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Article

Queues with two units connected in series with a multiserver bulk service with accessible and non accessible batch in unit II

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ABSTRACT

This paper analyses a queueing model consisting of two units I and II connected in series, separated by a finite buffer of size N. Unit I has only one exponential server capable of serving customers one at a time. Unit II consists of c parallel exponential servers and they serve customers in groups according to the bulk service rule. This rule admits each batch served to have not less than 'a' and not more than 'b' customers such that the arriving customers can enter service station without affecting the service time if the size of the batch being served is less than 'd' ($a \le d \le b$). The steady state probability vector of the number of customers waiting and receiving service in unit I and waiting in the buffer is obtained using the modified matrix-geometric method. Numerical results are also presented.

AMS Subject Classification number: 60k25 and 65k30

| Bulk service queues | Accessible and Non-accessible Batch service | Matrix geometric method | Steady state solution |

1. Introduction

Queueing models consisting of two units in series with an intermediate waiting room of finite capacity have been studied by several authors. A model with a finite waiting room in between the two units has been discussed by Neuts [1]. Unit I of this model contains one server with a general service time distribution and unit II consists of c parallel exponential servers. The transform method is used to find steady state probabilities.

The study of blocking in two or more units in service with a general service time distribution, without an intermediate buffer has been considered by Aviltzhak and Yadin [2]. Clarke [3] has investigated a tandem queuing model, wherein two servers are placed in series and each customer will receive service from one and only one server. The novel feature of this model is that a busy service unit prevents the access of new customers to servers further down the line. A departing customer may also be temporarily prevented from leaving by occupied service units down line.

Prabhu [4] has studied transient analysis in a tandem queue. Models of related type with finite total number of customers have been treated by Sharma [5]. Queues with two units connected in series with single service in unit I by a server and multiserver general bulk service in unit II has been studied by Krishna Reddy , Natarajan and Kandasamy [6].

In this paper we deal with a queuing model consisting of two units I and II connected in series with a finite intermediate waiting room of capacity N. Arrivals to unit I occur according to a Poisson process of rate λ. The queue in front of unit I can be of infinite length. Customers in unit I are served one at a time. The service time distribution of unit I is negative exponential with a parameter μ_1. In unit II customers are served by c servers in batches of size n ($a \le n \le b$). The service times, irrespective of batch size, are assumed to have independent identical negative exponential distributions with parameter μ_2. The service rule is assumed to operate as follows: the server starts service only when a minimum of 'a' customers is in the buffer, and the maximum capacity is 'b' customers. Such a rule for bulk service, first introduced by Neuts [7], may be called general bulk service rule. Here the general bulk service rule is further assumed to allow the late entries to join a batch, without affecting the service time, in course of ongoing service as long as the number of customers in that batch is less than 'd' ($a \le d \le b$, called maximum accessible limit).

For the model describe above, the steady state probability vector of the number of customers waiting and receiving service in unit I, waiting in the buffer and the stability condition are obtained by using the matrix-geometric method.

In our discussion, the following notations are used: a diagonal matrix of order n is denoted by $[a_1, a_2 \ldots .a_n]$ and if the all entries are equal to 'a' then this is denoted by $[a]_{n \times n}$. All the unmarked entries are *zeros.*

2. Steady State Probability Vector

This process can be formulated as a continuous time Markov chain with state space

$$
\begin{aligned}
S = \; & \{(i, j, k);\ i \ge 0,\ 0 \le j \le a-1,\ 0 \le k \le c-1\} \\
& \cup \{(i, j, k);\ i \ge 0,\ 0 \le j \le N,\ k = c\} \\
& \cup \{(i, j, q, k);\ i \ge 0,\ j = 0,\ a \le q \le d-1,\ 0 \le k \le c-1\}
\end{aligned}
$$

where i denotes number of customers in unit I (including the one being in the service station), j denotes the number of customers in the buffer, q denotes number of customers in the accessible service batch and k denotes the number of busy servers in unit II with number in each service batch is greater than or equal to maximum accessible limit (d). The infinitesimal generator **Q** of the continuous time Markov chain with the above state has the block partitioned structure.

$$
\mathbf{Q} = \begin{bmatrix}
D & A_2 & 0 & 0 & 0 & . & . & . \\
A_0 & A_1 & A_2 & 0 & 0 & . & . & . \\
0 & A_0 & A_1 & A_2 & 0 & . & . & . \\
0 & 0 & A_0 & A_1 & A_2 & . & . & . \\
0 & 0 & 0 & A_0 & A_1 & . & . & . \\
0 & 0 & 0 & 0 & A_0 & . & . & . \\
. & . & . & . & . & . & . & .
\end{bmatrix}
$$

The matrix A_o is given by

$$
A_0 = \begin{array}{cc}
 & \begin{array}{cccccccccccccccccc} \underline{0} & \underline{1} & \underline{2} & \underline{3} & . & . & . & \underline{c-2} & \underline{c-1} & \underline{c} & \underline{\underline{0}} & \underline{\underline{1}} & \underline{\underline{2}} & . & . & . & \underline{\underline{c-2}} & \underline{\underline{c-1}} \end{array} \\
\begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \underline{3} \\ . \\ . \\ . \\ \underline{c-2} \\ \underline{c-1} \\ \underline{c} \\ \underline{\underline{0}} \\ \underline{\underline{1}} \\ \underline{\underline{2}} \\ . \\ . \\ . \\ \underline{\underline{c-2}} \\ \underline{\underline{c-1}} \end{array} &
\left[\begin{array}{cccccccccccccccccc}
C_0 & & & & & & & & & & C_3 \\
 & C_0 & & & & & & & & & & C_3 \\
 & & C_0 & & & & & & & & & & C_3 \\
 & & & C_0 & & & & & & & & & & . \\
 & & & & . & & & & & & & & & & . \\
 & & & & & . & & & & & & & & & & . \\
 & & & & & & . \\
 & & & & & & & C_0 & & & & & & & & & C_3 \\
 & & & & & & & & C_0 & & & & & & & & & C_3 \\
 & & & & & & & & & C_1 \\
 & C_2 \\
 & & C_2 \\
 & & & C_2 \\
 & & & & . \\
 & & & & & . \\
 & & & & & & . \\
 & & & & & & & & C_2 \\
 & & & & & & & & & C_2
\end{array}\right]
\end{array}
$$

$$
C_0 = \begin{bmatrix} 0 & \Delta a-1 \\ 0 & 0 \end{bmatrix}_{a\times a} \qquad \Delta a-1 = [\mu_1]_{(a-1)\times(a-1)}
$$

$$
C_1 = \begin{bmatrix} 0 & \Delta N \\ 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} \qquad \Delta N = [\mu_1]_{N\times N}
$$

C_2 is a matrix of order (d-a)$\times$a with μ_1 in the place (d-a,1) and *zero* in the remaining places.

C_3 is a matrix of order a$\times$(d-a) with μ_1 in the place (d-a,1) and *zero* in the remaining places.

The matrix A_1 is given by

$$
A_1 = \begin{array}{c}
\\
\underline{0} \\ \underline{1} \\ \underline{2} \\ \underline{3} \\ \cdot \\ \cdot \\ \cdot \\ \underline{c-2} \\ \underline{c-1} \\ \underline{c} \\ \underline{\underline{0}} \\ \underline{\underline{1}} \\ \underline{\underline{2}} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\underline{c-2}} \\ \underline{\underline{c-1}}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccccc}
\underline{0} & \underline{1} & \underline{2} & \underline{3} & \cdot & \cdot & \cdot & \underline{c-2} & \underline{c-1} & \underline{c} & \underline{\underline{0}} & \underline{\underline{1}} & \underline{\underline{2}} & \cdot & \cdot & \cdot & \underline{\underline{c-2}} & \underline{\underline{c-1}}
\end{array} \\
\left[\begin{array}{cccccccccccccccccc}
d_0 & & & & & & & & & & & & & & & & & \\
d_{10} & d_1 & & & & & & & & & & & & & & & & \\
 & d_{21} & d_2 & & & & & & & & & & & & & & & \\
 & & d_{32} & d_3 & & & & & & & & & & & & & & \\
 & & & \cdot & \cdot & & & & & & & & & & & & & \\
 & & & & \cdot & \cdot & & & & & & & & & & & & \\
 & & & & & \cdot & \cdot & & & & & & & & & & & \\
 & & & & & & \cdot & d_{c-2} & & & & & & & & & & \\
 & & & & & & & d_{c-1c-2} & d_{c-1} & & & & & & & & & \\
 & & & & & & & & B & d & & & & & & & & E \\
C_{00} & & & & & & & & & & B_{00} & & & & & & & \\
 & C_{11} & & & & & & & & & B_{10} & B_{11} & & & & & & \\
 & & C_{22} & & & & & & & & & B_{21} & B_{22} & & & & & \\
 & & & \cdot & & & & & & & & & \cdot & \cdot & & & & \\
 & & & & \cdot & & & & & & & & & \cdot & \cdot & & & \\
 & & & & & \cdot & & & & & & & & & \cdot & \cdot & & \\
 & & & & & & \cdot & & & & & & & & & \cdot & \cdot & \\
 & & & & & & & C_{c-1c-1} & & & & & & & & & B_{c-1c-2} & B_{c-1c-1}
\end{array}\right]
\end{array}
$$

$$d_i = \left[-\lambda - \mu_1 - i\mu_2\right]_{a\times a}, \quad 0 \le i \le c-1$$

$$d_{i,i-1} = \left[i\mu_2\right]_{a\times a} \quad , \qquad 1 \le i \le c-1$$

$$B = \begin{array}{c} \\ 0c \\ 1c \\ 2c \\ . \\ . \\ . \\ ac \\ . \\ . \\ . \\ Nc \end{array} \begin{array}{c} \begin{array}{ccccccc} 0c-1 & 1c-1 & 2c-1 & . & . & . & ac-1 \end{array} \\ \left[\begin{array}{ccccccc} c\mu_2 & & & & & & \\ & c\mu_2 & & & & & \\ & & c\mu_2 & & & & \\ & & & . & & & \\ & & & & . & & \\ & & & & & . & \\ & & & & & & c\mu_2 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{array} \right]_{(N+1)\times(a+1)} \end{array}$$

$d = D_1 + D_2$

where D_1 is given by

$$D_1 = \begin{array}{cc} & \begin{array}{ccccccccccc} 0c & 1c & 2c & . & . & . & N-bc & . & . & . & Nc \end{array} \\ \begin{array}{c} 0c \\ 1c \\ 2c \\ . \\ . \\ . \\ dc \\ . \\ . \\ . \\ bc \\ . \\ . \\ . \\ Nc \end{array} & \left[\begin{array}{ccccccccccc} & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ c\mu_2 & & & & & & & & & & \\ . & & & & & & & & & & \\ . & & & & & & & & & & \\ . & & & & & & & & & & \\ c\mu_2 & & & & & & & & & & \\ & . & & & & & & & & & \\ & & . & & & & & & & & \\ & & & . & & & & & & & \\ & & & & & & c\mu_2 & & & & \end{array} \right]_{a\times(N+1)} \end{array}$$

$$D_2 = [-\lambda - \mu_1 - c\mu_2]_{(N+1)\times(N+1)}$$

$C_{i\,i}$ $(0 \le i \le c-1)$, is a matrix of order ((d-a) × a). with μ_2 in the first column and *zero* in the remaining entries.

$B_{i-1\,i-1} = [-\lambda - \mu_1 - i\mu_2]_{(d-a)\times(d-a)}$, $1 \le i \le c$

$B_{i\,i-1} = [i\mu_2]_{(d-a)\times(d-a)}$, $1 \le i \le c-1$

and E is given by

$$E = \begin{array}{cc} & \begin{array}{ccccccc} 0\,ac-1 & 0\,a+1\,c-1 & 0\,a+2\,c-1 & . & . & . & 0\,d-1\,c-1 \end{array} \\ \begin{array}{c} 0c \\ 1c \\ . \\ . \\ . \\ ac \\ a+1c \\ a+2c \\ . \\ . \\ . \\ d-1c \\ dc \\ . \\ . \\ . \\ Nc \end{array} & \begin{bmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ c\mu_2 & & & & & & \\ & c\mu_2 & & & & & \\ & & c\mu_2 & & & & \\ & & & . & & & \\ & & & & . & & \\ & & & & & . & \\ & & & & & & c\mu_2 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix}_{(N+1)\times(d-a)} \end{array}$$

The Matrix D is obtained from A_1 by putting $\mu_1 = 0$. The matrix A_2 is given by $A_2 = \lambda I$ where I is the unit matrix. All the four submatrices D, A_0, A_1 and A_2 are square matrices of order (ac+N+1+(d-a)c).

The stationary probability vector $\underline{X} = (\underline{X}_0 , \underline{X}_1 , \underline{X}_2 , \underline{X}_3 , ... , \underline{X}_i , . . .)$ if it exist, can be found by solving the following system.

$$\underline{\mathbf{X}}\mathbf{Q} = \underline{0} \quad \text{and} \quad \underline{\mathbf{X}}\,\underline{\mathbf{e}} = 1 \qquad \mathbf{(1,2)}$$

where $\underline{e}$ represents a column vector with each component equal to 1. The system of linear equations $\underline{\mathbf{X}}\mathbf{Q} = \underline{0}$ is solved using the matrix - geometric method [8] since the rate matrix **Q** has a special block tridiagonal structure.

Each $\underline{X}_i$ (i = 0,1,2, . . .) is a row vector of order (ac+N+1+(d-a)c) and

$$\underline{X}_i = (X_{i00} , X_{i10}, X_{i20}, . . . ,X_{ia-10}, X_{i01}, X_{i11}, X_{i21}, . . . , X_{ia-11}, . . ., X_{i0c-1}, X_{i1c-1} , X_{i2c-1} , . . . ,$$

$$X_{ia-1c-1}, X_{i0c}, X_{i1c}, X_{i2c}, . . . , X_{ia-1c} , . . . , X_{iNc}, X_{i0a0}, X_{i0a+10}, . . . , X_{i0d-10}, X_{i0a1}, X_{i0a+11}, . . . ,$$

$$X_{i0d-11} , . . . , X_{i0ac-1}, X_{i0a+1c-1}, . . . , X_{i\,0d-1c-1}) \qquad (3)$$

In the stable case, there exists the stationary probability vector

$$\underline{X}_i = \underline{X}_0 R^i \qquad i \geq 1 \tag{4}$$

The stationary probability vector $\underline{X}$ is called the matrix geometric probability vector. If a matrix geometric solution exists, the system $\underline{\mathbf{X}}\mathbf{Q} = \underline{0}$ becomes

$$\underline{X}_0 (D + RA_0) = \underline{0} \tag{5}$$

and

$$\underline{X}_0 R^i (A_2 + RA_1 + R^2 A_0) = \underline{0} \tag{6}$$

The vector $\underline{X}_0$ is uniquely determined by

$$\underline{X}_0 (D + RA_0) = \underline{0}$$

together with normalizing equation

$$\underline{X}_0 (I - R)^{-1} \underline{e} = 1. \tag{7}$$

The matrix R is the minimal solution to a matrix non linear equation

$$A_2 + RA_1 + R^2 A_0 = 0 \tag{8}$$

where $R \geq 0$, [9], and it is an irreducible, non - negative matrix with a spectral radius of less than one. An iterative approach may be used to compute R as follows:

$$R(0) = 0$$

$$R(n+1) = -A_2 A_1^{-1} - R^2(n) A_0 A_1^{-1}, n \geq 0 \tag{9}$$

For the Markov process with such a generator, Neuts [8] obtained the stability condition

$$\underline{\Pi} A_2 \underline{e} < \underline{\Pi} A_0 \underline{e}, \tag{10}$$

where the row vector $\underline{\Pi}$ is defined as follows:

Consider the infinitesimal generator $A = A_0 + A_1 + A_2$. It can be shown that A is irreducible and there is a unique row vector $\underline{\Pi} \geq 0$ such that

$$\underline{\Pi} A = \underline{0}$$

and

$$\underline{\Pi}\,\underline{e} = 1 \tag{11}$$

In our case

$$\underline{\Pi} = [\Pi_{00}, \Pi_{10}, \Pi_{20}, \ldots, \Pi_{a-1\,0}, \Pi_{01}, \Pi_{11}, \Pi_{21}, \ldots, \Pi_{a-11}, \ldots, \Pi_{o\,c-1}, \Pi_{1c-1},$$
$$\Pi_{2c-1}, \ldots, \Pi_{a-1\,c-1}, \Pi_{0\,c}, \Pi_{1\,c}, \Pi_{2\,c}, \ldots, \Pi_{a-1\,c}, \Pi_{a\,c}, \Pi_{a+1\,c}, \ldots, \Pi_{d\,c},$$
$$\Pi_{d+1\,c}, \ldots, \Pi_{b\,c}, \ldots, \Pi_{N\,c}, \Pi_{0\,a\,0}, \Pi_{0\,a+1\,0}, \Pi_{0\,a+2\,0}, \ldots, \Pi_{0\,d-1\,0}, \Pi_{0\,a\,1},$$

$\Pi_{0\ a+1\ 1}, \Pi_{0\ a+2\ 1}, \ldots, \Pi_{0\ d-1\ 1}, \ldots, \Pi_{0\ a\ c-1}, \Pi_{0\ a+1\ c-1}, \Pi_{0\ a+2\ c-1}, \ldots, \Pi_{0\ d-1\ c-1}]$.

Since

$$\underline{\Pi}\, A_0\, \underline{e} = \underline{\Pi} \begin{bmatrix} \mu_1 \\ \mu_1 \\ . \\ . \\ . \\ \mu_1 \\ 0 \end{bmatrix}$$

$$= \mu_1\ [1-\Pi_{N,c}]$$

and

$$\underline{\Pi}\, A_2\, \underline{e} = \lambda,$$

the stability condition (10) becomes

$$\lambda < \mu_1 [1 - \Pi_{N,c}] \qquad (12)$$

The system $\underline{\Pi}$ A = 0 takes the form

$$\mu_1 \Pi_{N-1,c} - c\mu_2 \Pi_{N,c} = 0,$$

$$\mu_1 \Pi_{N-r,c} - (\mu_1 + c\mu_2) \Pi_{N-r+1,c} = 0, \qquad 2 \le r \le b$$

$$\Pi_{N-r,c} + (\mu_1 + c\mu_2) \Pi_{N-r+1,c} + c\mu_2 \Pi_{N+b+1-r,c} = 0, \qquad b+1 \le r \le n$$

$$\mu_1 \Pi_{0,d-1,c-1} - (\mu_1 + c\mu_2) \Pi_{0,c} + c\mu_2 \sum_{j=d}^{b} \Pi_{j,c} = 0,$$

$$\mu_1 \Pi_{0,d-r,c-1} - (\mu_1 + c\mu_2) \Pi_{0,d-r+1,c-1} + c\mu_2 \Pi_{d-r+1,c} = 0, \qquad 2 \le r \le a$$

$$\mu_1 \Pi_{a-1,c-1} - (\mu_1 + c\mu_2) \Pi_{0,d-a,c-1} + c\mu_2 \Pi_{d-a,c} = 0,$$

$$\mu_1 \Pi_{a-r,c-1} - (\mu_1 + \overline{c-1}\mu_2) \Pi_{a-r+1,c-1} + c\mu_2 \Pi_{a-r+1,c} = 0, \qquad 2 \le r \le a$$

$$\mu_1 \Pi_{0,d-1,c-k-1} - \left(\mu_1 + \overline{c-k}\mu_2\right) \Pi_{0,c-k} + (c-k+1)\mu_2 \Pi_{0,c-k+1} + \mu_2 \sum_{j=a}^{d-1} \Pi_{0,j,c-k} = 0, \qquad 1 \le k \le c-1$$

$$\mu_1 \Pi_{0,d-r,c-k-1} - \left(\mu_1 + \overline{c-k}\mu_2\right) \Pi_{0,d-r+1,c-k-1} + (c-k)\mu_2 \Pi_{0,d-r+1,c-k} = 0, \quad 2 \le r \le a,\ 1 \le k \le c-1$$

$$\mu_1 \Pi_{a-1,c-k-1} - \left(\mu_1 + \overline{c-k}\mu_2\right) \Pi_{0,d-a,c-k-1} + (c-k)\mu_2 \Pi_{0,d-a,c-k} = 0, \qquad 1 \le k \le c\text{-}1$$

$$\mu_1 \Pi_{a-r,c-k-1} - \left(\mu_1 + \overline{c-k-1}\mu_2\right) \Pi_{a-r+1,c-k-1} + (c-k)\mu_2 \Pi_{a-r+1,c-k} = 0, \quad 2 \le r \le a,\ 1 \le k \le c-1$$

solving the above system of equations, we get

$$\Pi_{N-r,c} = \left((\mu_1 + c\mu_2)/\mu_1\right)(c\mu_2/\mu_1)\Pi_{N,c}, \qquad 1 \le r \le b$$

$$\Pi_{N-(b+r+1),c} = \Big[\left((\mu_1 + c\mu_2)/\mu_1\right)^{N-b} \times \left\{\{(\mu_1 + c\mu_2)/\mu_1\}^b - 1\right\}(c\mu_2/\mu_1) - r(c\mu_2/\mu_1)^2 (\mu_1 + c\mu_2)^{r-1}\Big]\Pi_{N,c}, \qquad 0 \le r \le N-b-1$$

$$\Pi_{0,d-1,c-1} = \Big[\{((\mu_1 + c\mu_2)/\mu_1)\}^{N-b} \times \left\{\{(\mu_1 + c\mu_2)/\mu_1\}^b - 1\right\}(c\mu_2/\mu_1) - (c\mu_2/\mu_1)^2 \times \{((\mu_1 + c\mu_2)/\mu_1)\}^{N-b-1} - (c\mu_2/\mu_1) \sum_{r=N-(2b+1)}^{N-(a+b+1)} \{(\mu_1 + c\mu_2)/\mu_1\}^r \times \left\{\{(\mu_1 + c\mu_2)/\mu_1\}^b - 1\right\}(c\mu_2/\mu_1) - r(c\mu_2/\mu_1)^2 \{(\mu_1 + c\mu_2)/\mu_1\}^{r-1}\Big]\Pi_{N,c}$$

Provided $N \ge 2b+1$.

Using the above results, the following relations are derived:

$$\Pi_{0,d-k,c-1} = \left((\mu_1 + c\mu_2)/\mu_1\right)^{k-1}\Pi_{0,d-1,c-1}$$
$$-\sum_{r=1}^{k-1}(c\mu_2/\mu_1)\ \left((\mu_1 + c\mu_2)/\mu_1\right)^{k-r-1}\Pi_{d-r,c}\ , \qquad 2 \le k \le a$$

$$\Pi_{a-1,c-1} = \left((\mu_1 + c\mu_2)/\mu_1\right)\Pi_{0,d-a,c-1} - (c\mu_2/\mu_1)\Pi_{a,c}$$

$$\Pi_{a-k,c-1} = \left((\mu_1 + (c-1)\mu_2)/\mu_1\right)^{k-1}\Pi_{a-1,c-1}$$
$$-\sum_{r=1}^{k-1}(c\mu_2/\mu_1)\left((\mu_1 + (c-1)\mu_2)/\mu_1\right)^{k-r-1}\Pi_{a-r,c}\ , \qquad 2 \le k \le a$$

$$\Pi_{0,d-1,c-2} = \left((\mu_1 + (c-1)\mu_2)/\mu_1\right)\Pi_{0,c-1} - (c\mu_2/\mu_1)\Pi_{0,c} - (\mu_2/\mu_1)\sum_{j=a}^{d-1}\Pi_{0,j,c-1}$$

$$\Pi_{0,d-k,c-2} = \left((\mu_1 + (c-1)\mu_2)/\mu_1\right)^{k-1}\Pi_{0,d-1,c-2}$$
$$-\sum_{r=1}^{k-1}\left((\mu_1 + (c-1))\mu_2/\mu_1\right)^{k-r-1}\left((c-1)\mu_2/\mu_1\right)\Pi_{0,d-r,c-1}\ , \qquad 2 \le k \le a$$

$$\Pi_{a-1,c-2} = \left((\mu_1 + (c-1))\mu_2/\mu_1\right)\Pi_{0,d-a,c-2} - \left((c-1)\mu_2/\mu_1\right)\Pi_{0,d-a,c-1}$$

$$\Pi_{a-k,c-2} = \left((\mu_1 + (c-2))\mu_2/\mu_1\right)\Pi_{a-1,c-2}$$
$$-\sum_{r=1}^{k-1}\left((c-1)\mu_2/\mu_1\right)\left((\mu_1 + (c-2))\mu_2/\mu_1\right)^{k-r-1}\Pi_{a-r,c-1}\,, \qquad 2 \le k \le a$$

$$\Pi_{0,d-1,0} = \left((\mu_1 + \mu_2)/\mu_1\right)\Pi_{0,1} - (2\mu_2/\mu_1)\Pi_{0,2} - (\mu_2/\mu_1)\sum_{j=a}^{d-1}\Pi_{o,j,c-2}$$

$$\Pi_{0,d-k,0} = \left((\mu_1 + \mu_2)/\mu_1\right)^{k-1}\Pi_{0,d-1,0}$$
$$-(\mu_2/\mu_1)\sum_{r=1}^{k-1}\left((\mu_1 + \mu_2)/\mu_1\right)^{k-r-1}\Pi_{0,d-r,1}\,, \qquad 2 \le k \le a$$

$$\Pi_{a-1,0} = \left((\mu_1 + \mu_2)/\mu_1\right)\Pi_{0,d-a,0} - (\mu_2/\mu_1)\Pi_{0,d-a,1}$$

$$\Pi_{a-k,0} = \Pi_{a-1,0} - (\mu_2/\mu_1)\sum_{r=1}^{k-1}\Pi_{a-r,1}\ , \qquad 2 \le k \le a$$

By using the normalizing condition $\underline{\Pi}\,\underline{e} = 1$, $\Pi_{N,C}$ can be determined.

3. Numerical Results

By theorem 1 of Latouche and Neuts [10], R is the limit of the sequence $\{R(n)\},\ n \geq 0$, of matrices defined by

$$R(0) = 0$$
$$R(n+1) = -A_2 A_1^{-1} - R^2(n) A_0 A_1^{-1}, \qquad n \geq 0$$

Corollary 1 of [9] also provides an accuracy check on the evaluation of R, since

$$RA_0 \underline{\mathbf{e}} = A_2 \underline{\mathbf{e}},$$

where $\underline{\mathbf{e}}$ is the column vector with all its components equal to one. For a = 3, b = 9, d = 6, c = 3, N=20, the matrices A_0, A_1, A_2 and D of order 39×39 by choosing the parameters $\lambda = 0.8, \mu_1 = 1$ and $\mu_2 = 8$ we get $A_2 = 0.8I$, where I is unit matrix of order 39×39.

The matrix R is given by $R = [C_1, C_2, C_3, \ldots, C_{39}]$, where

$$C_1 = \begin{bmatrix} 0.51370085 \\ 0.09350768 \\ 0.14928557 \\ 0.43206809 \\ 0.09350254 \\ 0.14897776 \\ 0.39537915 \\ 0.09349758 \\ 0.14870525 \\ 0.37262628 \\ 0.09349371 \\ 0.14884568 \\ 0.67416900 \\ 0.35513828 \\ 0.35507280 \\ 0.35176948 \\ 0.35176947 \\ 0.35176874 \\ 0.35146078 \\ 0.09349033 \\ 0.14922940 \\ 0.63913646 \\ 0.33660295 \\ 0.33651091 \\ 0.33263056 \\ 0.33263055 \\ 0.33262713 \\ 0.33177191 \\ 0.09349524 \\ 0.15549419 \\ 0.43489540 \\ 0.43489444 \\ 0.43484848 \\ 0.39811337 \\ 0.39811242 \\ 0.39806692 \\ 0.37528420 \\ 0.37528325 \\ 0.37523818 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0.16763336 \\ 0.51704703 \\ 0.09884556 \\ 0.16695337 \\ 0.43541425 \\ 0.09883901 \\ 0.16638685 \\ 0.39872529 \\ 0.09883271 \\ 0.16589656 \\ 0.37597289 \\ 0.09902809 \\ 0.32184286 \\ 0.16259426 \\ 0.16264194 \\ 0.16506449 \\ 0.16506449 \\ 0.16506502 \\ 0.16528576 \\ 0.35480922 \\ 0.09940030 \\ 0.31811573 \\ 0.16106187 \\ 0.16112851 \\ 0.16394525 \\ 0.16394526 \\ 0.16394765 \\ 0.16455002 \\ 0.33512925 \\ 0.10384987 \\ 0.16481992 \\ 0.16482062 \\ 0.16485446 \\ 0.16434442 \\ 0.16434511 \\ 0.16437850 \\ 0.16392848 \\ 0.16392917 \\ 0.16396214 \end{bmatrix} \quad C_3 = \begin{bmatrix} 0.10308160 \\ 0.16840449 \\ 0.51827802 \\ 0.10307027 \\ 0.16772450 \\ 0.43664522 \\ 0.10305952 \\ 0.16715798 \\ 0.39995624 \\ 0.10304925 \\ 0.16666801 \\ 0.37733285 \\ 0.20340640 \\ 0.10212813 \\ 0.10214448 \\ 0.10295443 \\ 0.10295443 \\ 0.10295463 \\ 0.10303488 \\ 0.16605842 \\ 0.35641451 \\ 0.20236398 \\ 0.10178079 \\ 0.10180412 \\ 0.10278175 \\ 0.10278176 \\ 0.10278270 \\ 0.10301531 \\ 0.16534178 \\ 0.35063466 \\ 0.10243171 \\ 0.10243195 \\ 0.10244309 \\ 0.10242301 \\ 0.10242325 \\ 0.10243438 \\ 0.10241466 \\ 0.10241490 \\ 0.10242601 \end{bmatrix} \quad C_4 = \begin{bmatrix} 0.00001459 \\ 0.00001581 \\ 0.00001660 \\ 0.08164735 \\ 0.00002095 \\ 0.00032441 \\ 0.07339247 \\ 0.00002573 \\ 0.00056163 \\ 0.06827320 \\ 0.00003031 \\ 0.00079366 \\ 0.08068477 \\ 0.04264169 \\ 0.04323631 \\ 0.06358760 \\ 0.06358760 \\ 0.06358742 \\ 0.06351114 \\ 0.00003678 \\ 0.00108283 \\ 0.07552495 \\ 0.03985245 \\ 0.04056544 \\ 0.05929510 \\ 0.05929509 \\ 0.05929424 \\ 0.05908140 \\ 0.00004554 \\ 0.00139630 \\ 0.00001481 \\ 0.00002591 \\ 0.00069457 \\ 0.03679684 \\ 0.03680736 \\ 0.03636264 \\ 0.04567314 \\ 0.04568318 \\ 0.04616235 \end{bmatrix} \quad C_5 = \begin{bmatrix} 0.00001371 \\ 0.00000144 \\ 0.00000146 \\ 0.00608136 \\ 0.08163422 \\ 0.00000801 \\ 0.00113439 \\ 0.07337936 \\ 0.00001407 \\ 0.00147226 \\ 0.06826012 \\ 0.00002106 \\ 0.00216098 \\ 0.00094995 \\ 0.00096574 \\ 0.00176006 \\ 0.00176006 \\ 0.00176023 \\ 0.00183379 \\ 0.06349810 \\ 0.00003108 \\ 0.00227629 \\ 0.00104814 \\ 0.00107034 \\ 0.00200382 \\ 0.00200383 \\ 0.00200464 \\ 0.00220870 \\ 0.05906842 \\ 0.00004307 \\ 0.00000137 \\ 0.00016000 \\ 0.00001270 \\ 0.00047688 \\ 0.00047711 \\ 0.00048809 \\ 0.00083325 \\ 0.00083348 \\ 0.00084435 \end{bmatrix}$$

$$C_6 = \begin{bmatrix} 0.00000013 \\ 0.00000013 \\ 0.00000013 \\ 0.00001145 \\ 0.00038012 \\ 0.08163293 \\ 0.00002164 \\ 0.00113315 \\ 0.07337809 \\ 0.00003094 \\ 0.00147103 \\ 0.06825890 \\ 0.00004961 \\ 0.00002065 \\ 0.00002103 \\ 0.00004055 \\ 0.00004055 \\ 0.00004056 \\ 0.00004325 \\ 0.00183255 \\ 0.06349696 \\ 0.00005725 \\ 0.00002446 \\ 0.00002503 \\ 0.00005030 \\ 0.00005030 \\ 0.00005033 \\ 0.00005900 \\ 0.00221033 \\ 0.06144905 \\ 0.00000013 \\ 0.00000013 \\ 0.00000036 \\ 0.00000883 \\ 0.00000883 \\ 0.00000907 \\ 0.00001684 \\ 0.00001684 \\ 0.00001709 \end{bmatrix} \quad C_7 = \begin{bmatrix} 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.04494382 \\ 0.00000018 \\ 0.00003529 \\ 0.04180821 \\ 0.00000049 \\ 0.00008555 \\ 0.02480675 \\ 0.01301278 \\ 0.01325992 \\ 0.03893821 \\ 0.03893821 \\ 0.03893810 \\ 0.03891360 \\ 0.00000113 \\ 0.00015707 \\ 0.02326977 \\ 0.01215070 \\ 0.01250769 \\ 0.03630898 \\ 0.03630898 \\ 0.03630845 \\ 0.03617802 \\ 0.00000223 \\ 0.00023051 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000057 \\ 0.00011348 \\ 0.01395287 \\ 0.01395383 \\ 0.01410525 \end{bmatrix} \quad C_8 = \begin{bmatrix} 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00011348 \\ 0.04494382 \\ 0.00000024 \\ 0.00022864 \\ 0.04180821 \\ 0.00000070 \\ 0.00019159 \\ 0.00007825 \\ 0.00008038 \\ 0.00032771 \\ 0.00032771 \\ 0.00032782 \\ 0.00037392 \\ 0.03889136 \\ 0.00000150 \\ 0.00002206 \\ 0.00009424 \\ 0.00009808 \\ 0.00041260 \\ 0.00041260 \\ 0.00041312 \\ 0.00054130 \\ 0.03617803 \\ 0.00000246 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000057 \\ 0.00005957 \\ 0.00005957 \\ 0.00006057 \end{bmatrix}$$

$$C_9 = \begin{bmatrix} 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000057 \\ 0.00011348 \\ 0.04494382 \\ 0.00000145 \\ 0.00022864 \\ 0.04180821 \\ 0.00000146 \\ 0.00000051 \\ 0.00000053 \\ 0.00000246 \\ 0.00000246 \\ 0.00000246 \\ 0.00000310 \\ 0.00037392 \\ 0.03889137 \\ 0.00000186 \\ 0.00000070 \\ 0.00000073 \\ 0.00000357 \\ 0.00000357 \\ 0.00000358 \\ 0.00000582 \\ 0.00054305 \\ 0.03763682 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000034 \\ 0.00000034 \\ 0.00000035 \end{bmatrix} \quad C_{10} = \begin{bmatrix} 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.03100775 \\ 0.00000000 \\ 0.00000043 \\ 0.00027075 \\ 0.00000042 \\ 0.00010408 \\ 0.02887917 \\ 0.02887917 \\ 0.02887909 \\ 0.02884442 \\ 0.00000001 \\ 0.00000141 \\ 0.00033605 \\ 0.00000116 \\ 0.00019388 \\ 0.02692915 \\ 0.02392915 \\ 0.02692876 \\ 0.02868320 \\ 0.00000002 \\ 0.00000260 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000090 \\ 0.00003727 \end{bmatrix}$$

$$C_{11}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00003727\\0.03100775\\0.00000000\\0.00000127\\0.00000000\\0.00000033\\0.00006946\\0.00006946\\0.00006954\\0.00010400\\0.02884442\\0.00000001\\0.00000178\\0.00000000\\0.00000078\\0.00009717\\0.00009717\\0.00009756\\0.00019349\\0.02683202\\0.00000002\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000009\end{bmatrix}\quad C_{12}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000009\\0.00003727\\0.03100275\\0.00000001\\0.00000000\\0.00000000\\0.00000021\\0.00000021\\0.00000021\\0.00000042\\0.00010400\\0.02884442\\0.00000001\\0.00000000\\0.00000000\\0.00000035\\0.00000035\\0.00000035\\0.00000117\\0.00019479\\0.02791396\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}\quad C_{13}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000009\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000042\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000110\\0.00017045\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}\quad C_{14}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000009\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000042\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000087\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}\quad C_{15}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000008\\0.00003225\\0.02683211\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000036\\0.00009000\\0.02496043\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000095\\0.00014750\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}$$

$$C_{16}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000027\\0.00016124\\0.00000009\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000090\\0.00020999\\0.00000042\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000168\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}\quad C_{17}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000081\\0.00000000\\0.00000009\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000126\\0.00000000\\0.00000042\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}\quad C_{18}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000009\\0.00037270\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000000\\0.00000000\\0.00000042\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}\quad C_{19}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000009\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000042\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}\quad C_{20}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000009\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000042\\0.00010400\\0.02884420\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}$$

$$C_{21}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000042\\0.00010540\\0.03000750\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}
C_{22}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000034\\0.00007358\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}
C_{23}=\begin{bmatrix}0.00000000\\0.00000009\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000022\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}
C_{24}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000008\\0.00003467\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000039\\0.00009674\\0.02683211\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000102\\0.00015856\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}
C_{25}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00003727\\0.03100775\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000042\\0.00010540\\0.03000750\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\end{bmatrix}$$

$$C_{26} = \begin{bmatrix} 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000021 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000035 \\ 0.00000000 \\ 0.00000009 \\ 0.00003727 \\ 0.03100775 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \end{bmatrix} \quad C_{27} = \begin{bmatrix} 0.00000000 \\ 0.00000009 \\ 0.00003727 \\ 0.03100775 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \end{bmatrix} \quad C_{28} = \begin{bmatrix} 0.00000000 \\ 0.00000009 \\ 0.00003727 \\ 0.03100775 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \end{bmatrix} \quad C_{29} = \begin{bmatrix} 0.00000000 \\ 0.00000009 \\ 0.00003727 \\ 0.03100775 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \end{bmatrix} \quad C_{30} = \begin{bmatrix} 0.00000000 \\ 0.00000009 \\ 0.00003727 \\ 0.03100775 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \\ 0.00000000 \end{bmatrix}$$

$$C_{31}=\begin{bmatrix}0.01389844\\0.01899467\\0.03102936\\0.01389831\\0.01898843\\0.03065498\\0.01389819\\0.01898259\\0.03035890\\0.01389806\\0.01897713\\0.03013632\\0.04873822\\0.01382389\\0.01382524\\0.01389134\\0.01389134\\0.01389136\\0.01389788\\0.01896977\\0.02989220\\0.04716419\\0.01379632\\0.01379825\\0.01387849\\0.01387849\\0.01387857\\0.01389763\\0.01896164\\0.03084092\\0.09547935\\0.01384672\\0.01384764\\0.05053543\\0.01384662\\0.01384754\\0.03659926\\0.01384652\\0.01384743\end{bmatrix}\quad C_{32}=\begin{bmatrix}0.01389844\\0.01899467\\0.03102936\\0.01389831\\0.01898843\\0.03065498\\0.01389819\\0.01898259\\0.03035890\\0.01389806\\0.01897713\\0.03013632\\0.04873822\\0.01382389\\0.01382524\\0.01389134\\0.01389134\\0.01389136\\0.01389788\\0.01896977\\0.02989220\\0.04716419\\0.01379632\\0.01379825\\0.01387849\\0.01387849\\0.01387857\\0.01389763\\0.01896164\\0.03084092\\0.09547935\\0.01384672\\0.01384764\\0.05053543\\0.01384662\\0.01384754\\0.03659926\\0.01384652\\0.01384743\end{bmatrix}\quad C_{33}=\begin{bmatrix}0.01389844\\0.01899467\\0.03102936\\0.01389831\\0.01898843\\0.03065498\\0.01389819\\0.01898259\\0.03035890\\0.01389806\\0.01897713\\0.03013632\\0.04873822\\0.01382389\\0.01382524\\0.01389134\\0.01389134\\0.01389136\\0.01389788\\0.01896977\\0.02989220\\0.04716419\\0.01379632\\0.01379825\\0.01387849\\0.01387849\\0.01387857\\0.01389763\\0.01896164\\0.03084092\\0.09547935\\0.01384672\\0.01384764\\0.05053543\\0.01384662\\0.01384754\\0.03659926\\0.01384652\\0.01384743\end{bmatrix}\quad C_{34}=\begin{bmatrix}0.00000001\\0.00000001\\0.00000001\\0.00000014\\0.00000624\\0.00037438\\0.00000026\\0.00001169\\0.00059218\\0.00000038\\0.00001657\\0.00074610\\0.02592823\\0.00000003\\0.00000027\\0.00000052\\0.00000052\\0.00000052\\0.00000055\\0.00002288\\0.00009047\\0.02411946\\0.00000032\\0.00000033\\0.00000066\\0.00000066\\0.00000066\\0.00000079\\0.00003080\\0.00108750\\0.00000001\\0.00000001\\0.00000001\\0.04494393\\0.00000011\\0.00000011\\0.02787235\\0.00000210\\0.00000021\end{bmatrix}\quad C_{35}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000006\\0.00000266\\0.00000000\\0.00000011\\0.00000486\\0.00000000\\0.00000017\\0.00000688\\0.00024929\\0.02592757\\0.00000000\\0.00000001\\0.00000010\\0.00000001\\0.00000001\\0.00000024\\0.00000961\\0.00036087\\0.02411867\\0.00000000\\0.00000001\\0.00000001\\0.00000001\\0.00000001\\0.00000035\\0.00001316\\0.00000000\\0.00000000\\0.00000000\\0.00011348\\0.04494382\\0.00000000\\0.00015243\\0.02787214\\0.00000000\end{bmatrix}$$

$$C_{36}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000002\\0.00000000\\0.00000000\\0.00000004\\0.00000000\\0.00000070\\0.00002703\\0.02243804\\0.00024928\\0.02592757\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000030\\0.00007534\\0.02087455\\0.00036087\\0.02411867\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000080\\0.00012347\\0.00000000\\0.00000000\\0.00000000\\0.00000057\\0.00011348\\0.04494382\\0.00000097\\0.00015243\\0.02787214\end{bmatrix}\quad C_{37}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000040\\0.00007829\\0.00000001\\0.00000100\\0.00015774\\0.02884443\\0.00000000\\0.00000000\\0.00000001\\0.00000001\\0.00000001\\0.00000002\\0.00000214\\0.00025798\\0.02683203\\0.00000000\\0.00000000\\0.00000002\\0.00000002\\0.00000002\\0.00000004\\0.00000397\\0.00036165\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.03100775\\0.00000000\\0.00000000\end{bmatrix}\quad C_{38}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000023\\0.00000000\\0.00000000\\0.00000063\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000146\\0.00019349\\0.02683202\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000002\\0.00000255\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00003727\\0.03100775\\0.00000000\end{bmatrix}\quad C_{39}=\begin{bmatrix}0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000023\\0.00000000\\0.00000000\\0.00000063\\0.00010400\\0.02884442\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000001\\0.00000146\\0.00019349\\0.02683202\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000002\\0.00000255\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00000000\\0.00003727\\0.03100775\\0.00000000\end{bmatrix}$$

By using the above matrix R, the equation $A_2 \underline{\mathbf{e}} = R\, A_0 \underline{\mathbf{e}}$, which provides us with internal accuracy check on the computation of R, is verified. The vector $\underline{X}_0 = \{X_{000}, X_{010}, X_{020}, X_{001}, X_{011}, X_{021}, X_{003}, X_{013}, X_{023}, \ldots, X_{0\,20\,3}, X_{0030}, X_{0040}, X_{0050}, X_{\,0030}, X_{\,0041}, X_{\,0051}, X_{0032}, X_{0042}, X_{0052})$ is the left eigen-vector of the matrix $D + RA_0$, corresponding to an eigenvalue of *zero*. Vector $\underline{X}_0$ is calculated from the relation $\underline{X}_0 (D + RA_0) = 0$ and is normalized by $\underline{X}_0(I - R)^{-1}\, e = 1$.

For the numerical parameters chosen above, the row vector X is given by

X 0 = [0.00000000 , 0.00000000, 0.00000000, 0.00000000 , 0.00000000 , 0.00000000
0.00000000 , 0.00000000, 0.00000000, 0.00000000 , 0.00000000 , 0.00000000
0.00000000 , 0.38500000, 0.00000000, 0.00000000 , 0.00000000 , 0.00000000
0.00000000 , 0.00000000, 0.00000000, 0.38500000 , 0.00000000 , 0.00000000
0.00000000 , 0.00000000, 0.00000000, 0.00000000 , 0.00000000 , 0.00000000
0.00000000 , 0.00000000, 0.00020000, 0.04950000 , 0.00000000 , 0.00900000
0.00000000 , 0.00000000, 0.00000000]

Further, the remaining vectors $\underline{X}_i$, $i \geq 1$ are evaluated using the relation $\underline{X}_i = \underline{X}_0 R^i$, $i \geq 1$. It may be noted that $\underline{X}_k \rightarrow 0$ as $k \rightarrow \infty$. For the chosen parameters, $X_{97} \rightarrow 0$, and the sum of the steady state probabilities is found to be one.

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