t-(i+1)}\right) + \frac{d(1-d)}{2}\left(Y_{t-2} - \sum_{i=1}^{p} \phi_i Y_{t-(i+2)}\right) + \ldots., \tag{5}$$

where $-1 \le \phi_i \le 1,\ i = 1, 2, \ldots, p$ are AR coefficients and the initial values of $Y_{t-1}, Y_{t-2}, \ldots, X_{1t}, X_{2t}, \ldots, X_{rt}$ are equal to 1.

The Proposed Control Chart

Substituting Equation (5) into Equation (1) results in

$$E_t = (1-\pi)E_{t-1} + \pi\left(\varepsilon_t + \sum_{j=1}^{r} \omega_j X_{jt} + \sum_{i=1}^{p} \phi_i Y_{t-i} + d\left(Y_{t-1} - \sum_{i=1}^{p} \phi_i Y_{t-(i+1)}\right) + \frac{d(1-d)}{2}\left(Y_{t-2} - \sum_{i=1}^{p} \phi_i Y_{t-(i+2)}\right) + \ldots\right), \tag{6}$$

When the initial time (*t*) = 1, $E_0 = \varphi$. Thus, the proposed EWMA statistic becomes

$$E_1 = (1-\pi)\varphi + \pi\left(\varepsilon_1 + \sum_{j=1}^{r} \omega_j X_{j1} + \sum_{i=1}^{p} \phi_i Y_{t-i} + d\left(Y_0 - \sum_{i=1}^{p} \phi_i Y_{t-(i+1)}\right) + \frac{d(1-d)}{2}\left(Y_{t-2} - \sum_{i=1}^{p} \phi_i Y_{t-(i+2)}\right) + \ldots\right), \tag{7}$$

The stopping time for the EWMA control chart to detect an out-of-control process is defined as

$$\tau_h = \inf\{t > 0;\ E_t \ge h\},$$

where h represents the pre-established UCL for the control chart.

Interval E_1 between the lower and upper control limits for the in-control process can be represented as 0 to h, which can be rewritten for ε_1 on the interval as follows:

$$\frac{-(1-\pi)\varphi-\pi H}{\pi}<\varepsilon_1<\frac{h-(1-\pi)\varphi-\pi H}{\pi},$$

where $H=\sum_{j=1}^{r}\omega_j X_{j1}+\sum_{i=1}^{p}\phi_i Y_{t-i}+d\left(Y_0-\sum_{i=1}^{p}\phi_i Y_{t-(i+1)}\right)+\frac{d(1-d)}{2}\left(Y_{t-2}-\sum_{i=1}^{p}\phi_i Y_{1-(i+2)}\right)+\dots$.

Methods For Defining the ARL of Changes in the Mean of an ARFIX(*p,d,r*) Process Running on a EWMA Control Chart

Assume that sequential observations $\varepsilon_t;\ t=1,2,\dots$, are i.i.d. random variables with distribution function $F(\varepsilon,\lambda)$, where λ represents a control parameter. Thus, $\lambda=\lambda_0$ can be the parameter value before the change-point time $(\nu\le\infty)$, and $\lambda_1>\lambda_0$ can be parameter value after the change-point time. We can designate $\mathrm{E}_\nu(.)$ as the expectation under distribution $F(\varepsilon,\lambda)$ for a fixed change point occurring at point ν, to more precisely define ARL.

In the case of $\nu=\infty$, a suitable control chart yields a large ARL, where $\lambda=\lambda_0$ indicates no change point at time *t*. Therefore, the ARL for the in-control process (ARL_0) representing the expected stopping time can be defined as

$$\mathrm{ARL}_0=\mathrm{E}_\infty(\tau_h)=\gamma. \tag{8}$$

On the other hand, the appropriate control chart should yield a small ARL when λ indicates the time point of the change from λ_0 to λ_1, denoted as $\lambda_1>\lambda_0$. As a result, the ARL is only assessed using a specific case when $\nu=1$. Therefore, the ARL for the process that is out-of-control (ARL_1) can be denoted as

$$\mathrm{ARL}_1=\mathrm{E}_1(\tau_h\,|\,\tau_h\ge 1) \tag{9}$$

For the $\mathrm{ARFIX}(p,d,r)$ process on a EWMA control chart, we employed solutions for integral equations to obtain the exact and approximate ARL by using explicit formulas and the NIE method, respectively, to assess the efficiency of the upper-sided EWMA control chart.

The Explicit Formulas Method

This is obtained from the solution of integral equations mathematically represented via a second type of Fredholm integral equation. Crowder [17] introduced this concept in the context of the EWMA control chart.

Let $\mathrm{L}(\varphi)$ be the analytical ARL for detecting changes in the mean of a long-memory $\mathrm{ARFIX}(p,d,r)$ process with exponential white noise $(\varepsilon_t\sim Exp(\lambda_\varepsilon))$ on a EWMA control chart when the initial value of is $E_0=\varphi;\ 0\le\varphi\le h$. Thus, the ARL is a unique solution for integral equations for $\mathrm{L}(\varphi)=\mathrm{E}_\infty(\tau_h)<\infty$. defined as

$$\mathrm{L}(\varphi)=\left\{1-P\left(\frac{(1-\pi)\varphi-\pi H}{\pi}<\varepsilon_1<\frac{h-(1-\pi)\varphi-\pi H}{\pi}\right)\right\}+\int_{\frac{-(1-\pi)\varphi-\pi H}{\pi}}^{\frac{h-(1-\pi)\varphi-\pi H}{\pi}}\left(1+\mathrm{L}((1-\pi)\varphi+\pi(H)\right)f(\varepsilon)d\varepsilon$$

$$\mathrm{L}(\varphi)=1+\int_{\frac{-(1-\pi)\varphi-\pi H}{\pi}}^{\frac{h-(1-\pi)\varphi-\pi H}{\pi}}\mathrm{L}((1-\pi)\varphi+\pi(H))f(\varepsilon)d\varepsilon.$$

By modifying the integral variable, we can derive the integral equations as

$$\mathrm{L}(\varphi)=1+\frac{1}{\pi}\int_0^h \mathrm{L}(s)f\left(\frac{s-(1-\pi)\varphi}{\pi}-H\right)ds \tag{10}$$

Thus, we obtain the integral equations in the following manner:

$$\mathrm{L}(\varphi)=1+\frac{1}{\pi\lambda}\int_0^h \mathrm{L}(s)e^{-\frac{s}{\pi\lambda}}.e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda}+\frac{H}{\lambda}\right)}ds. \tag{11}$$

Operator T is referred to as a contraction on the metric space of all continuous functions $(\mathrm{C}(I),\|\ \|_1)$, where I is a complete interval and the norm is defined as $\|\mathrm{L}\|=\underset{\varphi\in\mathrm{I}}{Sup}|\mathrm{L}(\varphi)|$ if $\eta\in[0,1)$ exists such that $\|\mathrm{T}(\mathrm{L}_1)-\mathrm{T}(\mathrm{L}_2)\|\le\eta\|\mathrm{L}_1-\mathrm{L}_2\|$ for all, $\mathrm{L}_1,\mathrm{L}_2\in[0,h]$. Subsequently, we verified the uniqueness and existence of the ARL computation using Banach's fixed-point theorem. [18-21].

Let $\mathrm{C}(I_1)$ represent the class of all continuous functions defined on compact interval $I_1=[0,h]$. In Equation (11), the right-hand side is continuous, so the integral is a continuous function. Hence, the integral equations become

$$\mathrm{T}\left(\mathrm{L}(\varphi)\right)=1+\frac{1}{\pi\lambda}\int_0^h \mathrm{L}(s)e^{-\frac{s}{\pi\lambda}}.e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda}+\frac{H}{\lambda}\right)}ds. \tag{12}$$

Consequently, the integral equations can be expressed as $\mathrm{T}\left(\mathrm{L}(\varphi)\right)=\mathrm{L}(\varphi)$.

Proposition 1: Operator T is a contraction on metric space $M=(\mathrm{C}(I),\|\ \|_1)$ with the norms

$$\|\mathrm{L}\|=\underset{\varphi\in\mathrm{I}}{Sup}|\mathrm{L}(\varphi)|.$$

Proof: T is a contraction based on inequality $\|\mathrm{T}(\mathrm{L}_1)-\mathrm{T}(\mathrm{L}_2)\|\le\eta\|\mathrm{L}_1-\mathrm{L}_2\|$ being valid for two variables: $\varphi\in I$ and $\mathrm{L}_1,\mathrm{L}_2\in\mathrm{C}(I)$, where $\eta\in[0,1)$ is a positive constant. As per Equation (12), it can be concluded that

$$\begin{aligned}\|\mathrm{T}(\mathrm{L}_1)-\mathrm{T}(\mathrm{L}_2)\| &= \underset{\varphi\in[0,h]}{Sup}\left|\mathrm{L}_1(s)-\mathrm{L}_2(s)\frac{1}{\pi\lambda}e^{\frac{(1-\pi)\varphi}{\pi\lambda}+\frac{H}{\lambda}}\int_0^h \mathrm{T}(s)e^{-\frac{s}{\pi\lambda}}ds\right| \\ &\le \underset{\varphi\in[0,h]}{Sup}\left|\|\mathrm{L}_1-\mathrm{L}_2\|\frac{1}{\pi\lambda}e^{\frac{(1-\lambda)\varphi}{\pi\lambda}+\frac{H}{\lambda}}(-\pi\lambda)(e^{-\frac{h}{\pi\lambda}}-1)\right| \\ &= \|\mathrm{L}_1-\mathrm{L}_2\|\underset{\varphi\in[0,h]}{Sup}\left[\frac{1}{\pi\lambda}e^{\frac{(1-\pi)\varphi}{\pi\lambda}+\frac{H}{\lambda}}.(1-e^{-\frac{h}{\pi\lambda}})\right] \\ &= \|\mathrm{L}_1-\mathrm{L}_2\|\underset{\varphi\in[0,h]}{Sup}\left[e^{\frac{(1-\lambda)\varphi}{\pi\lambda}+\frac{H}{\lambda}}.(1-e^{-\frac{h}{\pi\lambda}})\right]\end{aligned}$$

As a result, we acquire

$$\|\mathrm{T}(\mathrm{L}_1)-\mathrm{T}(\mathrm{L}_2)\|\ \le\ \eta\|\mathrm{L}_1-\mathrm{L}_2\|,\ \text{where}\ \eta=\underset{\varphi\in[0,h]}{Sup}\left[e^{\frac{(1-\lambda)\varphi}{\pi\lambda}+\frac{H}{\lambda}}.(1-e^{-\frac{h}{\pi\lambda}})\right].$$

We can implement triangular inequality by considering

$$|\mathrm{L}_1-\mathrm{L}_2|=\underset{\varphi\in[0,h]}{Sup}|\mathrm{L}_1-\mathrm{L}_2|=\|\mathrm{L}_1-\mathrm{L}_2\|.$$

Therefore, operator T is a contraction resulting in $\|\mathrm{T}(\mathrm{L}_1)-\mathrm{T}(\mathrm{L}_2)\|\ \le\ \eta\|\mathrm{L}_1-\mathrm{L}_2\|$. Fixed-point equation $\mathrm{T}\left(\mathrm{L}(\varphi)\right)=\mathrm{L}(\varphi)$ has a unique solution if T is a contraction, as per Banach's fixed-point theorem.

Now, we consider the integral equations expressed in Equation (11).

For $D(\varphi)=e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda}+\frac{H}{\lambda}\right)}$ and $g=\int_0^h \mathrm{L}(s)e^{-\frac{s}{\pi\lambda}}ds$, we can consequentially obtain

$$\mathrm{L}(\varphi) = 1 + \frac{D(\varphi)}{\pi\lambda} g, \;\; 0 \le \varphi \le h. \tag{13}$$

By solving for constant $g = \int_0^h \left(1 + \frac{D(s)}{\pi\lambda} g\right) . e^{-\frac{s}{\pi\lambda}} ds$, we can generate

$$\begin{aligned} g &= \int_0^h e^{-\frac{s}{\pi\lambda}} ds + g \int_0^h \frac{D(s)}{\pi\lambda} e^{-\frac{s}{\pi\lambda}} ds \\ &= \pi\lambda(1 - e^{-\frac{h}{\pi\lambda}}) + \frac{g}{\pi\lambda} e^{\left(\frac{H}{\lambda}\right)} . \int_0^h e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda} + -\frac{s}{\pi\lambda}\right)} ds \\ &= \pi\lambda(1 - e^{-\frac{h}{\pi\lambda}}) + \frac{g}{\pi} e^{\left(\frac{H}{\lambda}\right)} .(1 - e^{-\frac{h}{\lambda}}) \end{aligned}$$

$$\therefore g = \frac{\pi\lambda(1 - e^{-\frac{h}{\pi\lambda}})}{1 - \frac{1}{\pi} e^{\left(\frac{H}{\lambda}\right)} .(1 - e^{-\frac{h}{\lambda}})}.$$

By substituting constant g into Equation (13), $\mathrm{L}(\varphi)$ becomes

$$\begin{aligned} \mathrm{L}(\varphi) &= 1 + \frac{e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda} + \frac{H}{\lambda}\right)}}{\pi\lambda} . \frac{\pi\lambda(1 - e^{-\frac{h}{\pi\lambda}})}{1 - \frac{1}{\pi} e^{\left(\frac{H}{\lambda}\right)} .(1 - e^{-\frac{h}{\lambda}})} \\ &= 1 + e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda} + \frac{H}{\lambda}\right)} . \frac{\pi(1 - e^{-\frac{h}{\pi\lambda}})}{\pi - (1 - e^{-\frac{h}{\lambda}}).e^{\left(\frac{H}{\lambda}\right)}} \\ &= 1 - \frac{e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda} + \frac{H}{\lambda}\right)} \pi(e^{-\frac{h}{\pi\lambda}} - 1)}{\pi + e^{\left(\frac{H}{\lambda}\right)} (e^{-\frac{h}{\lambda}} - 1).}. \end{aligned}$$

Consequently, the explicit formulas for the ARL by solving the integral equations become

$$\mathrm{L}(\varphi) = 1 - \frac{e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda}\right)} \pi(e^{-\frac{h}{\pi\lambda}} - 1)}{\pi e^{\left(-\frac{H}{\lambda}\right)} + (e^{-\frac{h}{\lambda}} - 1).},$$

where $H = \sum_{j=1}^{r} \omega_j X_{j1} + \sum_{i=1}^{p} \phi_i Y_{1-i} + d\left(Y_0 - \sum_{i=1}^{p} \phi_i Y_{t-(i+1)}\right) + \frac{d(1-d)}{2}\left(Y_{t-2} - \sum_{i=1}^{p} \phi_i Y_{t-(i+2)}\right) + \ldots$.

For the in-control process with exponential parameter $\lambda = \lambda_0$, we can obtain the explicit formula for ARL$_0$ as

$$\mathrm{ARL}_0 = 1 - \frac{e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda_0}\right)} \pi(e^{-\frac{h}{\pi\lambda_0}} - 1)}{\pi e^{\left(-\frac{1}{\lambda_0}\left(\sum_{j=1}^{r} \omega_j X_{j1} + \sum_{i=1}^{p} \phi_i Y_{t-i} + d\left(Y_0 - \sum_{i=1}^{p} \phi_i Y_{t-(i+1)}\right) + \ldots\right)\right)} + (e^{-\frac{h}{\lambda_0}} - 1)}. \tag{14}$$

On the other hand, for the out-of-control process with exponential parameter $\lambda = \lambda_1$, where $\lambda_1 = \lambda_0(1+\delta)$, the explicit formula for ARL₁ can be written as

$$\mathrm{ARL}_1 = 1 - \frac{e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda_1}\right)}\pi(e^{-\frac{h}{\pi\lambda_1}}-1)}{\pi e^{\left(-\frac{1}{\lambda_1}\left(\sum_{j=1}^{r}\omega_j X_{j1}+\sum_{i=1}^{p}\phi_i Y_{t-i}+d\left(Y_0-\sum_{i=1}^{p}\phi_i Y_{t-(i+1)}\right)+\ldots\right)\right)}+(e^{-\frac{h}{\lambda_1}}-1)}. \quad (15)$$

The NIE method

This is commonly used to approximate the ARL for a process on a EWMA control chart (in our case, a long-memory ARFIX model with exponential white noise). To this end, the midpoint quadrature rule, denoted as $\hat{L}(\varphi)$., is used to resolve an *m* linear equation system within interval $[0,h]$. The technique is applied by dividing into $0 \le \upsilon_1 \le \upsilon_2 \le .... \le \upsilon_m \le h$ using constant weights $w_k = h/m$ as follows:

$$\int_0^h L(s)f(s)ds \approx \sum_{k=1}^{m} w_k f(\upsilon_k) \quad (16)$$

where $\upsilon_k = h/m(k-0.5)$ and $w_k = h/m;\ k = 1,2,...,m.$

Thus, by solving a system of algebraic linear equations, we obtain

$$\hat{L}(\upsilon_l) \approx 1 + \frac{1}{\pi}\sum_{k=1}^{m} w_k \hat{L}(\upsilon_k) f\left(\frac{\upsilon_k-(1-\pi)\upsilon_l}{\pi} - H\right)$$

Finally, for function $\hat{L}(\upsilon_l)$, υ_l is replaced by substituting with φ. Therefore, the approximate ARL can be represented as

$$\hat{L}(\varphi) \approx 1 + \frac{1}{\pi}\sum_{k=1}^{m} w_k \hat{L}(\upsilon_k) f\left(\frac{\upsilon_k-(1-\pi)\varphi}{\pi} - \left(\sum_{j=1}^{r}\omega_j X_{jt} + \sum_{i=1}^{p}\phi_i Y_{t-i} + d\left(Y_{t-1} - \sum_{i=1}^{p}\phi_i Y_{t-(i+1)}\right) + \frac{d(1-d)}{2}\left(Y_{t-2} - \sum_{i=1}^{p}\phi_i Y_{t-(i+2)}\right) + ...\right)\right) \quad (17)$$

The Procedure for Computing the ARL

For ARL₀ Using Explicit Formulas

(1) Compute the EWMA statistic (E_t)
- Set the desired ARL₀ value (370 or 500)
- Set the value of π = 0.03,0.01, or 0.1

(2) Solve the equations for observations from a long-memory $\mathrm{ARFIX}(p,\ d,\ r)$ process with exponential white noise defined as Y_t running on a EWMA control chart
- Set the initial values of the process: $Y_{t-1}, Y_{t-2}, ..., Y_{t-p}, Y_{t-(p+1)} = 1; X_{1t}, X_{2t}, ..., X_{rt} = 1$
- Set the values of process coefficients: $\phi_1 = 0.1,\ 0.2,\ 0.3;\ d = 1/3,\ 1/4, 1/5;\ \omega_1 = 0.3$
- Specify the mean for the exponential parameter $(\lambda = \lambda_0)$: $\lambda_0 = 1;$ for in-control mean shift size $(\delta) = 0$

(3) The design structure of the EWMA for long-memory $\mathrm{ARFIX}(p,\ d,\ r)$ process with exponential white noise.
- Set the smoothing parameter π = 0.03, 0.01, or 0.1
- Set the initial value of $\varphi = 1$

(4) Compute the UCL for LCL = 0
- Repeat steps (1)–(3) to compute the UCL or *h* in conjunction with each value of π using Equation (14). If the computed ARL₀ is the same as the specified ARL_0, then stop the procedure and go to the procedure for the out-of-control process. Otherwise, change the values of the control chart parameters and coefficients and repeat steps (1)–(3)

For ARL_1 Using Explicit Formulas

(5) Compute ARL_1 for the shift size using Equation (15).

- Specify shifts in the mean for the exponential parameter from λ_0 to λ_1 where $\lambda_1 = (1+\delta)\lambda_0$; $\delta =$ 0.025, 0.050, 0.075, 0.100, 0.125, 0.150, 0.175, or 0.200

(6) Repeat steps (1)–(6) for ARL_1 and record the computational time for the first out-of-control signal. After computing the ARL using the explicit formulas, we examined the accuracy of these calculations using Equation (17) by setting the number of division points as $m = 500$. Both methods were executed using the Eviews 10 statistical software program.

Performance Statistics

Standard Deviation Run Length (SDRL)

The run-length (RL) distribution's properties are frequently used to evaluate the sensitivity of a monitoring control chart. The RL represents the control chart's plotting statistics before the first out-of-control signal. Besides the ARL, the SDRL can be used as a performance measure computed as

$$\mathrm{SDRL}=\sqrt{\frac{1-\lambda}{\lambda^2}}, \tag{18}$$

where $\lambda = \lambda_0$ represents the Type I error indicating no change in the mean of a long-memory $\mathrm{ARFIX}(p, d, r)$ process when the process is in-control ($SDRL_0$). Here, the ARL_0 value is assigned as 370 or 500, resulting in λ_0 being calculated as 0.0027 and 0.002, respectively. On the other hand, $\lambda = \lambda_1$ represents the type II error.

The Expected ARL (EARL)

This can be used to assess the performances of the proposed methods for an unknown magnitude of the shift size. Consider the EARL for a comprehensive range of shift sizes where $\delta \in [\delta_{\min}, \delta_{\max}]$., where $\delta_{\min}$ and $\delta_{\max}$ represent the lower and upper bounds of the mean shift, respectively. EARL can be calculated as [22].

$$\mathrm{EARL} = \int_{\delta_{\min}}^{\delta_{\max}} f_\delta(\delta).\mathrm{ARL}(\delta)d\delta, \tag{19}$$

where $f_\delta(\delta)$ is the probability density function of the shift size and $\mathrm{ARL}(\delta)$ is the ARL function for the shift (δ). We assume that the shifts in the process mean all happen with an equal probability; i.e. $f_\delta(\delta) = \frac{1}{\delta_{\max} - \delta_{\min}}$. As a result, we choose the scheme with the smallest EARL that produces the most optimal performance within a certain range of shifts. We can express Equation (19) in equivalent form as follows:

$$\mathrm{EARL} = \frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} \mathrm{ARL}(\delta), \tag{20}$$

where Δ represents the increase from $\delta_{\min}$ to $\delta_{\max}$ in a Riemann sum. Note that the lower limit of δ is not included in the summation when $\delta_{\min} = 0$.

The Percentage Difference ($\%Diff_A$)

This can be expressed as

$$\%\mathrm{Diff}_A = \frac{\mathrm{EARL}_{Exp} - \mathrm{EARL}_{\mathrm{NIE}}}{\mathrm{EARL}_{Exp}} \times 100\%, \tag{21}$$

where EARL_{Exp} and $\mathrm{EARL}_{\mathrm{NIE}}$ represent the EARL values computed using the explicit formulas and NIE methods, respectively, calculated using Equation (20). A $\%Diff_A$ value of less than 0.25% means that the EARL results obtained from both methods are in excellent agreement.

Table 1. Comparison of the explicit formulas and NIE methods for a long-memory ARFIX(1, *d*, 1) process on a EWMA control chart for ϕ_1 = 0.1, ω_1 = 0.3, and ARL_0 = 370

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
π = 0.03	h = 1.57058×10^{-14}			h = 1.73078×10^{-14}			h =1.84403×10^{-14}		
δ	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL
0.025	157.338	157.355	156.837	157.759	157.752	157.258	157.978	157.978	157.477
	(< 0.001)	(4.406)		(< 0.001)	(5.503)		(< 0.001)	(5.035)	
0.050	69.984	69.983	69.482	70.314	70.314	69.812	70.525	70.525	70.023
	(< 0.001)	(6.016)		(< 0.001)	(6.175)		(< 0.001)	(4.985)	
0.075	32.602	32.598	32.098	32.819	32.818	32.315	32.957	32.955	32.453
	(< 0.001)	(5.656)		(< 0.001)	(7.909)		(< 0.001)	(4.563)	
0.100	15.990	15.988	15.482	16.124	16.124	15.616	16.212	16.212	15.704
	(< 0.001)	(7.297)		(< 0.001)	(5.628)		(< 0.001)	(5.102)	
0.125	8.347	8.345	7.831	8.427	8.427	7.911	8.479	8.479	7.963
	(< 0.001)	(6.859)		(< 0.001)	(4.378)		(< 0.001)	(6.035)	
0.150	4.711	4.711	4.181	4.760	4.760	4.231	4.791	4.791	4.262
	(< 0.001)	(6.453)		(< 0.001)	(7.113)		(< 0.001)	(6.896)	
0.175	2.929	2.929	2.378	2.959	2.959	2.408	2.977	2.977	2.426
	(< 0.001)	(5.109)		(< 0.001)	(6.642)		(< 0.001)	(4.456)	
0.200	2.031	2.031	1.447	2.048	2.048	1.465	2.059	2.059	1.477
	(< 0.001)	(5.093)		(< 0.001)	(6.753)		(< 0.001)	(4.265)	
EARL	11,757.31	11,757.63		11,808.40	11,808.08		11,839.12	11,839.04	
$\%Diff_A$	0.0027			0.0027			0.0007		
π = 0.05	h = 1.61638×10^{-8}			h = 1.78095×10^{-8}			h = 1.89751×10^{-8}		
δ	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL
0.025	217.466	217.466	216.965	217.978	217.978	217.477	218.314	218.314	217.813
	(< 0.001)	(4.356)		(< 0.001)	(5.551)		(< 0.001)	(4.102)	
0.050	131.176	131.176	130.675	131.778	131.778	131.277	132.173	132.173	131.672
	(< 0.001)	(5.945)		(< 0.001)	(6.333)		(< 0.001)	(4.987)	
0.075	81.113	81.113	80.611	81.657	81.657	81.155	82.014	82.014	81.512
	(< 0.001)	(7.265)		(< 0.001)	(7.021)		(< 0.001)	(5.031)	
0.100	51.377	51.377	50.875	51.823	51.823	51.321	52.116	52.116	51.614
	(< 0.001)	(6.548)		(< 0.001)	(7.724)		(< 0.001)	(5.986)	
0.125	33.321	33.321	32.817	33.672	33.672	33.168	33.902	33.902	33.398
	(< 0.001)	(5.656)		(< 0.001)	(8.474)		(< 0.001)	(4.125)	
0.150	22.131	22.131	21.625	22.399	22.399	21.893	22.578	22.578	22.072
	(< 0.001)	(6.865)		(< 0.001)	(7.161)		(< 0.001)	(4.986)	
0.175	15.061	15.061	14.552	15.265	15.265	14.757	15.401	15.401	14.893
	(< 0.001)	(6.824)		(< 0.001)	(6.421)		(< 0.001)	(5.218)	
0.200	10.512	10.512	10.000	10.667	10.667	10.155	10.769	10.769	10.257
	(< 0.001)	(7.055)		(< 0.001)	(5.817)		(< 0.001)	(5.632)	
EARL	22,486.28	22,486.28		22,609.56	22,609.56		22,690.68	22,690.68	
$\%Diff_A$	0.0000			0.0000			0.0000		
π = 0.10	h = 6.83290×10^{-4}			h = 7.53110×10^{-4}			h = 8.02590×10^{-4}		
δ	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL
0.025	276.431	276.431	275.931	277.113	277.113	276.613	277.560	277.560	277.060
	(< 0.001)	(5.124)		(< 0.001)	(5.113)		(< 0.001)	(4.598)	
0.050	209.388	209.388	208.887	210.394	210.394	209.893	211.055	211.055	210.554
	(< 0.001)	(6.469)		(< 0.001)	(6.896)		(< 0.001)	(5.985)	
0.075	160.659	160.659	160.158	161.786	161.786	161.285	162.528	162.528	162.027
	(< 0.001)	(7.326)		(< 0.001)	(7.552)		(< 0.001)	(6.035)	
0.100	124.765	124.765	124.264	125.903	125.903	125.402	126.653	126.653	126.152
	(< 0.001)	(7.783)		(< 0.001)	(5.177)		(< 0.001)	(5.136)	
0.125	97.995	97.995	97.494	99.084	99.084	98.583	99.803	99.803	99.302
	(< 0.001)	(5.408)		(< 0.001)	(7.802)		(< 0.001)	(4.983)	
0.150	77.796	77.795	77.294	78.807	78.807	78.305	79.475	79.475	78.973
	(< 0.001)	(7.065)		(< 0.001)	(6.427)		(< 0.001)	(4.563)	
0.175	62.387	62.387	61.885	63.309	63.309	62.807	63.920	63.920	63.418

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
	(< 0.001)	(4.421)		(< 0.001)	(8.229)		(< 0.001)	(4.958)	
0.200	50.512	50.511	50.010	51.343	51.343	50.841	51.895	51.895	51.393
	(< 0.001)	(4.674)		(< 0.001)	(10.099)		(< 0.001)	(5.679)	
EARL	42,387.32	42,387.24		42,709.56	42,709.56		42,915.56	42,915.56	
%Diff$_A$	0.0002			0.0000			0.0000		

Table 2. Comparison of the explicit formulas and NIE methods for a long-memory ARFIX(1, *d*, 1) process on a EWMA control chart for ϕ_1 = 0.2, ω_1 = 0.3, and ARL$_0$ = 370

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
π = 0.03	h = 1.49500×10^{-14}			h = 1.62960×10^{-14}			h = 1.72413×10^{-14}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	157.173	157.177	156.672	157.497	157.509	156.996	157.722	157.722	157.221
	(< 0.001)	(5.140)		(< 0.001)	(5.676)		(< 0.001)	(5.064)	
0.050	69.828	69.826	69.326	70.114	70.114	69.612	70.298	70.298	69.796
	(< 0.001)	(5.812)		(< 0.001)	(6.317)		(< 0.001)	(4.262)	
0.075	32.489	32.491	31.985	32.680	32.682	32.176	32.806	32.806	32.302
	(< 0.001)	(6.484)		(< 0.001)	(5.020)		(< 0.001)	(4.789)	
0.100	15.924	15.923	15.416	16.041	16.040	15.533	16.118	16.118	15.610
	(< 0.001)	(7.171)		(< 0.001)	(8.692)		(< 0.001)	(5.123)	
0.125	8.306	8.306	7.790	8.377	8.377	7.861	8.423	8.423	7.907
	(< 0.001)	(6.765)		(< 0.001)	(5.302)		(< 0.001)	(6.562)	
0.150	4.688	4.688	4.158	4.730	4.730	4.200	4.757	4.757	4.228
	(< 0.001)	(8.390)		(< 0.001)	(8.864)		(< 0.001)	(5.462)	
0.175	2.916	2.916	2.364	2.941	2.941	2.389	2.957	2.957	2.406
	(< 0.001)	(7.524)		(< 0.001)	(7.586)		(< 0.001)	(4.982)	
0.200	2.022	2.022	1.438	2.037	2.031	1.453	2.047	2.047	1.464
	(< 0.001)	(5.140)		(< 0.001)	(7.505)		(< 0.001)	(4.695)	
EARL	11,733.84	11,733.96		11,776.68	11,776.96		11,805.12	11,805.12	
%Diff$_A$	0.0010			0.0024			0.0000		
π = 0.05	h = 1.53850×10^{-8}			h = 1.67700×10^{-8}			h = 1.77420×10^{-8}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	217.205	217.205	216.704	217.663	217.663	217.162	217.959	217.959	217.458
	(< 0.001)	(5.889)		(< 0.001)	(5.161)		(< 0.001)	(7.123)	
0.050	130.869	130.869	130.368	131.406	131.406	130.905	131.755	131.755	131.254
	(< 0.001)	(7.576)		(< 0.001)	(5.801)		(< 0.001)	(5.235)	
0.075	80.837	80.837	80.335	81.320	81.320	80.818	81.636	81.636	81.134
	(< 0.001)	(6.264)		(< 0.001)	(6.520)		(< 0.001)	(4.263)	
0.100	51.151	51.151	50.649	51.546	51.546	51.044	51.805	51.805	51.303
	(< 0.001)	(5.889)		(< 0.001)	(6.583)		(< 0.001)	(5.896)	
0.125	33.145	33.145	32.641	33.454	33.454	32.950	33.658	33.658	33.154
	(< 0.001)	(5.483)		(< 0.001)	(7.708)		(< 0.001)	(4.132)	
0.150	21.995	21.995	21.489	22.233	22.233	21.727	22.390	22.390	21.884
	(< 0.001)	(4.217)		(< 0.001)	(6.614)		(< 0.001)	(5.032)	
0.175	14.958	14.958	14.449	15.138	15.138	14.629	15.257	15.257	14.749
	(< 0.001)	(4.528)		(< 0.001)	(6.214)		(< 0.001)	(5.841)	
0.200	10.434	10.434	9.921	10.571	10.571	10.059	10.661	10.661	10.149
	(< 0.001)	(4.904)		(< 0.001)	(5.302)		(< 0.001)	(6.021)	
EARL	22,423.76	22,423.76		22,533.24	22,533.24		22,604.84	22,604.84	
%Diff$_A$	0.0000			0.0000			0.0000		
π = 0.10	h = 6.50264×10^{-4}			h = 7.09000×10^{-4}			h = 7.50240×10^{-4}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	276.085	276.085	275.585	276.693	276.693	276.193	277.086	277.086	276.586
	(< 0.001)	(5.497)		(< 0.001)	(5.002)		(< 0.001)	(5.113)	
0.050	208.878	208.878	208.377	209.772	209.772	209.271	210.354	210.354	209.853
	(< 0.001)	(4.122)		(< 0.001)	(5.645)		(< 0.001)	(5.035)	
0.075	160.088	160.088	159.587	161.088	161.088	160.587	161.742	161.742	161.241

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
	(< 0.001)	(6.794)		(< 0.001)	(6.395)		(< 0.001)	(6.265)	
0.100	124.190	124.190	123.689	125.197	125.197	124.696	125.858	125.858	125.357
	(< 0.001)	(7.434)		(< 0.001)	(6.973)		(< 0.001)	(7.035)	
0.125	97.446	97.446	96.945	98.408	98.408	97.907	99.041	99.041	98.540
	(< 0.001)	(7.122)		(< 0.001)	(7.629)		(< 0.001)	(4.065)	
0.150	77.286	77.286	76.784	78.179	78.179	77.677	78.767	78.767	78.265
	(< 0.001)	(8.778)		(< 0.001)	(6.286)		(< 0.001)	(6.842)	
0.175	61.923	61.923	61.421	62.7358	62.7358	62.234	63.273	63.273	62.771
	(< 0.001)	(7.294)		(< 0.001)	(6.025)		(< 0.001)	(5.021)	
0.200	50.093	50.093	49.590	50.826	50.826	50.324	51.310	51.310	50.808
	(< 0.001)	(5.419)		(< 0.001)	(5.989)		(< 0.001)	(4.058)	
EARL	42,239.56	42,239.56		42,515.95	42,515.95		42,697.24	42,697.24	
%Diff$_A$	0.0000			0.0000			0.0000		

Table 3. Comparison of the explicit formulas and NIE methods for a long-memory ARFIX(1, *d*, 1) process on a EWMA control chart for ϕ_1 = 0.3, ω_1 = 0.3, and ARL$_0$ = 370

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
π = 0.03	h = 1.42310×10^{-14}			h = 1.53461×10^{-14}			h = 1.61200×10^{-14}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	157.022	157.004	156.521	157.289	157.295	156.788	157.472	157.472	156.971
	(< 0.001)	(5.451)		(< 0.001)	(4.721)		(< 0.001)	(4.956)	
0.050	69.666	69.670	69.164	69.918	69.918	69.416	70.068	70.068	69.566
	(< 0.001)	(5.217)		(< 0.001)	(5.409)		(< 0.001)	(5.156)	
0.075	32.389	32.386	31.885	32.552	32.552	32.048	32.657	32.657	32.153
	(< 0.001)	(5.920)		(< 0.001)	(5.018)		(< 0.001)	(4.856)	
0.100	15.856	15.857	15.348	15.961	15.959	15.453	16.026	16.026	15.518
	(< 0.001)	(6.592)		(< 0.001)	(6.596)		(< 0.001)	(5.956)	
0.125	8.267	8.266	7.751	8.329	8.328	7.813	8.368	8.368	7.852
	(< 0.001)	(5.248)		(< 0.001)	(7.206)		(< 0.001)	(6.024)	
0.150	4.664	4.664	4.134	4.702	4.702	4.172	4.725	4.725	4.195
	(< 0.001)	(7.905)		(< 0.001)	(7.799)		(< 0.001)	(5.056)	
0.175	2.902	2.902	2.349	2.923	2.923	2.371	2.938	2.938	2.386
	(< 0.001)	(5.842)		(< 0.001)	(7.215)		(< 0.001)	(5.012)	
0.200	2.014	2.014	1.429	2.027	2.027	1.443	2.035	2.035	1.451
	(< 0.001)	(4.702)		(< 0.001)	(8.408)		(< 0.001)	(4.895)	
EARL	11,711.20	11,710.52		11,748.04	11,748.16		11,771.56	11,771.56	
%Diff$_A$	0.0058			0.0010			0.0000		
π = 0.05	h = 1.46437×10^{-8}			h = 1.57907×10^{-8}			h = 1.65890×10^{-8}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	216.945	216.945	216.444	217.342	217.342	216.841	217.605	217.605	217.104
	(< 0.001)	(6.219)		(< 0.001)	(5.049)		(< 0.001)	(4.668)	
0.050	130.565	130.565	130.064	131.031	131.031	130.530	131.338	131.338	130.837
	(< 0.001)	(7.938)		(< 0.001)	(5.799)		(< 0.001)	(5.132)	
0.075	80.563	80.563	80.061	80.982	80.982	80.480	81.259	81.259	80.757
	(< 0.001)	(4.641)		(< 0.001)	(6.533)		(< 0.001)	(4.986)	
0.100	50.926	50.926	50.424	51.269	51.269	50.767	51.496	51.496	50.994
	(< 0.001)	(5.313)		(< 0.001)	(7.283)		(< 0.001)	(5.039)	
0.125	32.969	32.969	32.465	33.238	33.238	32.734	33.415	33.415	32.911
	(< 0.001)	(4.016)		(< 0.001)	(5.893)		(< 0.001)	(6.125)	
0.150	21.861	21.861	21.355	22.067	22.067	21.561	22.203	22.203	21.697
	(< 0.001)	(4.672)		(< 0.001)	(7.565)		(< 0.001)	(7.058)	
0.175	14.855	14.855	14.346	15.012	15.012	14.503	15.115	15.115	14.606
	(< 0.001)	(6.359)		(< 0.001)	(4.237)		(< 0.001)	(6.483)	
0.200	10.357	10.357	9.844	10.475	10.475	9.962	10.553	10.553	10.041
	(< 0.001)	(6.359)		(< 0.001)	(4.237)		(< 0.001)	(6.483)	

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
EARL	22,361.64	22,361.64		22,456.64	22,456.64		22,519.36	22,519.36	
$\%Diff_A$	0.0000			0.0000			0.0000		
π = 0.10	h = 6.18840×10^{-4}			h = 6.67466×10^{-4}			h = 7.01320×10^{-4}		
δ	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL
0.025	275.739	275.739	275.239	276.267	276.267	275.767	276.615	276.615	276.115
	(< 0.001)	(5.218)		(< 0.001)	(5.032)		(< 0.001)	(5.123)	
0.050	208.369	208.369	207.868	209.146	209.146	208.645	209.658	209.658	209.157
	(< 0.001)	(6.859)		(< 0.001)	(5.894)		(< 0.001)	(6.016)	
0.075	159.519	159.519	159.018	160.388	160.388	159.887	160.961	160.961	160.460
	(< 0.001)	(5.631)		(< 0.001)	(6.153)		(< 0.001)	(6.166)	
0.100	123.618	123.618	123.117	124.493	124.493	123.992	125.069	125.069	124.568
	(< 0.001)	(7.297)		(< 0.001)	(6.798)		(< 0.001)	(7.036)	
0.125	96.899	96.899	96.398	97.735	97.735	97.234	98.286	98.286	97.785
	(< 0.001)	(4.172)		(< 0.001)	(7.036)		(< 0.001)	(5.765)	
0.150	76.780	76.780	76.278	77.554	77.554	77.052	78.065	78.065	77.563
	(< 0.001)	(5.879)		(< 0.001)	(6.659)		(< 0.001)	(4.555)	
0.175	61.432	61.432	60.930	62.167	62.167	61.665	62.633	62.633	62.131
	(< 0.001)	(4.484)		(< 0.001)	(8.556)		(< 0.001)	(4.065)	
0.200	49.679	49.679	49.176	50.313	50.313	49.810	50.733	50.733	50.231
	(< 0.001)	(4.484)		(< 0.001)	(8.556)		(< 0.001)	(4.065)	
EARL	42,081.40	42,081.40		42,322.52	42,322.52		42,480.80	42,480.80	
$\%Diff_A$	0.0000			0.0027			0.0000		

Table 4. Comparison of the explicit formulas and NIE methods for a long-memory ARFIX(1, *d*, 1) process on a EWMA control chart for ϕ_1 = 0.1, ω_1 = 0.3, and ARL_0 = 500

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
π = 0.03	h = 2.12414×10^{-14}			h = 2.34030×10^{-14}			h = 2.49351×10^{-14}		
δ	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL
0.025	212.464	212.463	211.963	212.949	212.949	212.448	213.288	213.288	212.787
	(< 0.001)	(5.159)		(< 0.001)	(4.646)		(< 0.001)	(5.097)	
0.050	94.299	94.296	93.798	94.725	94.724	94.224	95.010	95.010	94.509
	(< 0.001)	(5.800)		(< 0.001)	(5.287)		(< 0.001)	(5.469)	
0.075	43.738	43.735	43.235	44.022	44.022	43.519	44.214	44.214	43.711
	(< 0.001)	(4.409)		(< 0.001)	(5.897)		(< 0.001)	(6.031)	
0.100	21.273	21.272	20.767	21.451	21.450	20.945	21.569	21.569	21.063
	(< 0.001)	(4.956)		(< 0.001)	(6.615)		(< 0.001)	(5.559)	
0.125	10.935	10.935	10.423	11.043	11.042	10.531	11.114	11.114	10.602
	(< 0.001)	(6.628)		(< 0.001)	(5.287)		(< 0.001)	(4.659)	
0.150	6.020	6.020	5.497	6.084	6.084	5.562	6.126	6.126	5.604
	(< 0.001)	(7.331)		(< 0.001)	(7.974)		(< 0.001)	(4.239)	
0.175	3.610	3.610	3.070	3.648	3.648	3.108	3.673	3.673	3.133
	(< 0.001)	(5.184)		(< 0.001)	(7.024)		(< 0.001)	(5.942)	
0.200	2.394	2.394	1.827	2.417	2.417	1.851	2.432	2.432	1.866
	(< 0.001)	(4.049)		(< 0.001)	(7.677)		(< 0.001)	(6.018)	
EARL	15,789.32	15.789.00		15,853.56	15,853.44		15,897.04	15,897.04	
$\%Diff_A$	0.000			0.0008			0.000		
π = 0.05	h = 2.18584×10^{-8}			h = 2.40840×10^{-8}			h = 2.56601×10^{-8}		
δ	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL	ARL_{Exp}	ARL_{NIE}	SDRL
0.025	293.727	293.727	293.227	294.422	294.422	293.922	294.874	294.874	294.374
	(< 0.001)	(5.721)		(< 0.001)	(4.473)		(< 0.001)	(4.035)	
0.050	177.037	177.037	176.536	177.852	177.852	177.351	178.386	178.386	177.885
	(< 0.001)	(4.377)		(< 0.001)	(5.161)		(< 0.001)	(5.119)	
0.075	109.337	109.337	108.836	110.073	110.073	109.572	110.556	110.556	110.055
	(< 0.001)	(5.081)		(< 0.001)	(6.817)		(< 0.001)	(6.059)	

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
0.100	69.124	69.124	68.622	69.728	69.728	69.226	70.125	70.125	69.623
	(< 0.001)	(5.704)		(< 0.001)	(6.520)		(< 0.001)	(6.113)	
0.125	44.708	44.708	44.205	45.182	45.182	44.679	45.494	45.494	44.991
	(< 0.001)	(4.314)		(< 0.001)	(8.177)		(< 0.001)	(5.598)	
0.150	29.576	29.576	29.072	29.940	29.940	29.436	30.180	30.180	29.676
	(< 0.001)	(6.924)		(< 0.001)	(8.833)		(< 0.001)	(4.789)	
0.175	20.014	20.014	19.508	20.291	20.291	19.785	20.474	20.474	19.968
	(< 0.001)	(5.218)		(< 0.001)	(7.219)		(< 0.001)	(4.528)	
0.200	13.863	13.863	13.354	14.073	14.073	13.564	14.211	14.211	13.702
	(< 0.001)	(5.517)		(< 0.001)	(4.489)		(< 0.001)	(4.465)	
EARL	30,295.44	30,295.44		30,462.44	30,462.44		30,572.00	30,572.00	
%Diff$_A$	0.0000			0.0000			0.0000		
π = 0.10	h = 9.10991×10^{-4}			h = 1.00420×10^{-3}			h = 1.07024×10^{-3}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	373.072	373.072	372.572	374.013	374.013	372.572	374.619	374.619	374.119
	(< 0.001)	(4.876)		(< 0.001)	(6.349)		(< 0.001)	(4.128)	
0.050	282.239	282.239	281.739	283.621	283.621	281.739	284.849	284.849	284.349
	(< 0.001)	(5.470)		(< 0.001)	(5.958)		(< 0.001)	(5.897)	
0.075	216.294	216.294	215.793	217.838	217.838	215.793	218.849	218.849	218.348
	(< 0.001)	(7.079)		(< 0.001)	(6.551)		(< 0.001)	(6.595)	
0.100	167.767	167.767	167.266	169.322	169.322	167.266	170.343	170.343	169.842
	(< 0.001)	(5.751)		(< 0.001)	(5.192)		(< 0.001)	(5.843)	
0.125	131.607	131.607	131.106	133.092	133.092	131.106	134.069	134.069	133.568
	(< 0.001)	(6.607)		(< 0.001)	(7.864)		(< 0.001)	(6.105)	
0.150	104.344	104.344	103.843	105.722	105.722	103.843	106.629	106.629	106.128
	(< 0.001)	(5.078)		(< 0.001)	(6.536)		(< 0.001)	(4.898)	
0.175	83.5624	83.5624	83.061	84.817	84.817	83.061	85.647	85.647	85.146
	(< 0.001)	(5.612)		(< 0.001)	(5.365)		(< 0.001)	(5.147)	
0.200	67.557	67.557	67.055	68.688	68.688	67.055	69.436	69.436	68.934
	(< 0.001)	(4.703)		(< 0.001)	(4.192)		(< 0.001)	(4.236)	
EARL	57,057.70	57,057.70		57,484.52	57,484.52		57,777.64	57,777.64	
%Diff$_A$	0.0000			0.0000			0.0000		

Table 5. Comparison of the explicit formulas and NIE methods for a long-memory ARFIX(1, *d*, 1) process on a EWMA control chart for ϕ_1 = 0.2, ω_1 = 0.3, and ARL$_0$ = 500

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
π = 0.03	h = 2.02200×10^{-14}			h = 2.20374×10^{-14}			h = 2.33164×10^{-14}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	212.238	212.238	211.737	212.646	212.646	212.145	212.963	212.963	212.462
	(< 0.001)	(54.009)		(< 0.001)	(4.743)		(< 0.001)	(4.556)	
0.050	94.091	94.087	93.590	94.453	94.453	93.952	94.714	94.714	94.213
	(< 0.001)	(5.793)		(< 0.001)	(5.396)		(< 0.001)	(5.255)	
0.075	43.591	43.592	43.088	43.845	43.845	43.342	44.016	44.016	43.513
	(< 0.001)	(5.481)		(< 0.001)	(6.005)		(< 0.001)	(4.592)	
0.100	21.183	21.183	20.677	21.339	21.339	20.833	21.445	21.445	20.939
	(< 0.001)	(4.044)		(< 0.001)	(6.599)		(< 0.001)	(5.103)	
0.125	10.881	10.881	10.369	10.975	10.975	10.463	11.040	11.040	10.528
	(< 0.001)	(6.653)		(< 0.001)	(5.192)		(< 0.001)	(4.596)	
0.150	5.988	5.988	5.465	6.044	6.044	5.521	6.081	6.081	5.559
	(< 0.001)	(5.263)		(< 0.001)	(7.833)		(< 0.001)	(5.123)	
0.175	3.591	3.591	3.050	3.624	3.624	3.084	3.647	3.647	3.107
	(< 0.001)	(5.364)		(< 0.001)	(7.124)		(< 0.001)	(5.891)	
0.200	2.383	2.382	1.815	2.403	2.403	1.836	2.416	2.416	1.850
	(< 0.001)	(4.857)		(< 0.001)	(6.426)		(< 0.001)	(6.053)	
EARL	15,757.84	15,757.68		15,813.16	15,813.16		15,852.88	15.852.88	
%Diff$_A$	0.0010			0.0000			0.0000		

π = 0.05	h = 2.08060×10^{-8}			h = 2.26780×10^{-8}			h = 2.39925×10^{-8}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	293.386	293.386	292.886	293.993	293.993	293.493	294.394	294.394	293.894
	(< 0.001)	(5.467)		(< 0.001)	(4.863)		(< 0.001)	(4.025)	
0.050	176.629	176.629	176.128	177.347	177.347	176.846	177.819	177.819	177.318
	(< 0.001)	(4.092)		(< 0.001)	(5.488)		(< 0.001)	(5.138)	
0.075	108.969	108.969	108.468	109.616	109.616	109.115	110.044	110.044	109.543
	(< 0.001)	(5.748)		(< 0.001)	(6.129)		(< 0.001)	(6.023)	
0.100	68.822	68.822	68.320	69.353	69.353	68.851	69.704	69.704	69.202
	(< 0.001)	(4.388)		(< 0.001)	(5.769)		(< 0.001)	(5.556)	
0.125	44.471	44.471	43.968	44.888	44.888	44.385	45.163	45.163	44.660
	(< 0.001)	(5.982)		(< 0.001)	(6.379)		(< 0.001)	(5.407)	
0.150	29.393	29.393	28.889	29.713	29.713	29.209	29.925	29.925	29.421
	(< 0.001)	(4.623)		(< 0.001)	(4.051)		(< 0.001)	(4.789)	
0.175	19.876	19.876	19.370	20.119	20.119	19.613	20.280	20.280	19.774
	(< 0.001)	(5.289)		(< 0.001)	(5.263)		(< 0.001)	(5.268)	
0.200	13.758	13.758	13.249	13.942	13.942	13.433	14.064	14.064	13.555
	(< 0.001)	(6.217)		(< 0.001)	(5.707)		(< 0.001)	(7.125)	
EARL	30,212.16	30,212.16		30,358.84	30,358.84		30,455.72	30,455.72	
%Diff$_A$	0.0000			0.0000			0.0000		
π = 0.10	h = 8.66915×10^{-4}			h = 9.45300×10^{-4}			h = 1.00035×10^{-3}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	372.596	372.596	372.096	373.428	373.428	372.928	373.890	373.890	373.390
	(< 0.001)	(5.639)		(< 0.001)	(4.270)		(< 0.001)	(4.566)	
0.050	281.542	281.542	281.042	282.763	282.763	282.263	283.563	283.563	283.063
	(< 0.001)	(4.295)		(< 0.001)	(5.879)		(< 0.001)	(5.633)	
0.075	215.514	215.514	215.013	216.879	216.879	216.378	217.774	217.774	217.273
	(< 0.001)	(5.967)		(< 0.001)	(4.488)		(< 0.001)	(4.896)	
0.100	166.982	166.982	166.481	168.355	168.355	167.854	169.258	169.258	168.757
	(< 0.001)	(4.608)		(< 0.001)	(7.145)		(< 0.001)	(5.039)	
0.125	130.858	130.858	130.357	132.169	132.169	131.668	133.032	133.032	132.531
	(< 0.001)	(6.311)		(< 0.001)	(5.754)		(< 0.001)	(5.987)	
0.150	103.651	103.651	103.150	104.865	104.865	104.364	105.666	105.666	105.165
	(< 0.001)	(4.998)		(< 0.001)	(6.379)		(< 0.001)	(6.023)	
0.175	82.932	82.932	82.430	84.036	84.036	83.535	84.766	84.766	84.265
	(< 0.001)	(5.641)		(< 0.001)	(5.267)		(< 0.001)	(5.992)	
0.200	66.989	66.989	66.487	67.984	67.984	67.482	68.642	68.642	68.140
	(< 0.001)	(5.685)		(< 0.001)	(4.035)		(< 0.001)	(6.113)	
EARL	56,842.56	56,842.56		57,219.16	57,219.16		57,463.64	57,463.64	
%Diff$_A$	0.0000			0.0000			0.0000		

Table 6. Comparison of the explicit formulas and NIE methods for a long-memory ARFIX(1, *d*, 1) process on a EWMA control chart for ϕ_1 = 0.3, ω_1 = 0.3, and ARL$_0$ = 500

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
π = 0.03	h = 1.92430×10^{-14}			h = 2.07518×10^{-14}			h = 2.18010×10^{-14}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	211.961	211.946	211.460	212.366	212.366	211.865	212.612	212.612	212.111
	(< 0.001)	(5.078)		(< 0.001)	(5.025)		(< 0.001)	(4.563)	
0.050	93.849	93.855	93.348	94.199	94.199	93.698	94.420	94.420	93.919
	(< 0.001)	(4.719)		(< 0.001)	(4.988)		(< 0.001)	(5.059)	
0.075	43.437	43.440	42.934	43.669	43.669	43.166	43.816	43.816	43.313
	(< 0.001)	(4.406)		(< 0.001)	(6.025)		(< 0.001)	(5.129)	
0.100	21.089	21.089	20.583	21.229	21.229	20.723	21.319	21.319	20.813
	(< 0.001)	(5.141)		(< 0.001)	(5.265)		(< 0.001)	(4.896)	
0.125	10.827	10.827	10.315	10.909	10.909	10.397	10.964	10.964	10.452
	(< 0.001)	(6.876)		(< 0.001)	(5.098)		(< 0.001)	(6.025)	
0.150	5.956	5.956	5.433	6.005	6.005	5.482	6.038	6.038	5.515

Long-memory	ARFIX(1, 1/3, 1)			ARFIX(1, 1/4, 1)			ARFIX(1, 1/5, 1)		
	(< 0.001)	(4.673)		(< 0.001)	(4.156)		(< 0.001)	(5.058)	
0.175	3.572	3.572	3.031	3.601	3.601	3.060	3.621	3.621	3.081
	(< 0.001)	(5.851)		(< 0.001)	(5.951)		(< 0.001)	(4.641)	
0.200	2.371	2.371	1.803	2.389	2.389	1.822	2.400	2.400	1.833
	(< 0.001)	(5.361)		(< 0.001)	(5.261)		(< 0.001)	(4.569)	
EARL	15,722.48	15,722.24		15,774.68	15,774.68		15,807.60	15,807.60	
%Diff$_A$	0.0015			0.0000			0.0000		
π = 0.05	h = 1.98030×10^{-8}			h = 2.13540×10^{-8}			h = 2.24331×10^{-8}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	293.027	293.027	292.527	293.563	293.563	293.063	293.913	293.913	293.413
	(< 0.001)	(5.923)		(< 0.001)	(4.651)		(< 0.001)	(4.459)	
0.050	176.214	176.214	175.713	176.843	176.843	176.342	177.255	177.255	176.754
	(< 0.001)	(4.579)		(< 0.001)	(4.595)		(< 0.001)	(5.028)	
0.075	108.595	108.595	108.094	109.161	109.161	108.660	109.533	109.533	109.032
	(< 0.001)	(5.267)		(< 0.001)	(5.057)		(< 0.001)	(5.996)	
0.100	68.517	68.517	68.015	68.981	68.981	68.479	69.285	69.285	68.783
	(< 0.001)	(4.956)		(< 0.001)	(5.988)		(< 0.001)	(6.098)	
0.125	44.232	44.232	43.729	44.595	44.595	44.092	44.835	44.835	44.332
	(< 0.001)	(6.658)		(< 0.001)	(6.026)		(< 0.001)	(6.156)	
0.150	29.210	29.210	28.706	29.489	29.489	28.985	29.673	29.673	29.169
	(< 0.001)	(5.330)		(< 0.001)	(6.138)		(< 0.001)	(5.456)	
0.175	19.737	19.737	19.231	19.948	19.948	19.442	20.088	20.088	19.582
	(< 0.001)	(5.127)		(< 0.001)	(5.492)		(< 0.001)	(5.371)	
0.200	13.653	13.653	13.143	13.813	13.813	13.304	13.919	13.919	13.410
	(< 0.001)	(4.111)		(< 0.001)	(4.098)		(< 0.001)	(4.591)	
EARL	30,127.40	30,127.40		30,255.72	30.255.72		30,340.04	30,340.04	
%Diff$_A$	0.0000			0.0000			0.0000		
π = 0.10	h = 8.24990×10^{-4}			h = 8.89880×10^{-4}			h = 9.35049×10^{-4}		
δ	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL	ARL$_{Exp}$	ARL$_{NIE}$	SDRL
0.025	372.126	372.126	371.626	372.849	372.849	372.349	373.322	373.322	372.822
	(< 0.001)	(5.861)		(< 0.001)	(4.036)		(< 0.001)	(4.779)	
0.050	280.849	280.849	280.349	281.912	281.912	281.412	282.608	282.608	282.108
	(< 0.001)	(4.564)		(< 0.001)	(5.368)		(< 0.001)	(5.035)	
0.075	214.739	214.739	214.238	215.926	215.926	215.425	216.705	216.705	216.204
	(< 0.001)	(5.236)		(< 0.001)	(5.036)		(< 0.001)	(5.296)	
0.100	166.204	166.204	165.703	167.397	167.397	166.896	168.181	168.181	167.680
	(< 0.001)	(5.876)		(< 0.001)	(6.048)		(< 0.001)	(6.056)	
0.125	130.116	130.116	129.615	131.253	131.253	130.752	132.002	132.002	131.501
	(< 0.001)	(6.533)		(< 0.001)	(5.984)		(< 0.001)	(5.469)	
0.150	102.964	102.964	102.463	104.016	104.016	103.515	104.711	104.711	104.210
	(< 0.001)	(5.221)		(< 0.001)	(4.654)		(< 0.001)	(4.978)	
0.175	82.307	82.307	81.805	83.264	83.264	82.762	83.896	83.896	83.395
	(< 0.001)	(5.476)		(< 0.001)	(4.628)		(< 0.001)	(4.514)	
0.200	66.428	66.428	65.926	67.288	67.288	66.786	67.857	67.857	67.355
	(< 0.001)	(4.908)		(< 0.001)	(4.365)	372.349	(< 0.001)	(4.998)	
EARL	56,929.32	56,929.32		56,956.20	56,956.20		57,171.28	57,171.28	
%Diff$_A$	0.0000			0.0000			0.0000		

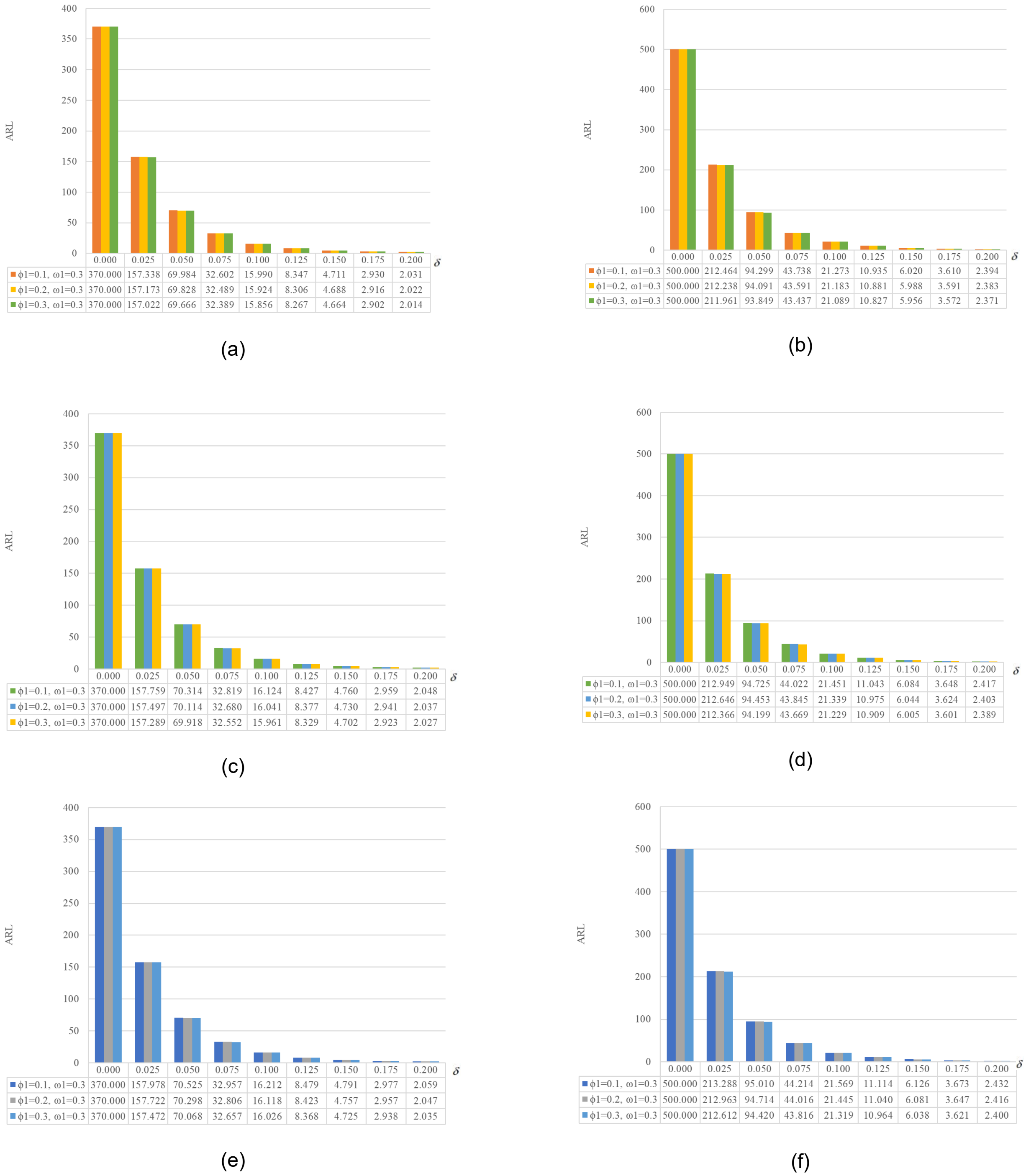

Figure 1. Graphical presentations of the ARL_1 results for various long-memory ARFIX processes with exponential white noise on a EWMA control chart when $\pi = 0.01$: (a) ARFIX(1, 1/3, 1) for $ARL_0 = 370$, (b) ARFIX(1, 1/3, 1) for $ARL_0 = 500$, (c) ARFIX(1, 1/4, 1) for $ARL_0 = 370$, (d) ARFIX(1, 1/4, 1) for $ARL_0 = 500$, (e) ARFIX(1, 1/5, 1) for $ARL_0 = 370$, and (f) ARFIX(1, 1/5, 1) for $ARL_0 = 500$

Results and Discussion

The ARL values acquired by using the explicit formula and the NIE methods are provided in Tables 1–6. These methods demonstrated sensitivity in promptly detecting slight shifts in the mean of various long-memory processes on an upper-side EWMA control chart. Notably, the ARL_1 values decreased rapidly as the shift size (δ) was incremented. The ARL_1 values for a long-memory ARFIX(1, *d*, 1) process for ϕ_1 = 0.1, ω_1 = 0.3, and ARL_0 = 370 for smoothing constant π = 0.05 and different shift values (δ = 0.025, 0.050, 0.075, 0.1, 0.125, 0.150, 0.175, or 0.200) are 217.466, 131.176, 81.113, 51.377, 33.321, 22.131, 15.061 and 10.512 and 217.466, 131.176, 81.113, 51.377, 33.321, 22.131, 15.061, and 10.512 for the NIE and explicit equations methods, respectively. It can be seen that the ARL_1 values obtained using both methods were similar and rapidly declined as the shift size was decreased and were affected by changing the smoothing constant (i.e., $\pi = 0.03,$ 0.05, or 0.10). For example, for ARL_0 = 370, π = 0.03, and δ = 0.025, the ARL_1 values were 157.338, 157.759, and 157.978 for ARFIX(1, 1/3, 1), ARFIX(1, 1/4, 1), and ARFIX(1, 1/5, 1), respectively. For the same value for δ (0.025) but varying π (0.05), the ARL_1 values were 217.466, 217.978, and 218.314, respectively. The $SDRL_1$ results were similar to the trend in ARL_1 albeit slightly lower for any δ value. Increasing the autocorrelation (ϕ_1) had a negative influence on the ARL_1 results, causing them to decline. Therefore, the methods both performed efficiently (Figure 1).

The results of $EARL_{Exp}$ and $EARL_{NIE}$, as determined by using Equation (20), exhibited strong similarity in each of the examined scenarios. It is evident that when ϕ_1 was increased, the $EARL_{Exp}$ and $EARL_{NIE}$ results also increased, yielding the same results as the ARL_1 trend. Moreover, the $\%Diff_A$ results were close to 0, indicating excellent agreement between the two methods.

In Tables 1–6, the values in parentheses representing the computing times for ARL_1 indicate that those using the explicit formulas were significantly shorter (only a fraction of a second) than those using the NIE method (around 4–7 seconds) for all of the scenarios examined. These results are in agreement with the research conducted by Sunthornwat and Areepong [23].

An Illustration of the Efficacy of the ARL Methods with Real Data

The ARL and SDRL for changes in the mean of the ARFIX(1, *d*, 1) processes with exponential white noise on a EWMA control were calculated using both methods. The dataset used satisfactorily met the requirements for this scenario.

A dataset of prices for the SCB Gold THB Hedged Open-Ended Fund (SCBGOLDH) is a mutual fund primarily investing in gold while hedging against fluctuations in the foreign exchange rate between Thai Baht (THB) and US dollars (USD) consisted of 55 observations from December 2019 to June 2024 (https://www.investing.com) (Table 7). The applicability of a long-memory ARFIX(*p, d, r*) process with the exchange rates as the exogenous variable was investigated using the Eviews 10 statistical software package. The dataset is evaluated for fit to the ARFIX model by the t-statistic.

Table 7. The parameter estimates for the long-memory ARFIX(*p, d, r*) process using the financial dataset

	Parameters		
	AR(1)	*d*	THB/USD
Estimate	0.639764	0.499998	190.0025
Std. Error	0.117598	8.46E-06	65.4223
t-Statistic	5.440269	59075.78	2.904242
p-value	0.0000*	0.0000*	0.0054*

*A significance level of 0.05.

Table 8. One-sample Kolmogorov test results for the financial dataset

Residual of ARFIX(1, 0.499998, 1) process	
Mean parameter [a]	0.3138
One-sample Kolmogorov-Smirnov test	0.9680
p-value	0.3050^{ns}

a. Test Distribution is Exponential.

ns A nonsignificance level of 0.05.

The last row of Table 7 shows the *p*-values of each parameter are calculated and it is found to be 0.0054, 0, 0, respectively (*p*-value < 0.05) indicating that parameter estimates are statistically significant in terms of long memory ARFIX(*p, d, r*) process. The estimate of long memory parameter (*d*) is found to be 0.499998. After confirming the presence of long memory ARFIX(*p, d, r*) process and coefficient parameter of ARFIX(1, 0.499998, 1) process is estimated as $\hat{\phi}_1$ = 0.639764, $\hat{\omega}$ = 190.0025, where λ_0 = 0.3138 is the mean parameter of exponential white noise was then tested by one-sample Kolmogorov-Smirnov test (*p*-value < 0.05), see Table 8.

Hence, the explicit formulas for the in-control ARL provided

$$\mathrm{ARL}_0 = 1 - \frac{e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda_0}\right)}\pi(e^{-\frac{h}{\pi\lambda_0}}-1)}{\pi e^{\left(-\frac{1}{\lambda_1}(\omega_1 X_{1t}+\phi_1 Y_{t-1}-0.499998\phi_1 Y_{t-2}-0.1249\phi_1 Y_{t-3}-0.0625\phi_1 Y_{t-4}+0.499998 Y_{t-1}+0.1249 Y_{t-2}+0.0625 Y_{t-3})\right)}+(e^{-\frac{h}{\lambda_0}}-1)}. \quad (22)$$

and for out-of-control ARL provided

$$\mathrm{ARL}_1 = 1 - \frac{e^{\left(\frac{(1-\pi)\varphi}{\pi\lambda_1}\right)}\pi(e^{-\frac{h}{\pi\lambda_1}}-1)}{\pi e^{\left(-\frac{1}{\lambda_1}(\omega_1 X_{1t}+\phi_1 Y_{t-1}-0.499998\phi_1 Y_{t-2}-0.1249\phi_1 Y_{t-3}-0.0625\phi_1 Y_{t-4}+0.499998 Y_{t-1}+0.1249 Y_{t-2}+0.0625 Y_{t-3})\right)}+(e^{-\frac{h}{\lambda_1}}-1)}. \quad (23)$$

where $\lambda_1 = \lambda_0(1+\delta)$, for δ = 0.025, 0.05, 0.075, 0.10, 0.125, 0.15, 0.2, 0.3, or 0.5.

Furthermore, using Equation (17), we approximated the ARL as

$$\hat{L}(\varphi) \approx 1 + \frac{1}{\pi}\sum_{k=1}^{m} w_k \hat{L}(\upsilon_k) f\left(\frac{\upsilon_k-(1-\pi)\varphi}{\pi} - \left(\omega_1 X_{1t} + \phi_1 Y_{t-1} - 0.499998\phi_1 Y_{t-2} - 0.1249\phi_1 Y_{t-3} - 0.0625\phi_1 Y_{t-4} + 0.499998 Y_{t-1} + 0.1249 Y_{t-2} + 0.0625 Y_{t-3}\right)\right) \quad (24)$$

Table 9. The ARL_1 results for the explicit formulas and NIE methods for a long-memory ARFIX(1, 0.499998, 1) process on an EWMA control chart: ϕ_1 = 0.639764 and ω_1 = 190.0025

λ = 0.10	ARL_0 = 370					ARL_0 = 500				
	h = 0.421					h = 0.585				
Methods	Explicit formula		NIE		SDRL	Explicit formula		NIE		SDRL
δ	ARL_{Exp}	Time	ARL_{NIE}	Time		ARL_{Exp}	Time	ARL_{NIE}	Time	
0.025	320.627	(< 0.001)	320.618	(16.346)	320.127	430.100	(< 0.001)	430.076	(41.205)	429.600
0.050	279.785	(< 0.001)	279.777	(22.986)	279.285	372.691	(< 0.001)	372.672	(42.441)	372.191
0.075	245.735	(< 0.001)	245.729	(20.768)	245.234	325.158	(< 0.001)	325.142	(45.121)	324.658
0.100	217.142	(< 0.001)	217.137	(25.455)	216.641	285.501	(< 0.001)	285.487	(46.939)	285.001
0.125	192.967	(< 0.001)	192.962	(27.624)	192.466	252.180	(< 0.001)	252.169	(50.180)	251.680
0.150	172.396	(< 0.001)	172.392	(29.970)	171.895	223.997	(< 0.001)	223.987	(52.192)	223.496
0.175	154.788	(< 0.001)	154.784	(33.263)	154.287	200.010	(< 0.001)	200.002	(53.580)	199.509
0.200	139.630	(< 0.001)	139.627	(34.542)	139.129	179.475	(< 0.001)	179.467	(54.591)	178.974
EARL ($0 < \delta \leq 0.2$)	68,922.80		68,921.04			90,764.46		90,760.08		
%DiffA	0.0026					0.0048				
0.225	126.513	(< 0.001)	126.512	(35.421)	126	161.797	(< 0.001)	161.794	(54.873)	161.296
0.250	115.106	(< 0.001)	115.105	(35.124)	114.6	146.500	(< 0.001)	146.498	(54.467)	145.999
0.275	105.138	(< 0.001)	105.137	(37.187)	104.6	133.198	(< 0.001)	133.196	(54.248)	132.697
0.300	96.389	(< 0.001)	96.389	(31.812)	95.88	121.578	(< 0.001)	121.576	(53.779)	121.077
0.400	70.390	(< 0.001)	70.390	(36.171)	69.88	87.398	(< 0.001)	87.397	(53.435)	86.897
0.500	53.793	(< 0.001)	53.793	(35.155)	53.28	65.922	(< 0.001)	65.921	(54.921)	65.421
1.000	21.977	(< 0.001)	21.972	(30.624)	21.47	25.978	(< 0.001)	25.978	(53.185)	25.477

The results in Table 9 reveal that both methods provided similar ARL_1 results for the ARFIX process involving the real dataset, both of which declined for small to moderate shift sizes. Moreover, the EARL values for both methods were similar and $\%Diff_A < 0.25$, demonstrating their equivalent ability for every shift size. Notably, the computation times for the explicit formulas method were appreciably quicker than for the NIE method. Finally, the results for ARL_0 = 370 and 500 were consistent with those in Tables 1–6.

Conclusions

We provided a method for calculating the ARL for a long-memory ARFIX process with exponential white noise on an EWMA control chart using explicit formulas and investigated its efficacy for various smoothing constant and shift level values. We compared the out-of-control ARL values obtained using the explicit formulas with those obtained using the NIE method, the results of which were in good agreement, a finding also reflected in the SDRL results. However, the computation times using the explicit formulas were much shorter than when using the NIE method. The empirical results indicate that the ARL calculated by using explicit formulas performed better on the EWMA control chart than on the NIE method for various processes. It is important to note that this study's applicability is limited to long-memory ARFIX processes with exponential white noise on an EWMA control chart. Future research avenues could establish the relevance of our method for enhancing the detection ability of process parameter shifts in other processes [24] on an EWMA control chart and hybrid EWMA control charts.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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