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Character Sums Associated with Non-homogeneous Beatty Sequences for Quadratic Common Difference

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Abstract Beatty sequence is a special type of sequence by taking the floor of the positive value of an irrational number. This paper will find an explicit bounds of character sums associated with non-homogeneous Beatty sequence in quadratic fields. Next, we obtained the cardinality of character sums associated with composite moduli. Finally, we found the estimated bound of character sums associated with composite moduli by using the discrepancy and the cardinality.

Keywords: Beatty sequence, Estimation of character sums, Cardinality, Multiplicative group, Discrepancy.

Introduction

Beatty sequences are divided into two types, namely homogeneous and non-homogeneous. In our discussion, we are focusing on non-homogeneous Beatty sequences given by

$$(\lfloor n\theta+\lambda\rfloor)_{n\le N} = \lfloor 1\theta+\lambda\rfloor, \lfloor 2\theta+\lambda\rfloor, \lfloor 3\theta+\lambda\rfloor, \dots, \lfloor N\theta+\lambda\rfloor$$

where θ is an irrational number and λ is a real number.

In [1], they used Maynard's methods to prove that there exist bounded gaps between primes in the homogeneous Beatty sequence. That is, $\#\{x\le n<2x$: there exist m distinct primes of the form $\lfloor\theta r\rfloor, r\in[n,n+\Delta_{\theta,m}]\}\gg x(\log x)^{-B}$. [2] found that the bound of multiplicative character sums of a non-homogeneous Beatty sequence is by obtaining the Diophantine properties of θ in order to obtain a sharper estimate and the results are as follows:

Theorem 1. Let θ be a fixed irrational number. Then, for all real numbers λ, integers a,g,m with $\gcd(ag,m)=1$, and positive integers $N\le t$, where t is the multiplicative order of g modulo m, the following bound holds: $S_m(\theta,\lambda,\chi;N)\ll m^{1/4}N^{1/2}+ND_{\theta,\lambda}(N)$.

Theorem 2. Let θ be a fixed irrational number. For any fixed $\delta>0$, there exists a constant $\eta>0$ such that for all real numbers λ, integers a, g and a prime p with $\gcd(ag,p)=1$ and positive integers $p^\delta<N\le t$, where t is the multiplicative order of g modulo p, the following bound holds: $S_p(\theta,\lambda,\chi;N)\ll Np^{-\eta}N^{1/2}+ND_{\theta,\lambda}(N)$.

Then, [3] estimated the summation of the number of prime divisors by using the method in [2]. They found that the number of prime divisors of an integer $n\neq 0$ with multiplicities defined by $\sigma(n)$ is given by $\sum_{n\le N}(-1)^{\sigma(\lfloor\theta n+\lambda\rfloor)}=o(N)$. [4] investigated the application of non-homogeneous Beatty sequences in invariant games. They used Sturmian word, which are infinite binary sequences that arises in combinatorics on sequences to characterize two pairs of complementary non-homogeneous Beatty

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sequences. [5] studied the number of primes in the intersection of two or more non-homogeneous Beatty sequences and give a few theorems for different compatibility conditions. [6] applied the complementary Beatty sequences in solving the trigonometric functions. They let $\theta = \sqrt{3}$ for tangent function and $\theta = \sqrt{6}$ for sine function in order to solve the trigonometric inequalities. [7] worked on the disjoint complementary Beatty sequence which covering the rational non-homogeneous Beatty sequence. [8] investigated the new approach in [2] to estimate the bound of character sums. They obtained an estimation of the bound associated with composite moduli over prime.

Theorem 3. Let θ be a fixed irrational number, λ be any real number and m is any composite number with primitive elements. Then, for positive prime $P \leq m$, set of prime $\mathcal{P}$ and non-trivial multiplicative characters $\chi \pmod m$, the following bound holds: $S_m(\theta,\lambda,\chi;P) \ll \phi(m)^{1/4}\#\mathcal{P}^{1/2} + \#\mathcal{P}D_{\theta,\lambda}(P)$.

Character sums can be used to find the number of solutions of equations over a given finite field. These sums can be obtained by one character or more. Here, we provided the propositions stated in [9] on the character sums associated with prime modulo. Suppose that p is an odd prime and F_p^* is a finite field of multiplicative group modulo p. Then,

Proposition 1. Let g be a primitive element of F_p^* with order $p-1$. For each fixed integer j where $0 \leq j \leq \phi(m)-1$, a multiplicative character of F_p^*, denoted by χ_j is given by

$$\chi_j(g^k) = e^{2\pi i\left(\frac{jk}{p-1}\right)} \text{ where } k = 0,1,\ \dots, p-1.$$

Proposition 2. For multiplicative character, if $a, b \in F_p^*$. Then,

$$\sum_{\chi} \chi_c(a)\overline{\chi_c(b)} = \begin{cases} p & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

where the sum is extended over all multiplicative character χ of F_p^*.

The other researches about Beatty sequences can refer to [10–13]. In this paper, we are going to study the problem 34 in [14] and give an explicit solution to the same problem. [15] purposed that $S_k(\theta,\lambda,\chi;N) = o(N)$. In our discussion, we let $\theta = \sqrt{k}$ where k is a non-perfect square integer and λ is an integer. We will explain more in the next section.

Multiplicative Character Sums

For any real number x, $\lfloor x \rfloor$ denotes the greatest integer less or equals to x. $\{x\}$ denotes the fractional part of x, that is $x - \lfloor x \rfloor$. Given functions J and K, the notations $J \ll K$, $K \gg J$ and $J = O(K)$ are all equivalent to $|J| \leq C|K|$ holds with some constant $C > 0$. By the definition of character sums, we have

$$S(\theta,\lambda,\chi;N) = \sum_{n \leq N} \chi(\lfloor n\theta + \lambda \rfloor)$$

where θ is an irrational number in quadratic field and χ is a non-trivial multiplicative character. Let λ be any integer. The fractional part of $n\theta + \lambda$ is the same as the fractional part of $n\theta$, that is $\{n\theta + \lambda\} = \{n\theta\}$. In this paper, we consider $\theta = \sqrt{k}$ with k is a non-perfect square integer at most 50, then

$$S(\theta,\lambda,\chi;N) = \chi(\lfloor 1\sqrt{k} + \lambda \rfloor) + \chi(\lfloor 2\sqrt{k} + \lambda \rfloor) + \chi(\lfloor 3\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor N\sqrt{k} + \lambda \rfloor). \quad (1)$$

Next, we grouped (1) into following manner.

$$\begin{aligned} S(\theta,\lambda,\chi;N) = {} & \chi(\lfloor 1\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor (m_1 - 1)\sqrt{k} + \lambda \rfloor) && (2) \\ & + \chi(\lfloor (m_1)\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor (m_1 + T)\sqrt{k} + \lambda \rfloor) && (3) \\ & + \chi(\lfloor (m_2)\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor (m_2 + T)\sqrt{k} + \lambda \rfloor) && (4) \\ & + \cdots && (5) \\ & + \chi(\lfloor (m_h)\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor (m_h + T)\sqrt{k} + \lambda \rfloor) && (6) \\ & + \chi(\lfloor (m_h + T + 1)\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor N\sqrt{k} + \lambda \rfloor) && (7) \end{aligned}$$

In (2), we grouped the beginning of the non-repeated series with cardinality $m_1 - 1$ since there is no pattern can be obtained. Then it follows by a few of the repeated series shown in (3) until (6). This is because (3) to (6) has the same cardinality which is $T + 1$. Lastly, the remaining series in which the cardinality less than $T + 1$, that is from (6) until the N-th term, the cardinality is $N - (m_1 + T)$. We then continued by extending (7) via adding in some terms to the series until the cardinality becomes $T + 1$, which is same to (3) until (6). It became

$$\begin{aligned} S(\theta,\lambda,\chi;N) \leq {} & \chi(\lfloor 1\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor (m_1 - 1)\sqrt{k} + \lambda \rfloor) \\ & + \chi(\lfloor (m_1)\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor (m_1 + T)\sqrt{k} + \lambda \rfloor) \end{aligned}$$

$$+\chi(\lfloor(m_2)\sqrt{k}+\lambda\rfloor)+\cdots+\chi(\lfloor(m_2+T)\sqrt{k}+\lambda\rfloor)$$
$$+\cdots$$
$$+\chi(\lfloor(m_h)\sqrt{k}+\lambda\rfloor)+\cdots+\chi(\lfloor(m_h+T)\sqrt{k}+\lambda\rfloor)$$
$$+\chi(\lfloor(m_h+T+1)\sqrt{k}+\lambda\rfloor)+\cdots+\chi(\lfloor N\sqrt{k}+\lambda\rfloor)$$
$$+\cdots+\chi(\lfloor(m_{h+1}+T)\sqrt{k}+\lambda\rfloor).$$

Then, (7) is now have the same cardinality as (3) until (6). Then, from (3) until extended (7) we have

$$S(\theta,\lambda,\chi;N)\le\chi(\lfloor 1\sqrt{k}+\lambda\rfloor)+\cdots+\chi(\lfloor(m_1-1)\sqrt{k}+\lambda\rfloor) \quad (8)$$
$$+M[\chi(\lfloor(m_M)\sqrt{k}+\lambda\rfloor)+\cdots+\chi(\lfloor(m_M+T)\sqrt{k}+\lambda\rfloor)] \quad (9)$$

where $M=h+1$.

Example 1. Let $k=3$, $\lambda=5$ and $N=100$. Then equation (1) becomes

$$S(\sqrt{3},5,\chi;100)=\chi(6)+\chi(8)+\chi(10)+\chi(11)+\chi(13)+\cdots+\chi(178).$$

We consider the Beatty sequence in each of the character above, we listed out all the 100 terms and grouped it into 8 groups as shown in the Table 1:

Table 1. The 100 terms in $S(\sqrt{3},5,\chi;100)$ without character χ.

6,8,10,	11,13,15,17,	18,20,22,24,	
25,27,29,	30,32,34,36,	37,39,41,43,	44,46,48,50,
51,53,55,	56,58,60,62,	63,65,67,69,	70,72,74,76,
77,79,81,	82,84,86,88,	89,91,93,95,	
96,98,100,	101,103,105,107,	108,110,112,114,	115,117,119,121,
122,124,126,	127,129,131,133,	134,136,138,140,	141,143,145,147,
148,150,152,	153,155,157,159,	160,162,164,166,	167,169,171,173,
174,176,178.			

By observing the pattern above, we can see that the most pattern appeared is 3 consecutive terms, follows by 3 times of 4 consecutive terms. For example, we take the second line in Table 1 which is $25,27,29,\ \ldots,46,48,50$. This pattern is called repeated sequence. When we consider back to the sum of character, it is known as repeated series. That is (3) until (6). The series before the repeated series begins will be written like in (2). For this example, it is the first line in Table 1. That is $6,8,10,\ \ldots,20,22,24$. On the other hand, the remaining series at the end of the sequence will be written like in (7). That is $174,176,178$ (last line in Table 1) which only consists of 3 terms. Lastly, we can see that the sequence below is not enough terms to be consider as the repeated sequence which is $77,79,81,\ \ldots,91,93,95$ (line 4 in Table 1). This will then extend by adding terms in order to fulfil the requirement for repeated sequence. Besides, we also extend the sequence $174,176,178$ (last line in Table 1) to fulfil the requirement for repeated sequence too, which mean all the lines in Table 1 (except first line) have 15 terms each. We let T_a where $a\in\mathbb{Z}^+$ be the terms to be added into the sequence. Take note that T_a is not the a-th term of the sequence that is T_6 is not the 6-th term of the sequence. Next, we consider the character sums as the following:

$$S(\sqrt{3},5,\chi;100)\le\chi(6)+\chi(8)+\chi(10)+\chi(11)+\cdots+\chi(89)+\chi(91)+\chi(93)+\chi(95)$$
$$+\chi(T_1)+\chi(T_2)+\chi(T_3)+\chi(T_4)+\chi(96)+\chi(98)+\chi(100)+\cdots$$
$$+\chi(174)+\chi(176)+\chi(178)+\chi(T_5)+\chi(T_6)+\chi(T_7)+\chi(T_8)$$
$$+\chi(T_9)+\chi(T_{10})+\chi(T_{11})+\chi(T_{12})+\chi(T_{13})+\chi(T_{14})+\chi(T_{15})$$
$$+\chi(T_{16}).$$

Then, we grouped the repeated series as shown in (8) and (9), we obtained

$$S(\sqrt{3},5,\chi;100)\le\chi(6)+\chi(8)+\chi(10)+\chi(11)+\chi(13)+\chi(15)+\cdots+\chi(22)+\chi(24)$$
$$+7[\chi(t_1)+\chi(t_2)+\chi(t_3)+\chi(t_4)+\chi(t_9)+\chi(t_{10})+\cdots+\chi(t_{13})+\chi(t_{14})+\chi(t_{15})]$$

where t_a represented the terms in the sequence.

In order to obtain the upper bound of the sum explicitly, we substituted the values of k and λ into (1) and observed the results as in (8) and (9). From the observation, we found that the series of $S(\theta,\lambda,\chi;N)$ have some different types of repeating pattern. We classified them into four different cases of k as follow for the values of $k\le 50$.

(i) Case 1: $k=\{3,8,14,15,23,24,34,35,47,48\}$
(ii) Case 2: $k=\{5,10,17,18,21,26,27,29,37,38,46,50\}$
(iii) Case 3: $k=\{2,6,7,11,12,13,19,20,22,30,32,33,39,40,41,42,44\}$
(iv) Case 4: $k=\{28,43\}$.

Now, we consider the Case 1. In this case, we discussed (8) has cardinality of C_1, then (9) has cardinality of $C_1 + 1$. The result is shown in Theorem 4 below.

Theorem 4. Let $\theta = \sqrt{k}$ be a fixed irrational number and λ be any integer. Suppose $k = \{3, 8, 14, 15, 23, 24, 34, 35, 47, 48\}$ and n is a positive integer. Then, the character sum of non-trivial multiplicative characters χ for the first N terms of a Beatty sequence is given by

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=k} \chi(\lfloor n\theta + \lambda \rfloor) + (hM - 1) \sum_{|n|=k+1} \chi(\lfloor n\theta + \lambda \rfloor)$$

where $|n|$ is the cardinality of series (8) and $h = \{3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$ with respect to the value of chosen k. $M = 1$ if N satisfies $1 \leq N < h(k + 1)$, otherwise M is a positive integer satisfies $h(k+1)(M-1) + k(M-2) \leq N < M(hk + h + k) - k$.

Proof Let $k = \{3, 8, 14, 15, 23, 24, 34, 35, 47, 48\}$ and M is a positive integer. Let $U \in \mathbb{R}^+$ be the least upper bound for $S(\theta,\lambda,\chi;N)$, such that

$$S(\theta,\lambda,\chi;N) \leq U. \tag{10}$$

From (1), we have

$$\chi(\lfloor 1\sqrt{k} + \lambda \rfloor) + \chi(\lfloor 2\sqrt{k} + \lambda \rfloor) + \chi(\lfloor 3\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor N\sqrt{k} + \lambda \rfloor) + O(N) = U, \tag{11}$$

where $O(N)$ is the number of terms to be added into (1) in order to get the equality as shown in (11). In order to find U, we are considering another two cases as follow depend on the value of k.

Case (i): In this case, we consider the set value of $k = \{3, 8\}$ as follows:
Case (i)(a). Suppose $k = 3$, by substituting in (1), we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor) + (3M - 1) \sum_{|n|=4} \chi(\lfloor n\theta + \lambda \rfloor).$$

Case (i)(b). Suppose $k = 8$, by substituting in (1), we have

$$\begin{aligned} S(\theta,\lambda,\chi;N) &\leq M \sum_{|n|=5} \chi(\lfloor n\theta + \lambda \rfloor) + (5M - 1) \sum_{|n|=6} \chi(\lfloor n\theta + \lambda \rfloor) \\ &\leq M \sum_{|n|=8} \chi(\lfloor n\theta + \lambda \rfloor) + (8M - 1) \sum_{|n|=9} \chi(\lfloor n\theta + \lambda \rfloor). \end{aligned}$$

From Case (i)(a) and Case (i)(b), we conclude that

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=h} \chi(\lfloor n\theta + \lambda \rfloor) + (hM - 1) \sum_{|n|=h+1} \chi(\lfloor n\theta + \lambda \rfloor) \tag{12}$$

where $h = 2m + 1$, $m = 1$ when $k = 3$ and $m = 2$ when $k = 8$. For the value of M, $M = 1$ if N satisfies $1 \leq N < h(k + 1)$, otherwise M is a positive integer satisfies $k[h(M-1) + (M-2)] + h(M-1) \leq N < M(hk + h + k) - k$.

Case (ii): $k = \{14, 15, 23, 24, 34, 35, 47, 48\}$. By using the similar argument in Case (i), the following results are obtained.
For $k = 14$, we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor) + (7M - 1) \sum_{|n|=4} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 15$, we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=7} \chi(\lfloor n\theta + \lambda \rfloor) + (7M - 1) \sum_{|n|=8} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 23$, we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=4} \chi(\lfloor n\theta + \lambda \rfloor) + (9M - 1) \sum_{|n|=5} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 24$, we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=9} \chi(\lfloor n\theta + \lambda \rfloor) + (9M - 1) \sum_{|n|=10} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 34$, we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=5} \chi(\lfloor n\theta + \lambda \rfloor) + (11M - 1) \sum_{|n|=6} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 35$, we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=11} \chi(\lfloor n\theta + \lambda \rfloor) + (11M - 1) \sum_{|n|=12} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 47$, we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=6} \chi(\lfloor n\theta+\lambda \rfloor) + (13M-1) \sum_{|n|=7} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 48$, we have

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=13} \chi(\lfloor n\theta+\lambda \rfloor) + (13M-1) \sum_{|n|=14} \chi(\lfloor n\theta+\lambda \rfloor).$$

Next, we take an upper bound for $k = \{14, 15, 23, 24, 34, 35, 47, 48\}$. We obtain the result for Case (ii) as follow.

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=\left\lfloor \frac{k}{2} \right\rfloor} \chi(\lfloor n\theta+\lambda \rfloor) + (aM-1) \sum_{|n|=\left\lfloor \frac{k}{2} \right\rfloor+1} \chi(\lfloor n\theta+\lambda \rfloor) \quad (13)$$

where $a = 2b+5$; $b = b_n \in N$, $b_{j-2} = k_j, k_{j+1}$ and j is odd. By comparing (12) and (13), we can see that the bound of $(13) < (12)$. Therefore, the cardinality of the bound for each cycle from h to k is

$$S(\theta,\lambda,\chi;N) \leq M \sum_{|n|=k} \chi(\lfloor n\theta+\lambda \rfloor) + (hM-1) \sum_{|n|=k+1} \chi(\lfloor n\theta+\lambda \rfloor)$$

where $M = 1$ if N satisfies $1 \leq N < h(k+1)$, otherwise M is a positive integer satisfies $k[h(M-1)+(M-2)] + h(M-1) \leq N < M(hk+h+k) - k$.

From (10), we have

$$U = M \sum_{|n|=k} \chi(\lfloor n\theta+\lambda \rfloor) + (hM-1) \sum_{|n|=k+1} \chi(\lfloor n\theta+\lambda \rfloor).$$

■

Now, we consider Case 2. By considering (8) and (9), the result as shown below.

Theorem 5. Let $\theta = \sqrt{k}$ be a fixed irrational number and λ be any integer. Suppose $k = \{5, 10, 17, 18, 21, 26, 27, 29, 37, 38, 46, 50\}$ and n is a positive integer. Then, the character sum of non-trivial multiplicative characters χ for the first N terms of a Beatty sequence is given by

$$S(\theta,\lambda,\chi;N) \leq 4M \sum_{|n|=k} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=k+1} \chi(\lfloor n\theta+\lambda \rfloor)$$

where $M = 1$ if N satisfies $1 \leq N < 5k+2$, otherwise M is a positive integer satisfies $M(4k+1) - 3k \leq N < M(4k+1) + k + 1$.

Proof Let $k = \{5, 10, 17, 18, 21, 26, 27, 29, 37, 38, 46, 50\}$ and M is a positive integer. Let $V \in \mathbb{R}^+$ be the least upper bound for $S(\theta,\lambda,\chi;N)$, that is

$$S(\theta,\lambda,\chi;N) \leq V. \quad (14)$$

From (1), we have

$$\chi(\lfloor 1\sqrt{k}+\lambda \rfloor) + \chi(\lfloor 2\sqrt{k}+\lambda \rfloor) + \chi(\lfloor 3\sqrt{k}+\lambda \rfloor) + \cdots + \chi(\lfloor N\sqrt{k}+\lambda \rfloor) + O(N) = V, \quad (15)$$

where $O(N)$ is the number of terms to be added into (1) in order to get the equality as shown in (15). In order to find V, we are considering 5 cases.

Case (iii): $k = \{5, 10, 17, 26, 37, 50\}$. By using the similar argument in Case (i), the following result obtained.
For $k = 5$, we have

$$S(\theta,\lambda,\chi;N) \leq 4M \sum_{|n|=4} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=5} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 10$, we have

$$S(\theta,\lambda,\chi;N) \leq 6M \sum_{|n|=6} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=7} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 17$, we have

$$S(\theta,\lambda,\chi;N) \leq 8M \sum_{|n|=8} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=9} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 26$, we have

$$S(\theta,\lambda,\chi;N) \leq 10M \sum_{|n|=10} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=11} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 37$, we have

$$S(\theta,\lambda,\chi;N) \leq 12M \sum_{|n|=12} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=13} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 50$, we have

$$S(\theta,\lambda,\chi;N) \le 14M \sum_{|n|=14} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=15} \chi(\lfloor n\theta+\lambda \rfloor).$$

Therefore, we conclude that from the above observation,

$$S(\theta,\lambda,\chi;N) \le hM \sum_{|n|=h} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=h+1} \chi(\lfloor n\theta+\lambda \rfloor) \quad (16)$$

where $h = 2m+2$; $m = 1,2,3,4,5,6$ and $M = 1$ if N satisfies $1 \le N < h^2+h+2$, otherwise M is a positive integer satisfies $M(h^2+1)+h-h^2 \le N < M(h^2+1)+h+1$.

Case (iv): $k = \{18, 27, 38\}$. By using the similar argument in Case (i), we have the following result.
For $k = 18$, we have

$$S(\theta,\lambda,\chi;N) \le 8M \sum_{|n|=4} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=5} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 27$, we have

$$S(\theta,\lambda,\chi;N) \le 10M \sum_{|n|=5} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=6} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 38$, we have

$$S(\theta,\lambda,\chi;N) \le 12M \sum_{|n|=6} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=7} \chi(\lfloor n\theta+\lambda \rfloor).$$

Therefore, we can conclude that

$$S(\theta,\lambda,\chi;N) \le 2gM \sum_{|n|=g} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=g+1} \chi(\lfloor n\theta+\lambda \rfloor) \quad (17)$$

where $g = m+3$; $m = 1,2,3$ and $M = 1$ if N satisfies $1 \le N < 2g^2+g+2$, otherwise M is a positive integer satisfies $M(2g^2+1)+g-2g^2 \le N < M(2g^2+1)+g+1$.

Case (v): $k = \{29, 46\}$. By using the similar argument in Case (i), we have the following result.
For $k = 29$, we have

$$S(\theta,\lambda,\chi;N) \le 3M \sum_{|n|=2} \chi(\lfloor n\theta+\lambda \rfloor) + 4M \sum_{|n|=3} \chi(\lfloor n\theta+\lambda \rfloor).$$

For $k = 46$, we have

$$S(\theta,\lambda,\chi;N) \le 3M \sum_{|n|=4} \chi(\lfloor n\theta+\lambda \rfloor) + 4M \sum_{|n|=5} \chi(\lfloor n\theta+\lambda \rfloor).$$

Therefore, we can conclude that

$$S(\theta,\lambda,\chi;N) \le 3M \sum_{|n|=f} \chi(\lfloor n\theta+\lambda \rfloor) + 4M \sum_{|n|=f+1} \chi(\lfloor n\theta+\lambda \rfloor) \quad (18)$$

where $f = 2m$; $m = 1,2$ and $M = 1$ if N satisfies $1 \le N < 7f+5$, otherwise M is a positive integer satisfies $M(4f+5)-(f+5) \le N < M(4f+5)+3f$.

Case (vi): $k = \{21\}$. By using the argument in Case (i), the following result obtained.

$$S(\theta,\lambda,\chi;N) \le 5M \sum_{|n|=r} \chi(\lfloor n\theta+\lambda \rfloor) + 3M \sum_{|n|=r+1} \chi(\lfloor n\theta+\lambda \rfloor) \quad (19)$$

where $r = 2$ and $M = 1$ if N satisfies $1 \le N < 8r+4$, otherwise M is a positive integer satisfies $M(5r+2)-2r \le N < M(5r+2)+3r+2$.

Case (vii): $k = \{31\}$. By using the argument in Case (i), we have

$$S(\theta,\lambda,\chi;N) \le 3M \sum_{|n|=j} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=j+1} \chi(\lfloor n\theta+\lambda \rfloor) \quad (20)$$

where $j = 2$ and $M = 1$ if N satisfies $1 \le N < 4j+2$, otherwise M is a positive integer satisfies $M(3j+1)-2j \le N < M(3j+1)+j+1$.

By observing the bounds, we found that all the bounds can be bounded by (16). We use $4M$ instead of hM in (16) in order to get the least upper bound and the cardinality of each repeated series h replaced by k that is $|n| = k$. Therefore,

$$S(\theta,\lambda,\chi;N) \le 4M \sum_{|n|=k} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=k+1} \chi(\lfloor n\theta+\lambda \rfloor)$$

where $M = 1$ if N satisfies $1 \le N < 5k+2$, otherwise M is a positive integer satisfies $M(4k+1)-3k \le N < M(4k+1)+k+1$.

From (14), we have

$$V = 4M \sum_{|n|=k} \chi(\lfloor n\theta + \lambda \rfloor) + M \sum_{|n|=k+1} \chi(\lfloor n\theta + \lambda \rfloor).$$

■

For Case 3, the cardinality of every repeated series is different by 1 alternately. For example, the cardinality of the first repeated series is 4, the second repeated series is 5, the third repeated series will return back to 4, the fourth repeated series is 5, and so on. Thus, the result is shown below.

Theorem 6. Let $\theta = \sqrt{k}$ be a fixed irrational number and λ be any integer. Suppose $k = \{2, 6, 7, 11, 12, 13, 19, 20, 22, 30, 32, 33, 39, 40, 41, 42, 44\}$ and n is a positive integer. Then, the character sum of non-trivial multiplicative characters χ for the first N terms of a Beatty sequence is given by

$$S(\theta, \lambda, \chi; N) \leq 14M \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + 13M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor)$$

where $M = 1$ if N satisfies $1 \leq N < 68$, otherwise M is a positive integer satisfies $67M - 66 \leq N < 67M + 1$.

Proof Let $k = \{2, 6, 7, 11, 12, 13, 19, 20, 22, 30, 32, 33, 39, 40, 41, 42, 44\}$ and $M \in \mathbb{Z}^+$. Let $W \in \mathbb{R}^+$ be the least upper bound for $S(\theta, \lambda, \chi; N)$, that is

$$S(\theta, \lambda, \chi; N) \leq W. \tag{21}$$

From (1), we have

$$\chi(\lfloor 1\sqrt{k} + \lambda \rfloor) + \chi(\lfloor 2\sqrt{k} + \lambda \rfloor) + \chi(\lfloor 3\sqrt{k} + \lambda \rfloor) + \cdots + \chi(\lfloor N\sqrt{k} + \lambda \rfloor) + O(N) = W, \tag{22}$$

where $O(N)$ is the number of terms to be added into (1) in order to get the equality as shown in (22). In order to find W, we are considering 8 cases.

Case (vii): $k = \{6, 12, 20, 30, 42\}$. By using the similar argument in Case (i), we have the following result. For $k = 6$, we have

$$S(\theta, \lambda, \chi; N) \leq 4M \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 12$, we have

$$S(\theta, \lambda, \chi; N) \leq 6M \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 20$, we have

$$S(\theta, \lambda, \chi; N) \leq 8M \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 30$, we have

$$S(\theta, \lambda, \chi; N) \leq 10M \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 42$, we have

$$S(\theta, \lambda, \chi; N) \leq 12M \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor).$$

Therefore, we can conclude that

$$S(\theta, \lambda, \chi; N) \leq hM \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor) \tag{23}$$

where $h = 2m + 2$; $m = 1, 2, 3, 4, 5$ and $M = 1$ if N satisfies $1 \leq N < 2h + 4$, otherwise M is a positive integer satisfies $M(2h + 3) - 2(h + 1) \leq N < M(2h + 3) + 1$.

Case (viii): $k = \{7, 19\}$. By using the similar argument in Case (i), we have the following result.

$$S(\theta, \lambda, \chi; N) \leq M \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + 4M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor) \tag{24}$$

where $M = 1$ if N satisfies $1 \leq N < 15$, otherwise M is a positive integer satisfies $14M - 14 \leq N < 14M$.

Case (ix): $k = \{32, 33\}$. By using the similar argument in Case (i), the following result obtained. For $k = 32$, we have

$$S(\theta, \lambda, \chi; N) \leq M \sum_{|n|=2} \chi(\lfloor n\theta + \lambda \rfloor) + 11M \sum_{|n|=3} \chi(\lfloor n\theta + \lambda \rfloor).$$

For $k = 33$, we have

$$S(\theta,\lambda,\chi;N) \le M\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor) + 11M\sum_{|n|=4}\chi(\lfloor n\theta+\lambda\rfloor).$$

Therefore, we can conclude that

$$S(\theta,\lambda,\chi;N) \le M\sum_{|n|=d}\chi(\lfloor n\theta+\lambda\rfloor) + 11M\sum_{|n|=d+1}\chi(\lfloor n\theta+\lambda\rfloor) \quad (25)$$

where $d = 2,3$ and $M = 1$ if N satisfies $1 \le N < 12d + 12$, otherwise M is a positive integer satisfies $M(12d + 11) - (12d + 10) \le N < M(12d + 11) + 1$.

Case (x): $k = \{39,40\}$. By using the similar argument in Case (i), the following result obtained.
For $k = 39$, we have

$$S(\theta,\lambda,\chi;N) \le 12M\sum_{|n|=4}\chi(\lfloor n\theta+\lambda\rfloor) + M\sum_{|n|=5}\chi(\lfloor n\theta+\lambda\rfloor).$$

For $k = 40$, we have

$$S(\theta,\lambda,\chi;N) \le 12M\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor) + M\sum_{|n|=4}\chi(\lfloor n\theta+\lambda\rfloor).$$

Therefore, we can conclude that

$$S(\theta,\lambda,\chi;N) \le 12M\sum_{|n|=b}\chi(\lfloor n\theta+\lambda\rfloor) + M\sum_{|n|=b+1}\chi(\lfloor n\theta+\lambda\rfloor) \quad (26)$$

where $b = 4,3$ and $M = 1$ if N satisfies $1 \le N < 13b + 2$, otherwise M is a positive integer satisfies $M(13b + 1) - 13 \le N < M(13b + 1) + 1$.

Case (xi): $k = \{11,22\}$. We have the following result.
For $k = 11$, we have

$$S(\theta,\lambda,\chi;N) \le 6M\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor) + M\sum_{|n|=4}\chi(\lfloor n\theta+\lambda\rfloor).$$

For $k = 22$, we have

$$S(\theta,\lambda,\chi;N) \le 4M\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor) + M\sum_{|n|=4}\chi(\lfloor n\theta+\lambda\rfloor).$$

Therefore, we can conclude that

$$S(\theta,\lambda,\chi;N) \le gM\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor) + M\sum_{|n|=4}\chi(\lfloor n\theta+\lambda\rfloor) \quad (27)$$

where $g = 6,4$ and $M = 1$ if N satisfies $1 \le N < 3g + 5$, otherwise M is a positive integer satisfies $M(3g + 4) - 3(g + 1) \le N < M(3g + 4) + 1$.

Case (xii): $k = \{2,41\}$. By using the similar argument in Case (i), we have the following.
For $k = 2$, we have

$$S(\theta,\lambda,\chi;N) \le 4M\sum_{|n|=2}\chi(\lfloor n\theta+\lambda\rfloor) + 3M\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor).$$

For $k = 41$, we have

$$S(\theta,\lambda,\chi;N) \le 14M\sum_{|n|=2}\chi(\lfloor n\theta+\lambda\rfloor) + 13M\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor).$$

Therefore, we can conclude that

$$S(\theta,\lambda,\chi;N) \le bM\sum_{|n|=2}\chi(\lfloor n\theta+\lambda\rfloor) + jM\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor) \quad (28)$$

where $b = 4,14$, $j = 3,13$ and $M = 1$ if N satisfies $1 \le N < 2b + 3j + 1$, otherwise M is a positive integer satisfies $M(2b + 3j) - (2b + 3j) + 1 \le N < M(2b + 3j) + 1$.

For the next two cases, we have to express them separately due to they cannot be classified into any category above.

Case (xiii): $k = \{13\}$. By using the similar argument in Case (i), we have the following result.

$$S(\theta,\lambda,\chi;N) \le 7M\sum_{|n|=2}\chi(\lfloor n\theta+\lambda\rfloor) + 8M\sum_{|n|=3}\chi(\lfloor n\theta+\lambda\rfloor) \quad (29)$$

where $M = 1$ if N satisfies $1 \le N < 39$, otherwise M is a positive integer satisfies $38M - 37 \le N < 38M + 1$.

Case (xiv): $k = \{44\}$. By using the similar argument in Case (i), we have the following result.

$$S(\theta,\lambda,\chi;N) \le M \sum_{|n|=2} \chi(\lfloor n\theta+\lambda \rfloor) + 3M \sum_{|n|=3} \chi(\lfloor n\theta+\lambda \rfloor) \quad (30)$$

where $M = 1$ if N satisfies $1 \le N < 12$, otherwise M is a positive integer satisfies $11M - 10 \le N < 11M + 1$.

Next, we choose a bigger bound to cover from (23) to (30). By observing the bounds, we can see that all the bounds can be bounded by (28) when we choose $b = 14$ and $j = 13$. Therefore, we have

$$S(\theta,\lambda,\chi;N) \le 14M \sum_{|n|=2} \chi(\lfloor n\theta+\lambda \rfloor) + 13M \sum_{|n|=3} \chi(\lfloor n\theta+\lambda \rfloor)$$

where $M = 1$ if N satisfies $1 \le N < 68$, otherwise M is a positive integer satisfies $67M - 66 \le N < 67M + 1$.

From (21), we have

$$W = 14M \sum_{|n|=2} \chi(\lfloor n\theta+\lambda \rfloor) + 13M \sum_{|n|=3} \chi(\lfloor n\theta+\lambda \rfloor).$$

■

Now, we are discussing a special case in which the different between any two consecutive terms is consistent since almost every repeated series has the same cardinality. The result as shown in the following theorem.

Theorem 7. Let $\theta = \sqrt{k}$ be a fixed irrational number and λ be any integer. Suppose $k = \{28, 43\}$ and n is a positive integer. The character sum of non-trivial multiplicative characters χ for the first N terms of a Beatty sequence is given by

$$S(\theta,\lambda,\chi;N) \le 4M \sum_{|n|=\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor} \chi(\lfloor n\theta+\lambda \rfloor) + 3M \sum_{|n|=\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor+1} \chi(\lfloor n\theta+\lambda \rfloor)$$

where $M = 1$ if N satisfies $1 \le N < 7\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor + 4$, otherwise M is a positive integer satisfies $M\left(7\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor + 3\right) - \left(7\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor + 2\right) \le N < M\left(7\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor + 3\right) + 1$.

Proof Let $k = \{28, 43\}$ and $M \in \mathbb{Z}^+$. Let $Y \in \mathbb{R}^+$ be the least upper bound for $S(\theta,\lambda,\chi;N)$, that is

$$S(\theta,\lambda,\chi;N) \le Y. \quad (31)$$

From (1), we have

$$\chi(\lfloor 1\sqrt{k}+\lambda \rfloor) + \chi(\lfloor 2\sqrt{k}+\lambda \rfloor) + \chi(\lfloor 3\sqrt{k}+\lambda \rfloor) + \cdots + \chi(\lfloor N\sqrt{k}+\lambda \rfloor) + O(N) = Y, \quad (32)$$

where $O(N)$ is the number of terms to be added into (1) in order to get the equality as shown in (32). In order to find Y, we are considering another 2 cases.

Case (xv): $k = \{28\}$. By using the similar argument in Case (i), the following result obtained.

$$S(\theta,\lambda,\chi;N) \le 4M \sum_{|n|=3} \chi(\lfloor n\theta+\lambda \rfloor) + 3M \sum_{|n|=4} \chi(\lfloor n\theta+\lambda \rfloor) \quad (33)$$

where $M = 1$ if N satisfies $1 \le N < 25$, otherwise M is a positive integer satisfies $24M - 23 \le N < 24M + 1$.

Case (xvi): $k = \{43\}$. By using the similar argument in Case (i), it follows by

$$S(\theta,\lambda,\chi;N) \le 3M \sum_{|n|=2} \chi(\lfloor n\theta+\lambda \rfloor) + M \sum_{|n|=3} \chi(\lfloor n\theta+\lambda \rfloor) \quad (34)$$

where $M = 1$ if N satisfies $1 \le N < 10$, otherwise M is a positive integer satisfies $9M - 8 \le N < 9M + 1$.

By comparing (33) and (34), we can see that $(34) < (33)$. Therefore, it is clear that (33) the generalised of the cardinality of each repeated series from 3 to $\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor$, is as follow.

$$S(\theta,\lambda,\chi;N) \le 4M \sum_{|n|=\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor} \chi(\lfloor n\theta+\lambda \rfloor) + 3M \sum_{|n|=\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor+1} \chi(\lfloor n\theta+\lambda \rfloor)$$

where $M = 1$ if N satisfies $1 \le N < 7\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor + 4$, otherwise M is a positive integer satisfies $M\left(7\left\lfloor \sqrt{\frac{k}{3}} \right\rfloor +\right.$

$3\Big) - \left(7\left\lfloor\sqrt{\frac{k}{3}}\right\rfloor + 2\right) \leq N < M\left(7\left\lfloor\sqrt{\frac{k}{3}}\right\rfloor + 3\right) + 1.$

From (31), we have

$$Y = 4M \sum_{|n|=\left\lfloor\sqrt{\frac{k}{3}}\right\rfloor} \chi(\lfloor n\theta + \lambda\rfloor) + 3M \sum_{|n|=\left\lfloor\sqrt{\frac{k}{3}}\right\rfloor+1} \chi(\lfloor n\theta + \lambda\rfloor).$$

■

From the observation of Theorem 4 to Theorem 7, it can be generalised that for the case $2 \leq k \leq 50$, we have the result as in Theorem 5. That is, the character sum of non-trivial multiplicative characters χ for the first N terms of a Beatty sequence is given by

$$S(\theta,\lambda,\chi;N) \leq 4M \sum_{|n|=k} \chi(\lfloor n\theta + \lambda\rfloor) + M \sum_{|n|=k+1} \chi(\lfloor n\theta + \lambda\rfloor)$$

where $M = 1$ if N satisfies $1 \leq N < 5k + 2$, otherwise M is a positive integer satisfies $M(4k+1) - 3k \leq N < M(4k+1) + k + 1$.

Cardinality of Double Character Sums Associated with Composite Moduli

In this section, we consider a finite group F_m for $m \in \{2, 4, p^t, 2p^t\}$ where p is an odd prime and $t \in N$. The multiplicative group F_m^* of F_m is a cyclic subgroup of F_m with order $\phi(m)$.The following lemma helps us to deliver our theorem later.

Lemma 1. Let χ be a nontrivial multiplicative character of F_m^*. If $a, b \in F_m^*$, then

$$\sum_{\chi} \chi(a)\overline{\chi(b)} = \begin{cases} \phi(m) & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

where the sum is extended over al multiplicative character χ of F_m.

Proof Let $g_k \in F_m^*$ be the primitive roots of order $\phi(m)$ with $0 \leq k \leq \phi(m) - 1$, we have

$$\chi_0(g_0^u), \chi_1(g_0^u), \dots, \chi_{\phi(m)-1}(g_0^u),$$
$$\chi_0(g_1^u), \chi_1(g_1^u), \dots, \chi_{\phi(m)-1}(g_1^u),$$
$$\dots$$
$$\chi_0\big(g_{\phi(m)-1}^u\big), \chi_1\big(g_{\phi(m)-1}^u\big), \dots, \chi_{\phi(m)-1}\big(g_{\phi(m)-1}^u\big).$$

In the case $a = b$,

$$\sum_{\chi} \chi(a)\overline{\chi(b)} = \phi(m).$$

For $a \neq b$, $\chi(a)\overline{\chi(b)}$ will vanish one another and end up equal to 0.

■

Then, the cardinality of the double character sums is as follows:

Theorem 8. Let θ be a fixed irrational number, λ be any real number and $m \in \{2, 4, p^t, 2p^t\}$ where p is an odd prime and $t \in N$. For any positive integers on the interval $N \geq m$ and non-trivial multiplicative characters $\chi \pmod m$, the following bound holds:

$$|W_1|^2 = \#\left\{\sum_{n\in\mathcal{N}} \sum_{k\in\mathcal{K}} \chi(\lfloor\theta(n+k)+\lambda\rfloor)\bar{\chi}(\lfloor\theta(n+k)+\lambda\rfloor)\right\} \ll N \cdot \phi(m) \cdot (\#\mathcal{K}).$$

Proof Let $k < N$, where N is a natural number and $\Delta \in (0,1)$ be a rational number. Suppose that $\gamma \in \mathbb{Q}$, we obtained the following fractional part.

$$\mathcal{N}_\gamma = \{1 \leq n \leq N : \{\theta n + \lambda - \gamma\} < 1 - \Delta\},$$
$$\mathcal{K}_\gamma = \{1 \leq k \leq K : \{\theta k + \gamma\} < \Delta\},$$
$$\mathcal{N}_\gamma^c = \{1, 2, 3, \dots, N\} \setminus \mathcal{N}_\gamma.$$

Fix $\gamma \in \mathbb{Q}$, the notation for intervals is as follows.

$$\mathcal{N} = \mathcal{N}_\gamma, \mathcal{K} = \mathcal{K}_\gamma \text{ and } \mathcal{N}^c = \mathcal{N}_\gamma^c.$$

Next, we insert γ in the following expression and obtain

$$\begin{aligned}\lfloor\theta(n+k)+\lambda\rfloor &= (\theta(n+k)+\lambda) - \{\theta(n+k)+\lambda\} \\ &= (\theta n + \lambda - \gamma) - \{\theta n + \lambda - \gamma\} + (\theta k + \gamma) - \{\theta k + \gamma\}.\end{aligned}$$

Therefore,

$$\lfloor \theta(n+k)+\lambda \rfloor = \lfloor \theta n+\lambda-\gamma \rfloor + \lfloor \theta k+\gamma \rfloor. \quad (35)$$

Let

$$W = \sum_{n\in\mathcal{N}} \sum_{k\in\mathcal{K}} \chi(\lfloor \theta(n+k)+\lambda \rfloor).$$

From (35), we have

$$W = \sum_{n\in\mathcal{N}} \sum_{k\in\mathcal{K}} \chi(\lfloor \theta n+\lambda-\gamma \rfloor + \lfloor \theta k+\gamma \rfloor).$$

Now, consider the cardinality of $\lfloor \theta n+\lambda-\gamma \rfloor$. Since $N \geq m$ and uniformly for all $r \in \mathbb{Z}$,

$$\#\{n \in N: \lfloor \theta n+\lambda-\gamma \rfloor \equiv r \pmod{m}\} = O(1).$$

By using Cauchy inequality, we obtain

$$\begin{aligned} |W|^2 &\ll N \sum_{n\in\mathcal{N}} \left| \sum_{k\in\mathcal{K}} \chi(\lfloor \theta n+\lambda-\gamma \rfloor + \lfloor \theta k+\gamma \rfloor) \right|^2 \\ &\ll N \sum_{r=1}^{m} \left| \sum_{k\in\mathcal{K}} \chi(r + \lfloor \theta k+\gamma \rfloor) \right|^2 \\ &= N \sum_{k,l\in\mathcal{K}} \sum_{r=1}^{m} \chi(r+\lfloor \theta k+\gamma \rfloor)\bar{\chi}(r+\lfloor \theta l+\gamma \rfloor). \end{aligned}$$

Now, consider the expression

$$\sum_{k,l\in\mathcal{K}} \sum_{r=1}^{m} \chi(r+\lfloor \theta k+\gamma \rfloor)\bar{\chi}(r+\lfloor \theta l+\gamma \rfloor).$$

We found that the maximum number of total elements is $(\#\mathcal{K}) \cdot (\#\mathcal{K}) \cdot \phi(m) = \phi(m)(\#\mathcal{K})^2$. By using the Lemma 1, the inner sum takes only two possible values as follows.

$$\sum_{r=1}^{m} \chi(r+\lfloor \theta k+\gamma \rfloor)\bar{\chi}(r+\lfloor \theta l+\gamma \rfloor) = \begin{cases} \phi(m) & \text{for } \lfloor \theta k+\gamma \rfloor \equiv \lfloor \theta l+\gamma \rfloor \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

The congruence of $\lfloor \theta k+\gamma \rfloor \equiv \lfloor \theta l+\gamma \rfloor \pmod{m}$ occurs for at most $O(\#\mathcal{K})$ pairs with $k,l \in \mathcal{K}$ since $K < m$. Then we have

$$\sum_{k,l\in\mathcal{K}} \sum_{r=1}^{m} \chi(r+\lfloor \theta k+\gamma \rfloor)\bar{\chi}(r+\lfloor \theta l+\gamma \rfloor) \ll \phi(m) \cdot (\#\mathcal{K}).$$

Therefore,

$$N \sum_{k,l\in\mathcal{K}} \sum_{r=1}^{m} \chi(r+\lfloor \theta k+\gamma \rfloor)\bar{\chi}(r+\lfloor \theta l+\gamma \rfloor) \ll N \cdot \phi(m) \cdot (\#\mathcal{K}).$$

We obtain

$$|W|^2 \ll N \cdot \phi(m) \cdot (\#\mathcal{K}).$$

■

Character Sums of Beatty Sequences Associated with Composite Moduli

Theorem 9. Let θ be a fixed irrational number, λ be any real number and $m \in \{2, 4, p^t, 2p^t\}$ where p is an odd prime and $t \in N$. If a positive integer $N \geq m$ and non-trivial multiplicative characters $\chi \pmod{m}$, then the following bound holds:

$$S_m(\theta, \lambda, \chi; N) \ll N^{\frac{1}{2}} \phi(m)^{\frac{1}{4}} + N D_{\theta,\lambda}(N).$$

Proof Let $k < N \in \mathbb{Z}^+$ and $\Delta \in (0,1]$ be a rational number, $\mathbb{Q}$. Suppose that $\gamma \in \mathbb{Q}$. Then, we obtained the following fractional part.

$$\begin{aligned} \mathcal{N}_\gamma &= \{1 \leq n \leq N: \{\theta n+\lambda-\gamma\} < 1-\Delta\}, \\ \mathcal{K}_\gamma &= \{1 \leq k \leq K: \{\theta k+\gamma\} < \Delta\}, \\ \mathcal{N}_\gamma^c &= \{1,2,3,\ldots,N\} \setminus \mathcal{N}_\gamma. \end{aligned}$$

From the definition of discrepancy, we have

$$D_{\theta,\lambda} = \sup_{I\subseteq[0,1)} \left| \frac{V(I,M)}{M} - |I| \right|.$$

Given that $V(I,M) = \#\mathcal{N}_\gamma^c$ and $|I| = |\Delta|$. Suppose $M \leq N$, we will obtain

$$D_{\theta,\lambda}(N) = \sup_{I\subseteq[0,1)} \left| \frac{\#\mathcal{N}_{\gamma}^{c}}{N} - |\Delta| \right|.$$

If supremum element from $I \subseteq [0,1)$, then the representation constants can be denoted as big O and vice-versa.

$$O\left(ND_{\theta,\lambda}(N)\right) = \left|\#\mathcal{N}_{\gamma}^{c} - N|\Delta|\right|.$$

Therefore,

$$\mathcal{N}_{\gamma}^{c} = O\left(ND_{\theta,\lambda}(N)\right) + N|\Delta|. \quad (36)$$

Fix $\gamma \in \mathbb{Q}$, the notation for interval are as follows.

$$\mathcal{N} = \mathcal{N}_{\gamma}, \mathcal{K} = \mathcal{K}_{\gamma} \text{ and } \mathcal{N}^{c} = \mathcal{N}_{\gamma}^{c}.$$

The fractional part of $\mathcal{K}_{\gamma}$ is less than Δ. Hence, the number of elements in the interval $\mathcal{K}_{\gamma}$ can be estimated by using the lemma in [2], we have

$$\#\mathcal{K}_{\gamma} \leq 0.5K\Delta. \quad (37)$$

Now, we have for every $k \in \mathcal{K}$,

$$\begin{aligned} S_m(\theta,\lambda,\chi;N) &= \sum_{n\leq N} \chi(\lfloor\theta(n+k)+\lambda\rfloor) + O(k) \\ &= \sum_{n\leq N} \chi(\lfloor\theta(n+k)+\lambda\rfloor) + O(K) \\ &= \sum_{n\in\mathcal{N}} \chi(\lfloor\theta(n+k)+\lambda\rfloor) + O(K+\#\mathcal{N}^{c}). \end{aligned}$$

Thus,

$$S_m(\theta,\lambda,\chi;N) = \frac{W_1}{\#\mathcal{K}} + O(K+\#\mathcal{N}^{c}), \quad (38)$$

where

$$W_1 = \sum_{n\in\mathcal{N}} \sum_{k\in\mathcal{K}} \chi(\lfloor\theta(n+k)+\lambda\rfloor).$$

The cardinality of double character sum of W_1 is given in Theorem 8. As a result, we will obtain the following inequalities.

$$|W_1|^2 \ll N \cdot \phi(m) \cdot \#\mathcal{K}.$$

Substituting this bound in (38) and by using (36) and (37), we have

$$S_m(\theta,\lambda,\chi;N) \ll \sqrt{\frac{\phi(m)N}{K\Delta}} + K + N\Delta + ND_{\theta,\lambda}(N)$$

with

$$K = \left\lceil N^{\frac{1}{2}}\phi(m)^{\frac{1}{4}} \right\rceil \text{ and } \Delta = \frac{\phi(m)^{\frac{1}{4}}}{N^{\frac{1}{2}}}.$$

Thus, the theorem holds.

■

Conclusions

The character sums of non-trivial multiplicative characters χ for first N terms of a Beatty sequence is given by

$$S(\theta,\lambda,\chi;N) \leq 4M \sum_{|n|=k} \chi(\lfloor n\theta+\lambda\rfloor) + M \sum_{|n|=k+1} \chi(\lfloor n\theta+\lambda\rfloor)$$

where $\theta = \sqrt{k}$, λ is an integer, $2 \leq k \leq 50$ is a square-free integer and $M = 1$ if N satisfies $1 \leq N < 5k + 2$, otherwise M is a positive integer satisfies $M(4k+1) - 3k \leq N < M(4k+1) + k + 1$.

Next, the cardinality of double character sums associated with composite moduli has the maximum value equals to $N \cdot \phi(m) \cdot (\#\mathcal{K})$. Finally, the estimated bound of character sums associated with composite moduli can be obtained by

$$S_m(\theta,\lambda,\chi;N) \ll N^{\frac{1}{2}}\phi(m)^{\frac{1}{4}} + ND_{\theta,\lambda}(N).$$

In the future, the non-homogeneous Beatty sequences can be extended to the floor of polynomial function which is beyond than a linear function.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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References

[1] Chua, L., Park, S. and Smith, G. D. (2015). Bounded Gaps between Primes in Special Sequences. *Proc. Amer. Math. Soc., 143,* 4597–4611.
[2] Banks, W. D. and Shparlinski, I. E. (2006). Non-residues and Primitive Roots in Beatty Sequences. *Bull. Austral. Math. Soc., 73,* 433–443.
[3] Banks, W. D. and Shparlinski, I. E. (2007). Prime Divisors in Beatty Sequences. *Journal of Number Theory, 123,* 413–425.
[4] Cassaigne, J., DuchÃłne, E. and Rigo, M. (2016). Nonhomogeneous Beatty Sequences Leading to Invariant Games. *SIAM Journal on Discrete Mathematics, 30,* 1798–1829.
[5] Harman, G. (2015). Primes in Intersections of Beatty Sequences. *Journal of Integer Sequences, 18.* (Article 15.7.3).
[6] Kimberling, C. (2016). Beatty Sequences and Trigonometric Functions. *Integers, 16*.
[7] Simpson, J. (2004). Disjoint Beatty Sequences. *Integers: Electronic Journal of Combinatorial Number Theory, 4*.
[8] Deraman, F., Sapar, S. H., Johari, M. A. M., Mohd Atan, K. A. and Rasedee, A. F. N. (2019). Extended Bounds of Beatty Sequence Associated with Primes. *International Journal of Engineering and Advanced Technology, 8,* 115–118.
[9] Lidl, R. and Niederreiter, H. (1986). *Introduction to Finites Fields and Their Applications*. Cambridge: Cambridge University Press.
[10] Russo, V. and Schwiebert, L. (2011). Beatty Sequences, Fibonacci Numbers, and the Golden Ratio. *Fibonacci Quart, 49,* 151–154.
[11] Shallit, J. (2022). Sumsets of Wythoff Sequences, Fibonacci Representation, and Beyond. *Period. Math. Hung., 84,* 37–46.
[12] Phunphayap, P. N., Pongsriiam, P. and Shallit, J. (2022). Sumsets Associated with Beatty Sequences. *Discrete Math, 345,* 112810.
[13] Khani, M. and Zarei, A. (2023). The Additive Structure of Integers with the Lower Wythoff. *Archive for Mathematical Logic, 62,* 225–237.
[14] Shparlinski, I. E. (2016). Open Problems on Exponential and Character Sums.
[15] Banks, W. D. and Shparlinski, I. E. (2009). Character Sums with Beatty Sequences on Burgess-type Intervals. *Analytic Number Theory,* 15–21.
[16] Venkov, B. A. (1970). *Elementary Number Theory.* Wolters-Noordhoff.
[17] Kuipers, L. and Niederreiter, H. (1974). *Uniform Distribution of Sequences*. New York, London, Sydney: Wiley-Interscience.
[18] Nathanson, M. B. (2000). *Elementary Methods in Number Theory.* Springer.
[19] Allouche, J.-P. and Shallit, J. (2003). *Automatic Sequences: Theory, Applications, Generalizations.* Cambridge University Press.
[20] Francis, F. (2023). *Short Character Sums and Their Applications.* Ph.D. Thesis. University of New South Wales Canberra.