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RESEARCH ARTICLE

# Beta and Gamma Products of Fuzzy Random Graphs with Hesitancy

**N. Sarala, R. Abirami***

Department of Mathematics, A.D.M. College for Women (Autonomous), Bharathidasan University, Nagapattinam-611 001, Tamil Nadu, India

Abstract Fuzzy random graphs offer a powerful framework for modeling uncertain and imprecise relationships in various real-world systems. This study introduces the concept of hesitancy fuzzy random graphs, which incorporate both fuzziness and randomness in edge and vertex memberships. Additionally, this study investigates the beta and gamma products within the context of hesitancy fuzzy random graphs. Leveraging the beta and gamma operations, this study investigates the application of combining and aggregating uncertain information from multiple sources represented by hesitancy fuzzy random graphs.

**Keywords**: Fuzzy random graph (FRG), hesitancy fuzzy random graph (hfrg), beta product, gamma product.

## Introduction

Lotfi A. Zadeh invented the fuzzy set theory, which had a profound impact on the field of multidisciplinary study. Rosenfeld, a pioneer in the subject like Euler, created fuzzy graph theory in 1975. In a different seminal study on fuzzy sets, Professor Atanassov proposed intuitionistic fuzzy sets. The next significant breakthrough came from T. Pathinathan, who came up with the new concept of the hesitant fuzzy graph [3]. Furthermore, N. Sarala and R. Abirami introduced the novel idea of a fuzzy random graph [4]. Also, Anil P.N. and Shashikala S. focused on defining and demonstrating particular fuzzy soft graph products. It pays attention to the analysis of the regular characteristics and vertex degrees of these FSG products under particular situations [5].

In addition, the direct sum of two fuzzy graphs, residue product, strong product, and lexicographic product was established as well. To help with more research into fuzzy graph operations, M. Vijaya talked about the conditions that must be met for the modular product of two fuzzy graphs to be totally regular under certain constraints. He gave formulas for total degree and information about the properties of totally regular fuzzy graphs so that more research into fuzzy graph operations could be done [7].

A. Nagoor Gani and B. Fathima Kani invented the idea of dividing a big fuzzy graph into smaller components to produce a beta and gamma product of fuzzy graphs [2]. The properties of HFGs, including the behavior of the β-product of various types of HFGs, are discussed by Sunil M.P. and Suresh Kumar in their work On beta product of hesitancy fuzzy graphs and Intuitionistic hesitancy fuzzy graphs. They also emphasize the benefits of HFGs in decision-making and problem-solving around mergers [6]. The reader should also refer to [1].

In this paper, we delve into the intricate properties of hesitancy fuzzy random graphs, particularly focusing on the beta and gamma products with examples. These products, derived from the fusion of hesitancy and randomness, offer a nuanced understanding of network dynamics, shedding light on how uncertainty influences connectivity patterns and structural properties.

***For correspondence:**
abiramielanchezhiyan@gmail.com

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# Preliminaries

**Definition 1**

Let $\mathbb{V}_{\mathcal{FR}} = \{v_i, i = 1,2, \ldots \ldots, n\}$ be a set of $n$ vertices has $n(n-1)/2$ possible edges $\mathbb{E}_{\mathcal{FR}}$ between them. Then there are two sets of edges

$$\mathbb{E}_{\mathcal{F}} = \{(v_i, v_j)/\ 1 \leq v_i < v_j \leq n\ ; i,j = 1,2, \ldots \ldots, n, i \neq j \text{ and} (v_i, v_j) \text{ are fuzzy edges}\},$$

$$\mathbb{E}_{\mathcal{R}} = \{(v_i, v_j)/\ 1 \leq v_i < v_j \leq n\ ; i,j = 1,2, \ldots \ldots, n, i \neq j \text{ and} (v_i, v_j) \text{ are random edges}\}$$

that are disjoint. Consider the mapping

$$\varphi_{\mathcal{G}}: \mathbb{V}_{\mathcal{FR}} \to [0,1] v_i \to \varphi_{\mathcal{G}}(v_i)$$
$$\psi_{\mathcal{G}}: \mathbb{V}_{\mathcal{FR}} \times \mathbb{V}_{\mathcal{FR}} \to [0,1]_P \times [0,1]_{f_A}$$
$$(v_i, v_j) \to \psi_{\mathcal{G}}(v_i, v_j) = (P(v_i, v_j), f_A\ (v_i, v_j))$$

with $P(v_i, v_j) = 0$ if and only if $f_A\ (v_i, v_j) = 0$. Then $\mathbb{G}_{\mathcal{FR}} = (\mathbb{V}_{\mathcal{FR}}, \mathbb{E}_{\mathcal{FR}})$ is called a fuzzy random graph (FRG) if $\psi_{\mathcal{G}}(v_i, v_j) \leq \min\{\varphi_{\mathcal{G}}(v_i), \varphi_{\mathcal{G}}(v_j)\}$. Where $(v_i, v_j)$ corresponds to the edge between $v_i$ and $v_j$, $P(v_i, v_j)$ and $f_A(v_i, v_j)$ represents the probability of the edge $(v_i, v_j)$ and the membership function for the edge $(v_i, v_j)$ within the fuzzy set $A$ in $X$ respectively.

**Definition 2**

In a fuzzy random graph $\mathbb{G}_{\mathcal{FR}} = (\mathbb{V}_{\mathcal{FR}}, \mathbb{E}_{\mathcal{FR}})$, the degree of a fuzzy random vertex $v_i$ is defined as

$$d_{\mathbb{G}_{\mathcal{FR}}}(v_i) = \sum_{v_i \neq v_j \in E_F} \psi_{\mathcal{G}}(v_i, v_j) + \sum_{v_i \neq v_j \in E_R} nP$$

for $(v_i, v_j) \in \mathbb{E}_{\mathcal{FR}}$ and $\psi_{\mathcal{G}}\ (v_i, v_j) = 0$ for $(v_i, v_j)$ not in $\mathbb{E}_{\mathcal{FR}}$. Where $\psi_{\mathcal{G}}(v_i, v_j) = P$ and $n$- number of vertices incident at a random edge. i.e.,$n = 2$

$$d_{\mathbb{G}_{\mathcal{FR}}}(v_i) = \sum_{v_i, v_j \in E_F} \psi_{\mathcal{G}}(v_i, v_j) + \sum_{v_i, v_j \in E_R} 2\ \psi_{\mathcal{G}}(v_i, v_j)$$

**Definition 3**

The fuzzy random graph $\mathbb{G}_{\mathcal{FR}} = (\mathbb{V}_{\mathcal{FR}}, \mathbb{E}_{\mathcal{FR}})$ is called a complete fuzzy random graph if $\psi_{\mathcal{G}}(v_i, v_j) = \min\{\varphi_{\mathcal{G}}(v_i), \varphi_{\mathcal{G}}(v_j)\}$ for all $(v_i, v_j) \in \mathbb{E}_{\mathcal{FR}}$.

**Definition 4**

A fuzzy random graph $\mathbb{G}_{\mathcal{FR}} = (\mathbb{V}_{\mathcal{FR}}, \mathbb{E}_{\mathcal{FR}})$ is connected if any two vertices are joined by a path.

**Definition 5**

A hesitancy fuzzy graph is of the form $G = (V, E)$ such that $\lambda_1: V \to [0,1]$, $\mu_1: V \to [0,1]$, $\nu_1: V \to [0,1]$ denote the degree of membership, non-membership and hesitancy of the vertex $v \in V$ respectively and $\lambda_1(v) + \mu_1(v) + \nu_1(v) = 1$ for every $v \in V$, where $\mu_1(v) = 1 - [\lambda_1(v) + \nu_1(v)]$ and $E \subseteq V \times V$ where $\lambda_2: E \to [0,1]$, $\mu_2: E \to [0,1]$,$\nu_2: E \to [0,1]$ such that

$$\lambda_2(u, v) \leq \min\{\lambda_1(u), \lambda_1(v)\};$$
$$\mu_2(u, v) \leq \max\{\mu_1(u), \mu_1(v)\};$$
$$\nu_2(u, v) \leq \min\{\nu_1(u), \nu_1(v)\} \text{ and}$$
$$0 \leq \lambda_2(u, v) + \mu_2(u, v) + \nu_2(u, v) \leq 1$$

# Hesitancy Fuzzy Random Graph

**Definition 6**

Consider a fuzzy random graph $\mathbb{G}_{\mathcal{FR}} = (\mathbb{V}_{\mathcal{FR}}, \mathbb{E}_{\mathcal{FR}})$. In cases that

*i.* $\varphi_{\alpha_1}: \mathbb{V}_{\mathcal{FR}} \to [0,1]$, $\psi_{\beta_1}: \mathbb{V}_{\mathcal{FR}} \to [0,1]$, $\eta_{\gamma_1}: \mathbb{V}_{\mathcal{FR}} \to [0,1]$ represents the amount of membership, non-membership and hesitancy of the vertex $v_i \in \mathbb{V}_{\mathcal{FR}}$ with

$$\varphi_{\alpha_1}(v_i) + \psi_{\beta_1}(v_i) + \eta_{\gamma_1}(v_i) = 1 \text{ and}$$

***ii.*** $\varphi_{\alpha_2}: \mathbb{E}_{\mathcal{FR}} \rightarrow [0,1]$, $\psi_{\beta_2}: \mathbb{E}_{\mathcal{FR}} \rightarrow [0,1]$, $\eta_{\gamma_2}: \mathbb{E}_{\mathcal{FR}} \rightarrow [0,1]$ such that

$$\varphi_{\alpha_2}(v_i, v_j) \leq \min\{\varphi_{\alpha_1}(v_i), \varphi_{\alpha_1}(v_j)\}$$
$$\psi_{\beta_2}(v_i, v_j) \leq \max\left\{\psi_{\beta_1}(v_i), \psi_{\beta_1}(v_j)\right\}$$
$$\eta_{\gamma_2}(v_i, v_j) \leq \min\{\eta_{\gamma_1}(v_i), \eta_{\gamma_1}(v_j)\} \text{ and}$$
$$0 \leq \varphi_{\alpha_2}(v_i) + \psi_{\beta_2}(v_i) + \eta_{\gamma_2}(v_i) \leq 1 \text{ for all } (v_i, v_j) \in \mathbb{E}_{\mathcal{FR}}$$

then $\mathbb{G}_{\mathcal{FR}}$ is referred to as hesitancy fuzzy random graph (HFRG) denoted by $\mathbb{G}_{\mathcal{HFR}} = (\mathbb{V}_{\mathcal{HFR}}, \mathbb{E}_{\mathcal{HFR}})$

**Example 1**

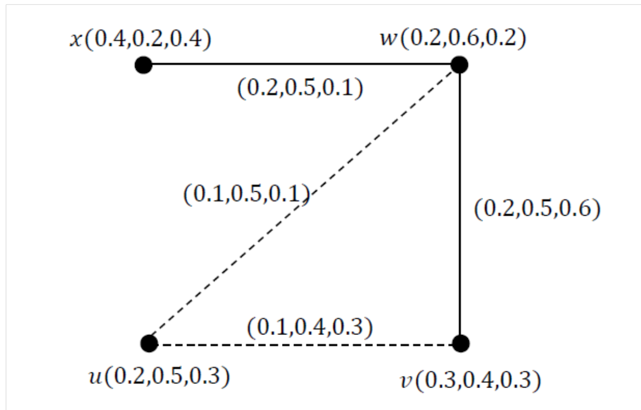


**Figure 1.** Hesitancy fuzzy random graph

**Note:**
In HFRG $\mathbb{G}_{\mathcal{HFR}} = (\mathbb{V}_{\mathcal{HFR}}, \mathbb{E}_{\mathcal{HFR}})$ the membership, non membership and hesitancy of the vertex $v_i \in \mathbb{V}_{\mathcal{HFR}}$ are indicated by $\varphi_{\alpha_{1i}}, \psi_{\beta_{1i}}, \eta_{\gamma_{1i}}$ respectively. Moreover, each of these indicators represent edge $\mathbb{E}_{\mathcal{HFR}}$, degree of membership (if it is fuzzy edge) or probability (if it is random edge) $\varphi_{\alpha_{2i}}$, degree of non-membership $\psi_{\beta_{2i}}$and hesitancy of the edge $\eta_{\gamma_{2i}}$ respectively.

**Definition 7**

If $\alpha_{2ij} = \min(\alpha_{1i}, \alpha_{2j})$ for all $v_i \in \mathbb{V}_{\mathcal{HFR}}$ then the HFRG is considered as an $\alpha$ strong HFRG.

**Example 2**

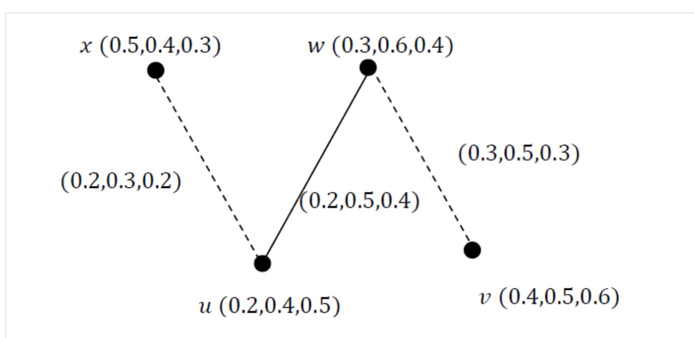


**Figure 2.** $\alpha$ strong Hesitancy fuzzy random graph

**Definition 8**

If $\beta_{2ij} = \min(\beta_{1i}, \beta_{2j})$ for all $v_i \in \mathbb{V}_{\mathcal{HFR}}$ then the HFRG is considered as an $\beta$ strong HFRG.

**Example 3**

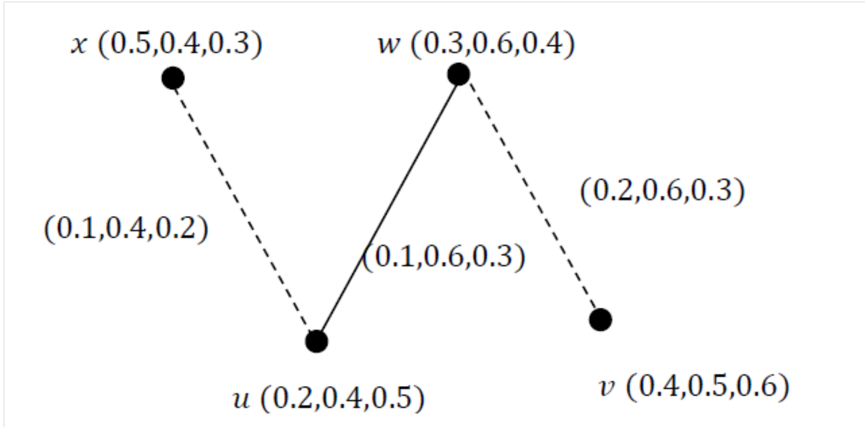


**Figure 3.** $\beta$ strong Hesitancy fuzzy random graph

**Definition 9**

If $\gamma_{2ij} = \min(\gamma_{1i}, \gamma_{2j})$ for all $v_i \in \mathbb{V}_{\mathcal{HFR}}$ then the HFRG is considered as an $\gamma$ strong HFRG.

**Example 4**

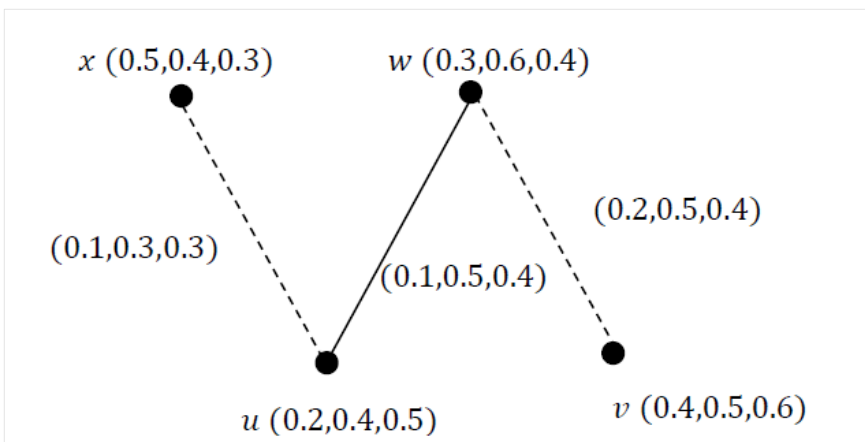


**Figure 4.** $\gamma$ strong Hesitancy fuzzy random graph

**Definition 10**

A strong HFRG can be defined by

$$\alpha_{2ij} = \min(\alpha_{1i}, \alpha_{2j})$$
$$\beta_{2ij} = \min(\beta_{1i}, \beta_{2j})$$
$$\gamma_{2ij} = \min(\gamma_{1i}, \gamma_{2j})$$

**Example 5**

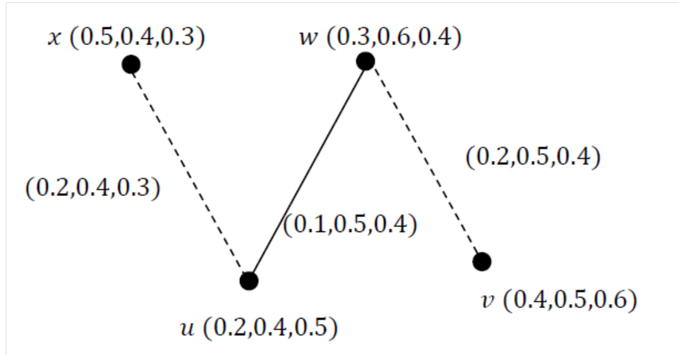


**Figure 5.** Strong Hesitancy fuzzy random graph

**Definition 11**

The complement of the HFRG is a HFRG where $V^C = V$ means that ${\alpha_{2ij}}^C = \alpha_{2ij}$; ${\beta_{2ij}}^C = \beta_{2ij}$; ${\gamma_{2ij}}^C = \gamma_{2ij}$ and

$$
\begin{gathered}
{\alpha_{2ij}}^C = \min(\alpha_{1i}, \alpha_{2j}) - \alpha_{2ij} \\
{\beta_{2ij}}^C = \min(\beta_{1i}, \beta_{2j}) - \beta_{2ij} \\
{\gamma_{2ij}}^C = \min(\gamma_{1i}, \gamma_{2j}) - \gamma_{2ij}
\end{gathered}
$$

**Example 6**

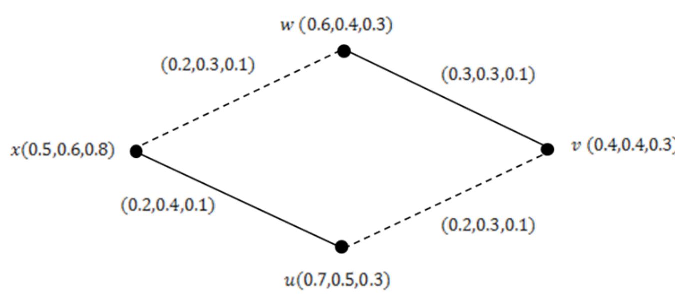


**Figure 6.** Hesitancy fuzzy random graph

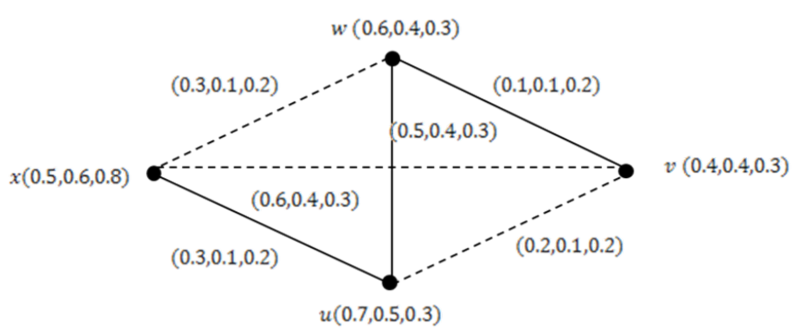


**Figure 7.** Complement of Hesitancy fuzzy random graph

**Theorem 1**

If $\mathbb{G}_{\mathcal{HFR}} = (\mathbb{V}_{\mathcal{HFR}}, \mathbb{E}_{\mathcal{HFR}})$ is a strong HFRG, then $\mathbb{G}_{\mathcal{HFR}}{}^{C} = (\mathbb{V}_{\mathcal{HFR}}{}^{C}, \mathbb{E}_{\mathcal{HFR}}{}^{C})$ is also a strong HFRG.

**Proof:**

There are two cases arise.

***Case (i)***

If $(v_i, v_j) \in \mathbb{E}_{\mathcal{FR}}$, then $\alpha_{2ij}{}^{C} = \min(\alpha_{1i}, \alpha_{2j}) - \alpha_{2ij} = \min(\alpha_{1i}, \alpha_{2j}) - \min(\alpha_{1i}, \alpha_{2j}) = 0$

$$\beta_{2ij}{}^{C} = \min(\beta_{1i}, \beta_{2j}) - \beta_{2ij} = \min(\beta_{1i}, \beta_{2j}) - \min(\beta_{1i}, \beta_{2j}) = 0$$
$$\gamma_{2ij}{}^{C} = \min(\gamma_{1i}, \gamma_{2j}) - \gamma_{2ij} = \min(\gamma_{1i}, \gamma_{2j}) - \min(\gamma_{1i}, \gamma_{2j}) = 0$$

***Case (ii)***

If $(v_i, v_j) \notin \mathbb{E}_{\mathcal{FR}}$, then $\alpha_{2ij}{}^{C} = \min(\alpha_{1i}, \alpha_{2j}) - \alpha_{2ij} = \min(\alpha_{1i}, \alpha_{2j})$

$$\beta_{2ij}{}^{C} = \min(\beta_{1i}, \beta_{2j}) - \beta_{2ij} = \min(\beta_{1i}, \beta_{2j})$$
$$\gamma_{2ij}{}^{C} = \min(\gamma_{1i}, \gamma_{2j}) - \gamma_{2ij} = \min(\gamma_{1i}, \gamma_{2j})$$

Thus if $\mathbb{G}_{\mathcal{HFR}}$ is a strong HFRG, then $\mathbb{G}_{\mathcal{HFR}}{}^{C}$ is also a strong HFRG.

# Results and Discussion

## Beta product of HFRG

**Definition 12**

Let $\mathbb{G}_{\mathcal{HFR}}{}^{1} = (\mathbb{V}_{\mathcal{HFR}}{}^{1}, \mathbb{E}_{\mathcal{HFR}}{}^{1})$, $\mathbb{G}_{\mathcal{HFR}}{}^{2} = (\mathbb{V}_{\mathcal{HFR}}{}^{2}, \mathbb{E}_{\mathcal{HFR}}{}^{2})$ be two HFRGs. The $\beta$ - product of two HFRGs $\mathbb{G}_{\mathcal{HFR}}{}^{1}$ and $\mathbb{G}_{\mathcal{HFR}}{}^{2}$denoted by

$$\mathbb{G}_{\mathcal{HFR}} = \mathbb{G}_{\mathcal{HFR}}{}^{1}(\times_\beta)\mathbb{G}_{\mathcal{HFR}}{}^{2} = (\mathbb{V}_{\mathcal{HFR}}{}^{1}(\times_\beta)\mathbb{V}_{\mathcal{HFR}}{}^{2}, \mathbb{E}_{\mathcal{HFR}}{}^{1}(\times_\beta)\mathbb{E}_{\mathcal{HFR}}{}^{2})$$

Where $\left(\varphi_{\alpha_1}{}^{1}(\times_\beta)\varphi_{\alpha_1}{}^{1}\right)(v) = \varphi_{\alpha_1}{}^{1}(v) \wedge \varphi_{\alpha_1}{}^{1}(v)$

$$\left(\psi_{\beta_1}{}^{1}(\times_\beta)\psi_{\beta_1}{}^{1}\right)(v) = \psi_{\beta_1}{}^{1}(v) \vee \psi_{\beta_1}{}^{1}(v)$$
$$\left(\eta_{\gamma_1}{}^{1}(\times_\beta)\eta_{\gamma_1}{}^{1}\right)(v) = 1 - \{\eta_{\gamma_1}{}^{1}(v) \wedge \eta_{\gamma_1}{}^{1}(v) + \psi_{\beta_1}{}^{1}(v) \vee \psi_{\beta_1}{}^{1}(v)$$

and $(\mathbb{E}_{\mathcal{HFR}}{}^{1}(\times_\beta)\mathbb{E}_{\mathcal{HFR}}{}^{2})(u_1v_1, u_2v_2)$ is defined by,

$$\left(\varphi_{\alpha_2}{}^{1}(\times_\beta)\varphi_{\alpha_2}{}^{2}\right)(u_1v_1, u_2v_2) = \begin{cases} \varphi_{\alpha_2}{}^{1}(u_1u_2) \wedge \varphi_{\alpha_2}{}^{2}(v_1v_2)\,; \text{if } u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{1} \text{ and } v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{2} \\ \varphi_{\alpha_1}{}^{2}(v_1) \wedge \varphi_{\alpha_1}{}^{2}(v_2) \wedge \varphi_{\alpha_2}{}^{1}(u_1u_2)\,; \text{if } v_1 \neq v_2 and u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{1} \\ \varphi_{\alpha_1}{}^{1}(u_1) \wedge \varphi_{\alpha_1}{}^{1}(u_2) \wedge \varphi_{\alpha_2}{}^{2}(v_1v_2)\,; \text{if } u_1 \neq u_2 and v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{2} \end{cases}$$

$$\left(\psi_{\beta_2}{}^{1}(\times_\beta)\psi_{\beta_2}{}^{2}\right)(u_1v_1, u_2v_2) = \begin{cases} \psi_{\beta_2}{}^{1} \wedge \psi_{\beta_2}{}^{2}\,; \text{if } u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{1} \text{ and } v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{2} \\ \psi_{\beta_1}{}^{2}(v_1) \wedge \psi_{\beta_1}{}^{2}(v_2) \wedge \psi_{\beta_2}{}^{1}(u_1u_2)\,; \text{if } v_1 \neq v_2 and u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{1} \\ \psi_{\beta_1}(u_1) \wedge \psi_{\beta_1}{}^{1}(u_2) \wedge \psi_{\beta_2}{}^{2}(v_1v_2)\,; \text{if } u_1 \neq u_2 and v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{2} \end{cases}$$

$$\left(\eta_{\gamma_2}{}^{1}(\times_\beta)\eta_{\gamma_2}{}^{2}\right)(u_1v_1, u_2v_2) = \begin{cases} \eta_{\gamma_2}{}^{1} \wedge \eta_{\gamma_2}{}^{2}\,; \text{if } u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{1} \text{ and } v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{2} \\ \eta_{\gamma_1}{}^{2}(v_1) \wedge \eta_{\gamma_1}{}^{2}(v_2) \wedge \eta_{\gamma_2}{}^{1}(u_1u_2)\,; \text{if } v_1 \neq v_2 and u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{1} \\ \eta_{\gamma_1}{}^{1}(u_1) \wedge \eta_{\gamma_1}{}^{1}(u_2) \wedge \eta_{\gamma_2}{}^{2}(v_1v_2)\,; \text{if } u_1 \neq u_2 and v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{2} \end{cases}$$

**Example 7**

Consider the HFRGs given by,

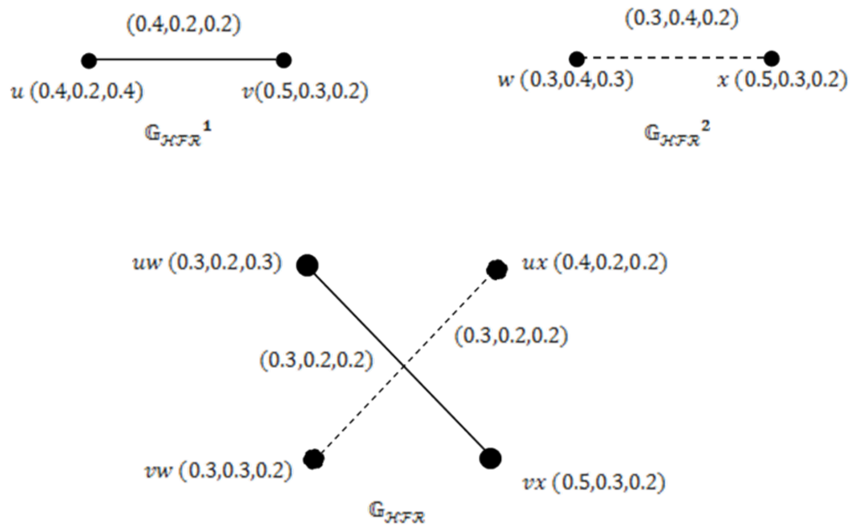


**Figure 8.** Beta product of Hesitancy fuzzy random graphs

**Example 8**

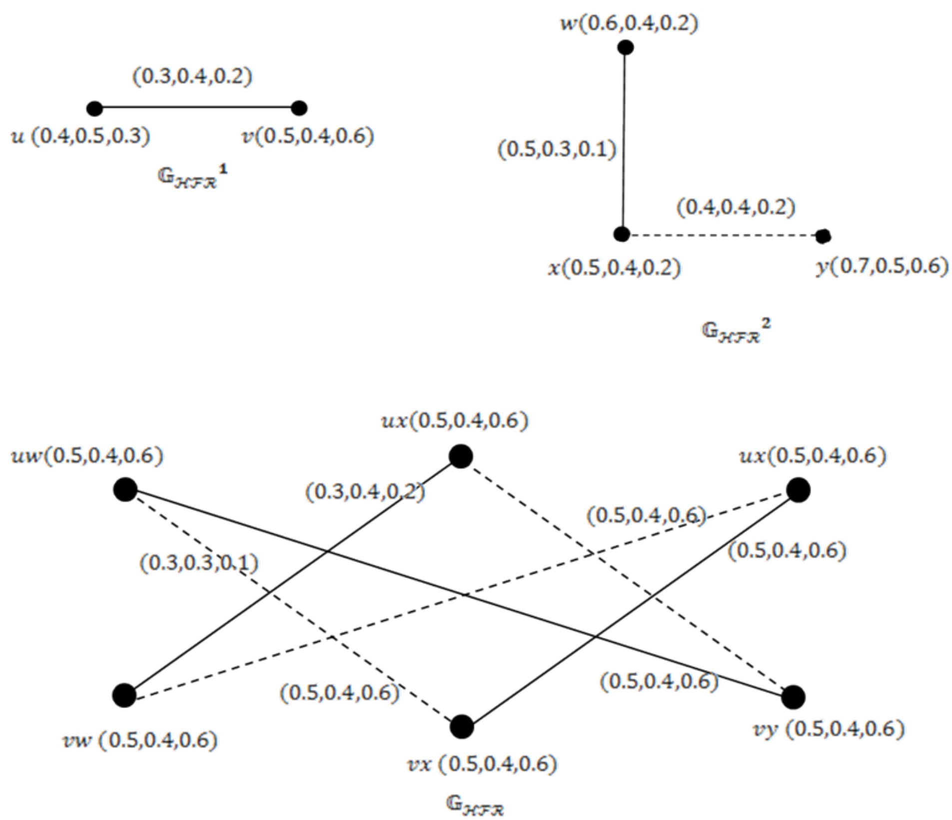


**Figure 9.** Beta product of Hesitancy fuzzy random graphs

**Theorem 2**

If $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}$ and $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ are two strong HFRGs, then their $\beta$- product $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}(\times_\beta)\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ is also a strong HFRG.

**Proof:**
Let $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}} = (\mathbb{V}_{\mathcal{HFR}}{}^{\mathbf{1}}, \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}})$, $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}} = (\mathbb{V}_{\mathcal{HFR}}{}^{\mathbf{2}}, \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}})$ be two HFRGs.
Then for $u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}}$ and $v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}}$,

$$\varphi_{\alpha_2}{}^{\mathbf{1}}(u_1u_2) = \varphi_{\alpha_1}{}^{\mathbf{1}}(u_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(u_2); \qquad \varphi_{\alpha_2}{}^{\mathbf{2}}(v_1v_2) = \varphi_{\alpha_1}{}^{\mathbf{2}}(v_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_2)$$
$$\psi_{\beta_2}{}^{\mathbf{1}}(u_1u_2) = \psi_{\beta_1}{}^{\mathbf{1}}(u_1) \vee \psi_{\beta_1}{}^{\mathbf{1}}(u_2); \qquad \psi_{\beta_2}{}^{\mathbf{2}}(v_1v_2) = \psi_{\beta_1}{}^{\mathbf{2}}(v_1) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_2)$$
$$\eta_{\gamma_2}{}^{\mathbf{1}}(u_1u_2) = \eta_{\gamma_1}{}^{\mathbf{1}}(u_1) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(u_2); \qquad \eta_{\gamma_2}{}^{\mathbf{2}}(v_1v_2) = \eta_{\gamma_1}{}^{\mathbf{2}}(v_1) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_2)$$

***Case (i)***: When if $u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}}$ and $v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}}$

$$\begin{aligned}\left(\varphi_{\alpha_2}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) &= \varphi_{\alpha_2}{}^{\mathbf{1}}(u_1u_2) \wedge \varphi_{\alpha_2}{}^{\mathbf{2}}(v_1v_2)\\ &= \varphi_{\alpha_1}{}^{\mathbf{1}}(u_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(u_2) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_2)\\ &= \left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_1}{}^{\mathbf{2}}\right)(u_1v_1) \wedge \left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_1}{}^{\mathbf{2}}\right)(u_2v_2)\end{aligned}$$

$$\begin{aligned}\left(\psi_{\beta_2}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) &= \psi_{\beta_2}{}^{\mathbf{1}}(u_1u_2) \wedge \psi_{\beta_2}{}^{\mathbf{2}}(v_1v_2)\\ &= \psi_{\beta_1}{}^{\mathbf{1}}(u_1) \vee \psi_{\beta_1}{}^{\mathbf{1}}(u_2) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_1) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_2)\\ &= \left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_1}{}^{\mathbf{2}}\right)(u_1v_1) \vee \left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_1}{}^{\mathbf{2}}\right)(u_2v_2)\end{aligned}$$

$$\begin{aligned}\left(\eta_{\gamma_2}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) &= \eta_{\gamma_2}{}^{\mathbf{1}}(u_1u_2) \wedge \eta_{\gamma_2}{}^{\mathbf{2}}(v_1v_2)\\ &= \eta_{\gamma_1}{}^{\mathbf{1}}(u_1) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(u_2) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_1) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_2)\\ &= \left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_1}{}^{\mathbf{2}}\right)(u_1v_1) \wedge \left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_1}{}^{\mathbf{2}}\right)(u_2v_2)\end{aligned}$$

***Case (ii)***: When if $v_1 \neq v_2$ *and* $u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}}$

$$\begin{aligned}\left(\varphi_{\alpha_2}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) &= \varphi_{\alpha_1}{}^{\mathbf{2}}(v_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_2) \wedge \varphi_{\alpha_2}{}^{\mathbf{1}}(u_1u_2)\\ &= \varphi_{\alpha_1}{}^{\mathbf{2}}(v_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_2) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(u_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(u_2)\\ &= \left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_1}{}^{\mathbf{2}}\right)(u_1v_1) \wedge \left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_1}{}^{\mathbf{2}}\right)(u_2v_2)\end{aligned}$$

$$\begin{aligned}\left(\psi_{\beta_2}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) &= \psi_{\beta_1}{}^{\mathbf{2}}(v_1) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_2) \vee \psi_{\beta_2}{}^{\mathbf{1}}(u_1u_2)\\ &= \psi_{\beta_1}{}^{\mathbf{2}}(v_1) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_2) \vee \psi_{\beta_1}{}^{\mathbf{1}}(u_1) \vee \psi_{\beta_1}{}^{\mathbf{1}}(u_2)\\ &= \left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_1}{}^{\mathbf{2}}\right)(u_1v_1) \vee \left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_1}{}^{\mathbf{2}}\right)(u_2v_2)\end{aligned}$$

$$\begin{aligned}\left(\eta_{\gamma_2}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) &= \eta_{\gamma_1}{}^{\mathbf{2}}(v_1) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_2) \wedge \eta_{\gamma_2}{}^{\mathbf{1}}(u_1u_2)\\ &= \eta_{\gamma_1}{}^{\mathbf{2}}(v_1) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_2) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(u_1) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(u_2)\\ &= \left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_1}{}^{\mathbf{2}}\right)(u_1v_1) \wedge \left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_1}{}^{\mathbf{2}}\right)(u_2v_2)\end{aligned}$$

***Case (iii)***: When if $u_1 \neq u_2$ *and* $v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}}$

$$\begin{aligned}\left(\varphi_{\alpha_2}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) &= \varphi_{\alpha_1}{}^{\mathbf{1}}(u_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(u_2) \wedge \varphi_{\alpha_2}{}^{\mathbf{2}}(v_1v_2)\\ &= \varphi_{\alpha_1}{}^{\mathbf{1}}(u_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(u_2) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_2)\\ &= \left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_1}{}^{\mathbf{2}}\right)(u_1v_1) \wedge \left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_1}{}^{\mathbf{2}}\right)(u_2v_2)\end{aligned}$$

$$\begin{aligned}\left(\psi_{\beta_2}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) &= \psi_{\beta_1}(u_1) \wedge \psi_{\beta_1}{}^{\mathbf{1}}(u_2) \wedge \psi_{\beta_2}{}^{\mathbf{2}}(v_1v_2)\\ &= \psi_{\beta_1}(u_1) \wedge \psi_{\beta_1}{}^{\mathbf{1}}(u_2) \wedge \psi_{\beta_1}{}^{\mathbf{2}}(v_1) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_2)\end{aligned}$$

$$= \left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_1}{}^{\mathbf{2}}\right)(u_1v_1) \vee \left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_1}{}^{\mathbf{2}}\right)(u_2v_2)$$

$$\left(\eta_{\gamma_2}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) = \eta_{\gamma_1}{}^{\mathbf{1}}(u_1) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(u_2) \wedge \eta_{\gamma_2}{}^{\mathbf{2}}(v_1v_2)$$
$$= \eta_{\gamma_1}{}^{\mathbf{1}}(u_1) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(u_2) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_1) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_2)$$
$$= \left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_1}{}^{\mathbf{2}}\right)(u_1v_1) \wedge \left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_1}{}^{\mathbf{2}}\right)(u_2v_2)$$

Thus the $\beta$- product $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}(\times_\beta)\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ is also a strong HFRG.

**Theorem 3**

If $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}$ and $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ are two HFRGs such that their $\beta$- product $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}(\times_\beta)\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ is a strong HFRG , then atleast one of $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}$ or $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ will be strong.

**Proof:**
Assume that the two HFRGs $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}$and $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ are not strong. Then there exists at least one $u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}}$ and $v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}}$,

$$\varphi_{\alpha_2}{}^{\mathbf{1}}(u_1u_2) < \varphi_{\alpha_1}{}^{\mathbf{1}}(u_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(u_2); \qquad \varphi_{\alpha_2}{}^{\mathbf{2}}(v_1v_2) < \varphi_{\alpha_1}{}^{\mathbf{2}}(v_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_2)$$
$$\psi_{\beta_2}{}^{\mathbf{1}}(u_1u_2) < \psi_{\beta_1}{}^{\mathbf{1}}(u_1) \vee \psi_{\beta_1}{}^{\mathbf{1}}(u_2); \qquad \psi_{\beta_2}{}^{\mathbf{2}}(v_1v_2) < \psi_{\beta_1}{}^{\mathbf{2}}(v_1) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_2)$$
$$\eta_{\gamma_2}{}^{\mathbf{1}}(u_1u_2) < \eta_{\gamma_1}{}^{\mathbf{1}}(u_1) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(u_2); \qquad \eta_{\gamma_2}{}^{\mathbf{2}}(v_1v_2) < \eta_{\gamma_1}{}^{\mathbf{2}}(v_1) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_2)$$

Let if $u_1u_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}}$ and $v_1v_2 \in \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}}$

$$\left(\varphi_{\alpha_2}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) = \varphi_{\alpha_2}{}^{\mathbf{1}}(u_1u_2) \wedge \varphi_{\alpha_2}{}^{\mathbf{2}}(v_1v_2)$$
$$< \varphi_{\alpha_1}{}^{\mathbf{1}}(u_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(u_2) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_1) \wedge \varphi_{\alpha_1}{}^{\mathbf{2}}(v_2)$$
*ie.,* $$\left(\varphi_{\alpha_2}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) < \left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_1}{}^{\mathbf{2}}\right)(u_1v_1) \wedge \left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\beta)\varphi_{\alpha_1}{}^{\mathbf{2}}\right)(u_2v_2)$$

$$\left(\psi_{\beta_2}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) = \psi_{\beta_2}{}^{\mathbf{1}}(u_1u_2) \wedge \psi_{\beta_2}{}^{\mathbf{2}}(v_1v_2)$$
$$< \psi_{\beta_1}{}^{\mathbf{1}}(u_1) \vee \psi_{\beta_1}{}^{\mathbf{1}}(u_2) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_1) \vee \psi_{\beta_1}{}^{\mathbf{2}}(v_2)$$
*ie.,* $$\left(\psi_{\beta_2}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) < \left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_1}{}^{\mathbf{2}}\right)(u_1v_1) \vee \left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\beta)\psi_{\beta_1}{}^{\mathbf{2}}\right)(u_2v_2)$$

$$\left(\eta_{\gamma_2}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) = \eta_{\gamma_2}{}^{\mathbf{1}}(u_1u_2) \wedge \eta_{\gamma_2}{}^{\mathbf{2}}(v_1v_2)$$
$$< \eta_{\gamma_1}{}^{\mathbf{1}}(u_1) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(u_2) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_1) \wedge \eta_{\gamma_1}{}^{\mathbf{2}}(v_2)$$
*ie.,* $$\left(\eta_{\gamma_2}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_2}{}^{\mathbf{2}}\right)(u_1v_1, u_2v_2) < \left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_1}{}^{\mathbf{2}}\right)(u_1v_1) \wedge \left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\beta)\eta_{\gamma_1}{}^{\mathbf{2}}\right)(u_2v_2)$$

This implies that the $\beta$- product $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}(\times_\beta)\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ is not strong. So at least one of $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}$ or $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ will be strong.

## Gamma Product of HFRGs:

**Definition 13**
The $\gamma$ - product of two HFRGs $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}} = (\mathbb{V}_{\mathcal{HFR}}{}^{\mathbf{1}}, \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}})$, $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}} = (\mathbb{V}_{\mathcal{HFR}}{}^{\mathbf{2}}, \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}})$ is defined as a HFRG $\mathbb{G}_{\mathcal{HFR}} = \mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}(\times_\gamma)\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}} = \left(\mathbb{V}_{\mathcal{HFR}}{}^{\mathbf{1}}(\times_\gamma)\mathbb{V}_{\mathcal{HFR}}{}^{\mathbf{2}}, \mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}}(\times_\gamma)\mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}}\right)$

Where $\left(\varphi_{\alpha_1}{}^{\mathbf{1}}(\times_\gamma)\varphi_{\alpha_1}{}^{\mathbf{1}}\right)(v) = \varphi_{\alpha_1}{}^{\mathbf{1}}(v) \wedge \varphi_{\alpha_1}{}^{\mathbf{1}}(v)$

$$\left(\psi_{\beta_1}{}^{\mathbf{1}}(\times_\gamma)\psi_{\beta_1}{}^{\mathbf{1}}\right)(v) = \psi_{\beta_1}{}^{\mathbf{1}}(v) \vee \psi_{\beta_1}{}^{\mathbf{1}}(v)$$

$$\left(\eta_{\gamma_1}{}^{\mathbf{1}}(\times_\gamma)\eta_{\gamma_1}{}^{\mathbf{1}}\right)(v) = 1 - \{\eta_{\gamma_1}{}^{\mathbf{1}}(v) \wedge \eta_{\gamma_1}{}^{\mathbf{1}}(v) + \psi_{\beta_1}{}^{\mathbf{1}}(v) \vee \psi_{\beta_1}{}^{\mathbf{1}}(v)$$

and $(\mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{1}}(\times_\gamma)\mathbb{E}_{\mathcal{HFR}}{}^{\mathbf{2}})(u_1v_1, u_2v_2)$ is defined by,

$$\left(\varphi_{\alpha_2}{}^{1}(\times_{\gamma})\varphi_{\alpha_2}{}^{2}\right)(u_1v_1,u_2v_2)=\begin{cases}\varphi_{\alpha_2}{}^{1}\wedge\varphi_{\alpha_2}{}^{2}\,;\text{if } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\\ \varphi_{\alpha_1}{}^{2}(v_2)\wedge\varphi_{\alpha_2}{}^{1}(u_1u_2)\,;\text{if } v_1=v_2 \text{ and } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\\ \varphi_{\alpha_1}{}^{1}(u_1)\wedge\varphi_{\alpha_2}{}^{2}(v_1v_2)\,;\text{if } u_1=u_2 \text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\\ \varphi_{\alpha_1}{}^{1}(u_1)\wedge\varphi_{\alpha_1}{}^{1}(u_2)\wedge\varphi_{\alpha_2}{}^{2}(v_1v_2)\,;\text{if } u_1\neq u_2 \text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\\ \varphi_{\alpha_1}{}^{2}(v_1)\wedge\varphi_{\alpha_1}{}^{2}(v_2)\wedge\varphi_{\alpha_2}{}^{1}(u_1u_2)\,;\text{if } v_1\neq v_2 \text{ and } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\end{cases}$$

$$\left(\psi_{\beta_2}{}^{1}(\times_{\gamma})\psi_{\beta_2}{}^{2}\right)(u_1v_1,u_2v_2)=\begin{cases}\psi_{\beta_2}{}^{1}\wedge\psi_{\beta_2}{}^{2}\,;\text{if } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\\ \psi_{\beta_1}{}^{2}(v_2)\wedge\psi_{\beta_2}{}^{1}(u_1u_2)\,;\text{if } v_1=v_2 \text{ and } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\\ \psi_{\beta_1}{}^{1}(u_1)\wedge\psi_{\beta_2}{}^{2}(v_1v_2)\,;\text{if } u_1=u_2 \text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\\ \psi_{\beta_1}{}^{2}(v_1)\wedge\psi_{\beta_1}{}^{2}(v_2)\wedge\psi_{\beta_2}{}^{1}(u_1u_2)\,;\text{if } v_1\neq v_2 \text{ and } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\\ \psi_{\beta_1}{}^{2}(u_1)\wedge\psi_{\beta_1}{}^{1}(u_2)\wedge\psi_{\beta_2}{}^{2}(v_1v_2)\,;\text{if } u_1\neq u_2 \text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\end{cases}$$

$$\left(\eta_{\gamma_2}{}^{1}(\times_{\gamma})\eta_{\gamma_2}{}^{2}\right)(u_1v_1,u_2v_2)=\begin{cases}\eta_{\gamma_2}{}^{1}\wedge\eta_{\gamma_2}{}^{2}\,;\text{if } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\\ \eta_{\gamma_1}{}^{2}(v_2)\wedge\eta_{\gamma_2}{}^{1}(u_1u_2)\,;\text{if } v_1=v_2 \text{ and } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\\ \eta_{\gamma_1}{}^{1}(u_1)\wedge\eta_{\gamma_2}{}^{2}(v_1v_2)\,;\text{if } u_1=u_2 \text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\\ \eta_{\gamma_1}{}^{2}(v_1)\wedge\eta_{\gamma_1}{}^{2}(v_2)\wedge\eta_{\gamma_2}{}^{1}(u_1u_2)\,;\text{if } v_1\neq v_2 \text{ and } u_1u_2\in\mathbb{E}_{\mathcal{HFR}}{}^{1}\\ \eta_{\gamma_1}{}^{1}(u_1)\wedge\eta_{\gamma_1}{}^{1}(u_2)\wedge\eta_{\gamma_2}{}^{2}(v_1v_2)\,;\text{if } u_1\neq u_2 \text{ and } v_1v_2\in\mathbb{E}_{\mathcal{HFR}}{}^{2}\end{cases}$$

**Example 9**

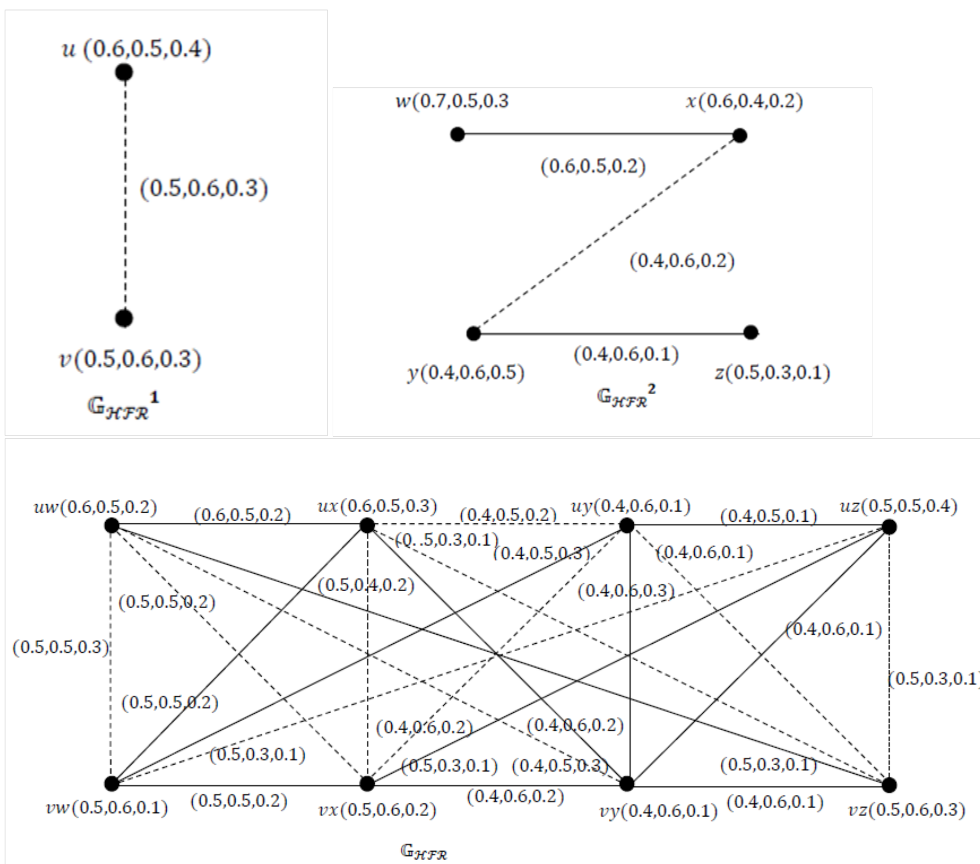


**Figure 10.** Gamma product of Hesitancy fuzzy random graphs

**Result:**
If $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{1}}$ and $\mathbb{G}_{\mathcal{HFR}}{}^{\mathbf{2}}$ are two strong HFRGs, then their $\gamma$- product need not to be a strong HFRG.

**Comparison of beta and gamma products of HFRG:**
Comparing beta and gamma products of hesitancy fuzzy random graphs is a relatively niche area of research in fuzzy graph theory. In these types of graphs, hesitancy arises when there is uncertainty or hesitation in assigning membership values to elements, reflecting hesitation between a degree of membership and non-membership.

- **Beta Product:** A product where the vertex and edge membership functions are combined using a beta-type operation. This can involve, for instance, taking a specific combination (like a product or a sum) of the hesitancy values of individual graphs.
- **Gamma Product:** Similar, but the operation is defined using gamma-type rules, which may differ in how they handle hesitancy values.

**Table 1.** Comparison of beta and gamma product of HFRG

| Comparison | Beta Product | Gamma Product |
|---|---|---|
| **Handling of Hesitancy** | Beta product tends to be more flexible and can use operations like weighted averages, arithmetic products, or sums to combine hesitancy values | Gamma product might use stricter or more conservative rules, such as max-min combinations or other non-linear compositions that handle the uncertainty more cautiously. |
| **Combination Rules** | In the beta product, the combination might focus on aggregating the hesitancy values from different graphs in a more flexible manner (e.g., using linear combinations, averaging methods). | In the gamma product, the combination might be stricter, focusing on selecting the most uncertain or conservative outcomes from the hesitancy values, often using max-min or non-linear combinations. |
| **Behavior Under Randomness** | Beta product could involve smoother handling of randomness and hesitancy, where the combined hesitancy values adjust more gradually based on the randomness in the graph. | Gamma product might provide more conservative outcomes under randomness, choosing the most uncertain (worst-case) combination when randomness introduces ambiguity. |
| **Computational Complexity** | Beta product might involve multiple operations, such as weighted averages, linear combinations, or more complex aggregation rules that increase the complexity. | Gamma product might involve even more computationally intensive operations like max-min or other non-linear combinations, especially when the hesitancy and randomness are handled together. |
| **Flexibility vs. Conservatism:** | Beta product offers more flexibility, allowing for smoother or less restrictive combinations of the hesitancy and randomness. | Gamma product, in contrast, is more conservative and cautious, focusing on preserving the maximum uncertainty or hesitation, making it better suited for scenarios where risk or extreme cases must be considered. |

The beta product and gamma product offer different methods of combining hesitancy fuzzy random graphs, with the beta product being more flexible and computationally lighter, while the gamma product tends to be more rigid but possibly more robust in high-uncertainty or risk scenarios. The choice between them depends on the specific needs of the problem, such as whether flexibility or strictness is desired in handling hesitancy and randomness.
The beta and gamma products in hesitancy fuzzy random graphs are specific types of operations designed to handle the combination of multiple graphs under conditions of uncertainty (hesitancy). These operations differ from traditional or existing operations in fuzzy graphs in several ways, primarily in how they handle uncertainty and hesitancy within the graph structures. A detailed comparison of how the beta and gamma products vary from existing operations in HFRG is given below.

**Table 2.** Comparison of the beta and gamma products with the existing operations of HFRG

| Comparison | Existing Operations | Beta and Gamma Products |
|---|---|---|
| **Handling of Hesitancy** | Traditional operations in fuzzy graphs, such as union, intersection, or complement, do not explicitly account for hesitancy. They are primarily designed for graphs where membership values (of vertices and edges) are known or deterministic. For example, the intersection of two fuzzy graphs might take the minimum of the membership values for corresponding vertices and edges, while the union might take the maximum. | In hesitancy fuzzy random graphs, the membership of an element (vertex or edge) is not a single value but can be represented by an interval or a set of possible values reflecting the uncertainty. Both beta and gamma products aim to combine these intervals or sets in ways that explicitly handle hesitancy. |
| **Combination Rules** | In regular fuzzy graph theory, combination operations like the Cartesian product or direct product involve straightforward combinations of membership values (often using basic arithmetic operations such as min, max, or product). These operations are designed for graphs where there's a known membership value for each edge and vertex. | These introduce more complex combination rules to handle the additional uncertainty introduced by hesitancy. |
| **Behavior Under Randomness** | In traditional fuzzy random graphs, randomness typically affects the presence or membership value of edges or vertices based on a probability distribution. Operations in these graphs deal with combining randomness and fuzziness, but hesitancy is not explicitly modeled | Both the beta and gamma products are designed to handle the triple combination of fuzziness, randomness, and hesitancy. This means they do not only deal with probabilistic uncertainty but also with hesitation about the membership values. |
| **Computational Complexity** | Basic operations like union, intersection, or complement in fuzzy graphs are generally computationally simpler because they involve straightforward arithmetic or logical combinations (min, max, complement, etc.) between the membership values. | The computational complexity of the beta and gamma products can be higher due to the more sophisticated nature of the operations. |
| **Flexibility vs. Conservatism** | Fuzzy graph operations like union and intersection are usually fixed in how they combine membership values, typically focusing on combining certainty in a deterministic or probabilistic way. | Offer a balance between flexibility and conservatism in the way they manage the hesitancy. |

## Applications

Modeling uncertain relationships or affiliations in social networks where connections are not clearly defined. Understanding uncertain interactions or dependencies in biological systems where data is inherently noisy or incomplete. Assessing reliability or vulnerability of infrastructure networks such that power grids, transportation networks under uncertain conditions. These applications highlight how beta and gamma products, along with concepts like hesitancy in fuzzy random graphs, can be utilized across various fields to model and analyze uncertainty and probabilistic relationships with complex systems. Beta and gamma products are more suited for problems where there is uncertainty not just in the structure (randomness) but also in the values themselves (hesitancy). They are useful in areas such as risk analysis, decision-making under uncertainty, and modeling scenarios where there is significant hesitation in assigning values to graph elements.

## Conclusion

This study proposes the innovative concept of a Hesitancy fuzzy random graph. The strong Hesitancy fuzzy random graph and its complement are also discussed, with examples and theorems. Furthermore, examples with related theorems are given to discover the beta and gamma products of two Hesitancy fuzzy random graphs.

## Conflicts of Interest

The author declare that there is no conflict of interest regarding the publication of this paper.

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