MJFAS
Malaysian Journal of Fundamental and Applied Sciences

Rafiullah *et al.* | Malaysian Journal of Fundamental and Applied Sciences, Vol. 20 (2024) 1392-1397

RESEARCH ARTICLE

Some Mathematical Properties of Sombor Indices for Regular Graphs

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Abstract In 2021, Gutman introduced Sombor index (SO) of a graph G and defined as $SO(G) = \sum_{uv\in E(G)} \sqrt{deg(u)^2 + deg(v)^2}$. In this paper, we have calculated the Sombor index of *r*-regular graph G_r, line graph of G_r, $L(G_r)$ and complement graph of G_r, $\overline{G_r}$. We have also discussed a particular case of regular graphs, generalized Petersen graph $P(s,t)$, its line graph $L((P(s,t))$ and complement graph $\overline{P(s,t)}$ for $s > 4$. We have proved the relation between these graphs and categorized them on the base of the Sombor index.

Keywords: Topological indices, Sombor index, regular graph, Petersen graph.

Introduction

Categorization of objects, structures and networks is essential in science, technology and especially in daily life. In this process, a category (or property) is assigned to an object and further, this object is recognized by the assigned category or property. The *r*-regular graphs or networks are graphs in which every vertex has the same degree (number of edges connected to a vertex). There are many regular graphs such as 0-regular graph which is a graph without edges, 1-regular graph is a disconnected graph ($\frac{n}{2}$ components if n is even, only a pair of vertices connected but not with any other), 2-regular graph is a cycle graph, 3-regular graph referred as a cubic graph, and $(n-1)$-regular graph is a complete graph, K_n. The Petersen graph and Balaban 10-cage graph are also 3-regular graphs.

Consider a connected graph G having vertices and edges set which can be represented by $=V(G)$ and $E(G)$, respectively. The cardinality of these sets can referred as $n = |V(G)|$ and $m = |E(G)|$, respectively. The vertices x and y connected by an edge e (where $x \neq y$) are shown as $e = xy$. The number of edges connected to a vertex x is known as degree and it is shown as $\deg(x)$. The minimum degree, $\delta(G)$ and maximum degree of G, $\Delta(G)$ follow the relation $0 \leq \delta(G) \leq deg(v) \leq \Delta(G) \leq n-1$. A regular graph G holds a relation $\delta(G) = \Delta(G)$. If $deg(v) = r$ then the graph is called an *r*-regular graph and $0 \leq r \leq n-1$.

Theorem 1: [1] Consider *r* and *n* to be two integers holding the relation $0 \leq r \leq n-1$. If *r* or *n* is even then there exists an *r*-regular graph of order *n*.

A line graph *H* of graph *G* can be constructed by replacing all edges with vertices, further joining this set of vertices of *H* by considering the corresponding edges of *G* and their incident edges. The inverse graph *H* of graph *G* is a graph where the pair of different vertices of *H* joined together if and only if they are not joined in *G*. The second name of the inverse graph is complementing graph. The line graph and complement graph of *G* are referred to respectively as $L(G)$ and $\bar{G}$ (or G'). More detail about line graph reader can reads in [2 – 6].

The Petersen graph $P(5,2)$ [7 - 9] is a simple connected graph with order 10 and size 15, named after the Danish mathematician Julius Petersen. In 1969, Watkins defined the generalized Petersen graph in [9] as, for integers *s* and *t* satisfying $1 \leq t \leq s-1$, $2t \neq s$, one states the generalized Petersen graph $P(s,t)$ with vertex set $v(P(s,t)) = \{x_0, x_1, \dots, x_{s-1}, y_0, y_1, \dots, y_{s-1}\}$, and the set of edges $E(P(s,t))$ consisting of all those of the form $[y_i, y_{i+t}]$, $[x, y_i]$, $[x_i, x_{i+1}]$, where i is an integer.

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Received: 03 Sept. 2024
Accepted: 22 Oct. 2024

For readers we present the graphs of the Petersen graph $P(5,2)$ in Figure 1, its line graph $L(P(5,2))$ in Figure 2 and complement graph $\overline{P(5,2)}$ in Figure 3.

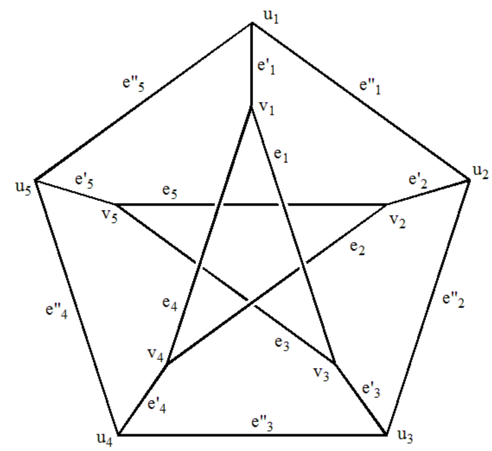

Figure 1. Petersen graph $P(5,2)$

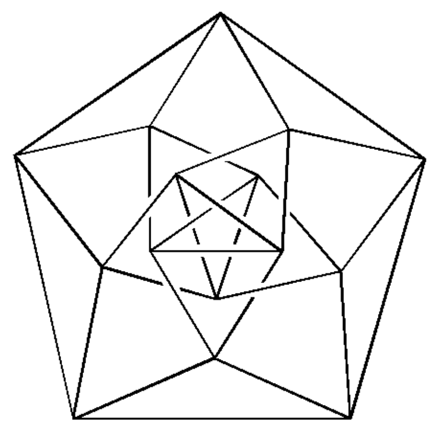

Figure 2. Line graph of Petersen $L(P(5,2))$

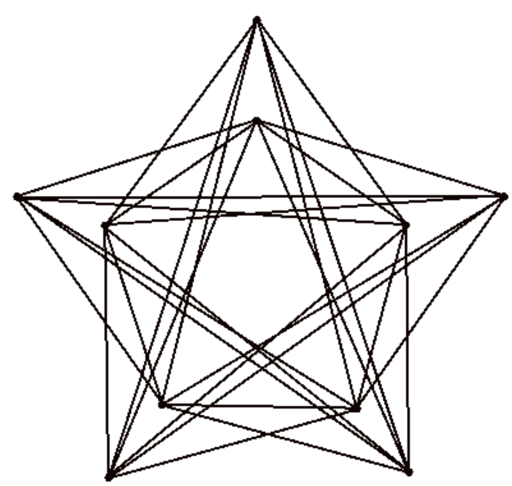

Figure 3. Complement graph of Petersen $\overline{P(5,2)}$

In 2021 [10], Gutman proposed the Sombor index of graph G which is described as,

$$SO(G) = \sum_{uv\in E(G)} \sqrt{deg(u)^2 + deg(v)^2} \quad (1)$$

Gutman also discussed some Mathematical properties of the Sombor index in the same paper as:

Theorem 2: [10] Let P_n, K_n and S_n be a path graph, complete graph, and Star graph of order n, respectively. Then for any graph G or tree graph T of order n hold the following conditions,
1. $SO(\overline{K_n}) \leq SO(G) \leq SO(K_n)$ if and only if $G \cong (\overline{K_n})$ or $G \cong K_n$,
2. $SO(P_n) \leq SO(G) \leq SO(K_n)$ if and only if $G \cong P_n$ or $G \cong K_n$,
3. $SO(P_n) \leq SO(T) \leq SO(S_n)$ if and only if $T \cong P_n$ or $T \cong S_n$.

Recently, Liu *et al*.[11] have provided minimum Sombor indices of tetracyclic graphs. Alidadi *et al*. [12] computed the minimum Sombor index for unicyclic graphs with constant diameter. Aashtab *et al*. [13] studied some minimum and maximum properties relating to the degree of graph. Fang *et al*. [14] have characterized the physicochemical properties of polycyclic aromatic compounds by using the Sombor index. Aguilar-Sanchez *et al*. [15] have proposed Shannon entropy with the Sombor index. Shang [16] has studied simplicial networks using the Sombor index. Ulker *et al*. [17] have found a relationship between energy and the Sombor index. Rather and Imran [18] have calculated bounds for the Sombor energy of graph G. Rather *et al*. have studied commaximal grapghs of commutative rings with the help of Sombor index and eigenvalues and they have provided bounds [19]. We suggest some published articles on the emerging theory of Sombor indices and their usage [10, 20 – 27]

In this paper, we will discuss Sombor index of an *r*-regular graph G_r, the line graph of G_r, $L(G_r)$ and the complement of graph G_r, $\overline{G_r}$. We also present a particular case of *r*-regular graph known as the generalized Petersen graph $P(s,t)$, the line graph of $P(s,t)$, $L(P(s,t))$ and the complement of $P(s,t)$, $\overline{P(s,t)}$ for $s \geq 3$. We calculate their bounds and categorize them.

Main Results

The Sombor index Eq. (1) can be written for *r*-regular graphs with m number of edges

$$SO(G_r) = m\sqrt{r^2 + r^2} = mr\sqrt{2} \quad (2)$$

Now, we prove some theorems for *r*-regular graphs using the Sombor index formula.

Theorem 3: Consider a connected *r*-regular graph G_r of order $n \geq 3$. Then for any graph G_r we have:
1. $SO(G_r) = nr^2 \times \frac{\sqrt{2}}{2}$,
2. $SO(L(G_r)) = nr(2r-2)^2 \times \frac{\sqrt{2}}{4}$,
3. $SO(\overline{G_r}) = (n-r-1)(2n^2-4n) \times \frac{\sqrt{2}}{2}$.

Proof:

1. We know the number of vertices, $n = |V(G_r)|$, the degree of any vertex v of G_r is $deg(v) = r$ and number of edges $m = |E(G_r)| = \frac{rn}{2}$. The graph G_r is *r*-regular graph. By using the formula of Sombor index Eq. (2), we have

$$SO(G_r) = m\sqrt{r^2 + r^2} = mr\sqrt{2} = \frac{nr^2\sqrt{2}}{2}.$$

2. The number of vertices $n = |V(L(G_r))| = \frac{rn}{2}$, the degree of every vertex of $L(G_r)$ is $deg(v) = 2r-2$ and number of edges $m = |E(L(G_r))| = \frac{rn}{2}(2r-2)$. The line graph $L(G_r)$ is $(2r-r)$-regular graph. By applying Eq. (2), we have

$$SO(L(G_r)) = m\sqrt{(2r-2)^2 + (2r-2)^2}$$
$$= (2r-2)\sqrt{2} = \frac{rn(2r-2)^2\sqrt{2}}{4}.$$

3. The number of vertices $n = |V(\overline{G_r})|$, the degree of every vertex of $\overline{G_r}$ is $deg(v) = n - r - 1$ and number of edges $m = |E(\overline{G_r})| = 2n^2 - 4n$. The complement graph $\overline{G_r}$ is $(n - r - 1)$-regular graph. By considering Eq. (2), we have

$$SO(\overline{G_r}) = m\sqrt{(n-r-1)^2 + (n-r-1)^2}$$
$$= (n-r-1)(2n^2-4n)\frac{\sqrt{2}}{2} = \frac{(n-r-1)(2n^2-4n)\sqrt{2}}{2}. \quad \square$$

Now we show the relation between $G_r, L(G_r)$ and $\overline{G_r}$ in the following theorem.

Theorem 4: Let G_r is be *r*-regular graph of n vertices where $n \geq 4$ and $2 \leq r \leq \left\lfloor\frac{n}{3}\right\rfloor$. Then inequalities hold $SO(G_r) \leq SO(L(G_r)) \leq SO(\overline{G_r})$.

Proof:
First we prove the relation $SO(G_r) \leq SO(L(G_r))$.

$$nr^2 \times \frac{\sqrt{2}}{2} \leq nr(2r-2)^2 \times \frac{\sqrt{2}}{4} \Rightarrow nr^2 \leq nr\big(2(r-1)\big)^2 \times \frac{1}{2}$$
$$\Rightarrow nr^2 \leq nr4(r-1)^2 \times \tfrac{1}{2} \Rightarrow r^2 < 2r(r-1)^2 \Rightarrow r^2 \leq 2(r^2-2r+1)r$$
$$\Rightarrow r \leq 2(r^2-2r+1) \Rightarrow 0 \leq 2r^2 - 5r + 2,$$

Condition hold for $r > 2$, which is required.

Here we can observe that the Sombor index of a line graph often include more dense relationships between edges than the Sombor index of the original graph, which usually leads to a greater or equal value. This reflects additional connectivity and structure in line graph.

Now we show the remaining part of this relation $SO(L(G_r)) \leq SO(\overline{G_r})$.

$$\Rightarrow nr(2r-2)^2 \times \tfrac{\sqrt{2}}{4} \leq (n-r-1)(2n^2-4n) \times \tfrac{\sqrt{2}}{2}$$
$$\Rightarrow 2rn(r-1)^2 \leq (n-r-1)(2n^2-4n)$$
$$\Rightarrow 2r^3n - 4r^2n + 2rn \leq 2n^3 - 4n^2 - 2rn^2 + 4rn - 2n^2 + 4n$$
$$\Rightarrow 2n(r^3 - 2r^2 - r) \leq 2n(n^2 - 3n - rn + 2)$$
$$\Rightarrow r^3 - (2r^2 + r) \leq n^2 - n(3+r) + 2$$
$$\Rightarrow r^3 + n(3+r) \leq n^2 + (2r^2 + r) + 2,$$

This condition hold for $n \geq 4$ and $2 \leq r \leq \left\lfloor\frac{n}{3}\right\rfloor$.

This shows that the Sombor index of complement of a graph may be greater than or equal to the line graph, because the complement graph has more strong relationship which is not present in the original graph. The structure of complement graph can be dense, potentially resulting in a high Sombor index. $\square$

Note: If G_r is a *r*-regular graph of n vertices and m edges where $2 \leq r \leq n-1$ the condition holds $SO(G_r) < SO(L(G_r))$. The special case is $(n-1)$-regular graph (K_n).

Now we consider a particular case of *r*-regular graph, generalized Petersen graphs $P(s,t)$ which is a 3-regular graph.

Theorem 5: Let $P(s,t)$ be the generalized Petersen graphs for $s > 4$ and $t = 1, \ldots, \left\lfloor\frac{n-1}{2}\right\rfloor$. Then for any $P(s,t)$ graph has

1. $SO\big(P(s,t)\big) = 9s\sqrt{2}$,
2. $SO(L(P(s,t))) = 24s\sqrt{2}$,
3. $SO\big(\overline{P(s,t)}\big) = (12s^2 - 24s)\sqrt{2}$.

Proof:
1. The Petersen graph has $n = \big|V\big(P(s,t)\big)\big| = 2s$ number of vertices and $m = \big|E\big(P(s,t)\big)\big| = 3s$ number of edges. The generalized Petersen graph is a 3-regular graph, in which every vertex has degree 3. By using the formula of Sombor index Eq. (2), we have,

$$SO(P(s,t)) = m\sqrt{3^2 + 3^2}$$
$$= 3m\sqrt{2} = 9s\sqrt{2}.$$

2. The number of vertices $n = \left|V\left(L(P(s,t))\right)\right| = 3s$ and number of edges $m = \left|E\left(L(P(s,t))\right)\right| = 6s$. The line graph of generalized Petersen graphs is a 4-regular graph; every vertex has a degree 4. By applying Eq. (2), we have,

$$SO(LP(s,t)) = m\sqrt{4^2 + 4^2}$$
$$= 4m\sqrt{2} = 24s\sqrt{2}.$$

3. The number of vertices $n = |V(\overline{P(s,t)})| = 2s$ and number of edges $m = |E(\overline{P(s,t)})| = 2s^2 - 4s$. The complement of generalized Petersen graphs is 6-regular graph, every vertex has a degree 6. By considering Eq. (2), we have,

$$SO(\overline{P(s,t)}) = m\sqrt{6^2 + 6^2}$$
$$= 6m\sqrt{2} = (12s^2 - 24s)\sqrt{2}.$$

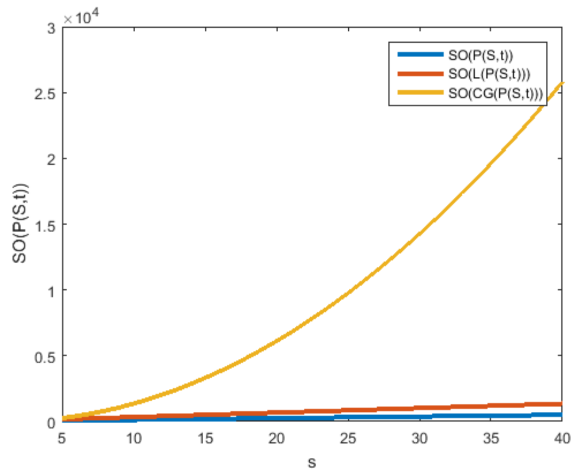

Figure 4. Sombor Index of $P(s,t)$

Now we show the relation between $SO(P(s,t))$, $SO(L(P(s,t)))$ and $SO(\overline{P(s,t)})$ in Figure 4. The behavior of the Sombor index of the complement of the Petersen graph, $SO(\overline{P(s,t)})$ is always higher than other graphs and grows faster on the *s*-axis. The behavior of graphs $SO(L(P(s,t)))$ and $SO(P(s,t))$ is linear and they are growing with slope $24\sqrt{2}$ and $9\sqrt{2}$, respectively. The graph of $SO(P(s,t))$ remains above the graph of $SO(L(P(s,t)))$. The graphs in Figure 4 hold the relation $SO(P(s,t)) \leq SO\left(L(P(s,t))\right) \leq SO(\overline{P(s,t)})$. The following theorem also satisfies this relation.

Theorem 6: Let $P(s,t)$ be the generalized Petersen graphs for $s \geq 3$ and $t = 1, \ldots, \left\lfloor\frac{n-1}{2}\right\rfloor$, then $SO(P(s,t)) < SO(L(P(s,t))) < SO(\overline{P(s,t)})$.

Proof

We can easily verify that $P(s,t) < L(P(s,t)) \Rightarrow 9s\sqrt{2} < 24s\sqrt{2} \Rightarrow 3 < 8$, inequality hold. Now we prove second part which says that $L(P(s,t)) < \overline{P(s,t)}$.

So $L(P(s,t)) < \overline{P(s,t)} \Rightarrow 24s\sqrt{2} < (12s^2 - 24s)\sqrt{2} \Rightarrow 48 < 12s$ for $s > 4$, inequality is true. □

Conclusions

In this paper we have calculated the Sombor indices of the *r*-regular graph G_r, line graph of G_r and the complement of G_r. We have also presented a special case of G_r, generalized Petersen graph $P(s,t)$. We further established that $SO(G_r) < SO(L(G_r)) < SO(\overline{G_r})$ for *r*-regular graphs. The sequence of these inequalities show that as we move from the *r*-regular graph G_r to its line graph $L(G_r)$ and then to its complement $\overline{G_r}$, the intricacy and connectivity represented by the Sombor index amplify. The increasing structural magnificence of these graphs increases or remains stable.

Conflicts of Interest

There is no conflict of interest declared by the authors.

Acknowledgement

This research was supported by Ministry of Higher Education (MOHE) through the Fundamental Research Grant Scheme (FRGS/1/2022/STG06/UMT/03/4).

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