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Interval valued Fuzzy *n*-normed linear space

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ABSTRACT

The aim of this paper is to introduce the notion of interval-valued fuzzy subspace with its flags and interval-valued fuzzy n-normed linear space. We define the operations intersection, sum, directsum and tensor product of interval-valued fuzzy subspaces and obtain their corresponding flags. Further we provide some results on interval-valued fuzzy n-normed linear space.

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| Interval valued fuzzy subspace | Operations | Flags | Interval valued fuzzy n-normed linear space |

1. Introduction

The theory of 2-norm and n-norm on a linear space has been introduced by G*ä*hler in [9, 10] and subsequently studied by several authors [11, 14, 19, 20]. A detailed theory of fuzzy norm on a linear space can be viewed in [4, 5, 6, 7, 8, 13, 15, 23]. Recently we have introduced the notion of fuzzy n-normed linear space [22]. After the introduction of fuzzy subspace of a vector space by Katsaras and Liu′s [12], many researchers [1, 2, 3, 16, 18, 21] engaged themselves in the development of fuzzy subspace of a vector space. Recently, G.Lubczonok and V.Murali introduced an interesting theory of flags and fuzzy subspaces of vector spaces [17]. Zadeh in his pioneering work introduced the theory of interval-valued fuzzy subset in [24].

In this paper we introduce the notions of interval-valued fuzzy subspace and interval-valued fuzzy n-normed linear space. We also define some operations on interval-valued fuzzy subspaces and in each case we obtain the corresponding flags. Further we establish some properties of interval-valued fuzzy n-normed linear space.

2. Preliminaries

In the following we provide the essential definitions and results necessary for the development of our theory.

Definition 2.1[24]. An interval number on $[0,1]$, say $\overline{a}$, is a closed sub interval of $[0,1]$ of the form $\overline{a}=[a^-,a^+]$, where $0\leq a^-\leq a^+\leq 1$. Let $D[0,1]$ denote the family of all closed sub-intervals of $[0,1]$, that is,
$D[0,1]=\{\overline{a}=[a^-,a^+]:a^-\leq a^+ \text{ and } a^-,a^+\in[0,1]\}$

Definition 2.2[24]. Let $\overline{a_i}=[a_i^-,a_i^+]\in D[0,1]$ for all $i\in\Omega$, Ω an index set. Define
(a)$\inf^i\{\overline{a_i}:i\in\Omega\}=[\inf_{i\in\Omega}a_i^-,\inf_{i\in\Omega}a_i^+]$
(b)$\sup^i\{\overline{a_i}:i\in\Omega\}=[\sup_{i\in\Omega}a_i^-,\sup_{i\in\Omega}a_i^+]$.
In particular, whenever $\overline{a}=[a^-,a^+]$, $\overline{b}=[b^-,b^+]$ in $D[0,1]$, we define
(i)$\overline{a}\leq\overline{b}$ if and only if $a^-\leq b^-$ and $a^+\leq b^+$
(ii)$\overline{a}=\overline{b}$ if and only if $a^-=b^-$ and $a^+=b^+$
(iii)$\overline{a}<\overline{b}$ if and only if $a^-<b^-$ and $a^+<b^+$.
(iv)$\min^i\{\overline{a},\overline{b}\}=[\min\{a^-,b^-\},\min\{a^+,b^+\}]$
(v)$\max^i\{\overline{a},\overline{b}\}=[\max\{a^-,b^-\},\max\{a^+,b^+\}]$.

Definition 2.3 [24]**.** Let X be a set. A mapping $\overline{A}:X\to D[0,1]$ is called an interval-valued fuzzy subset(briefly, an i-v fuzzy subset) of X, where
$\overline{A}(x)=[A^-(x),A^+(x)]$, and A^- and A^+ are fuzzy subsets in X such that $A^-(x)\leq A^+(x)$ for all $x\in X$.

Definition 2.4 [24]. Let $\overline{A}$ be an interval-valued fuzzy subset of X and $[t_1,t_2]\in D[0,1]$. Then the set
$\overline{U}(\overline{A};[t_1,t_2])=\{x\in X:\overline{A}(x)\geq[t_1,t_2]\}$ is called an upper level subset of $\overline{A}$. Note that
$\overline{U}(\overline{A};[t_1,t_2])=U(A^-;t_1)\cap U(A^+;t_2)$ where
$U(A^-;t_1)=\{x\in X:A^-(x)\geq t_1\}$ and
$U(A^+;t_2)=\{x\in X:A^+(x)\geq t_2\}$.

Definition 2.5 [12]. Let V denote a vector space of dimension n over a field F. A fuzzy subspace is a fuzzy subset μ of V such that
$\mu(\alpha x+\beta y)\geq\mu(x)\bigwedge\mu(y)$, $x,y\in V$, $\alpha,\beta\in F$(field), where $\bigwedge$ stands for intersection.

We recall some properties of fuzzy subspace [16]

Remark 2.6 [16]. (a)$V^\alpha=\mu^{-1}([\alpha,1])$ is a subspace of V for $\alpha\in[0,1]$, known as the α-cut of μ.
(b)$V^0=V$
(c) μ takes at the most $k+1$ values in $[0,1]$, say
$$1>\alpha_1>...>\alpha_k>0,\ k\leq n \tag{1}$$

and
$V \supset V^{\alpha_k} \supset ... \supset V^{\alpha_1} \supset \{0\}$ (2)
We can view (2) as a flag or chain of subspaces with a weight attached to each subspace.

Remark 2.7 [16]. A subset $\{x_1, x_2, ..., x_m\}$ of V is fuzzy linearly independent if it is linearly independent and $\mu(\sum_{i=1}^{m} \alpha_i x_i) = \bigwedge_{i=1}^{m} \mu(x_i)$ for all $\alpha_i \in F$, $\alpha_i \neq 0$, $i = 1, , ..., m$, $\bigwedge$ stands for intersection.

Definition 2.8 [11]. Let $n \in$N(natural numbers) and X be a real vector space of dimension $d \geq n$. A real valued function $||\bullet, ..., \bullet||$ on $\underbrace{X \times ... \times X}_{n} = X^n$ satisfying the following four properties:

1. $||x_1, x_2, ..., x_n|| = 0$ if any only if $x_1, x_2, ..., x_n$ are linearly dependent
2. $||x_1, x_2, ..., x_n||$ is invariant under any permutation of $x_1, x_2, ..., x_n$
3. $||x_1, x_2, ..., \alpha x_n|| = |\alpha| \; ||x_1, x_2, ..., x_n||$, for any $\alpha \in$R(set of real numbers)
4. $||x_1, x_2, , x_{n-1}, y + z|| \leq ||x_1, x_2, ..., x_{n-1}, y|| + ||x_1, x_2, ..., x_{n-1}, z||$,

is called an n-norm on X and the pair ($X, ||\bullet, ..., \bullet||$) is called an n-normed linear space.

Definition 2.9 [22]. Let X be a linear space over a real field F. A fuzzy subset N of $X^n \times R$ is called a fuzzy n-norm on X if and only if:

$(N1)$ For all $t \in$R with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$.
$(N2)$ For all $t \in$R with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.
$(N3)$ $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
$(N4)$ For all $t \in$R with $t > 0$,
$N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0 \in F$(field).
$(N5)$ For all $s, t \in R$,
$$N(x_1, x_2, ..., x_n + x_n', s + t) \geq \min\Big\{\mathrm{N}(x_1, x_2, ..., x_n, s), \mathrm{N}(x_1, x_2, ..., x_n', t)\Big\}.$$
$(N6)$ $N(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1$.

Then (X, N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Remark 2.10.

(i)Let V be a vector space over a field F and let $U_1, U_2, .., U_n$ be the subspaces of V. V is said to be direct sum of $U_1, U_2, .., U_n$ if every element $v \in V$ can be written in one and only one way $v = u_1 + u_2 + ... + u_n$ where $u_i \in U_i$ and denoted as $V = U_1 \oplus U_2 \oplus ... \oplus U_n$.

(ii)Let V and W be vector spaces over a field F. Then the tensor product of two vectors is denoted by $V \otimes W$and given $t \in V \otimes W$, t can be uniquely written as , $t = \sum_{i,j} t_{ij} x_i \otimes y_j$ where the sum is taken over all i, j for which $t_{ij} \neq 0$.

3. Interval-valued fuzzy subspaces and their flags

By generalizing the Definition 2.5 we obtain a new notion of interval-valued fuzzy subspace as follows:

Definition 3.1. Let V denote a vector space over a field F. Let $\overline{A} : X \to D[0,1]$ be an interval-valued fuzzy subset of V. Then $\overline{A}$ is said to be an interval-valued fuzzy subspace (or shortly i-v fuzzy subspace) if,
$\overline{A}(\alpha x * \beta y) \geq \min^i\{\overline{A}(x), \overline{A}(y)\}$, $x, y \in V$ and $\alpha, \beta \in F$(field).

Example 3.2. Let $V = \{e, x, y, z\}$ be the Klien 4-group defined by the binary operation $*$ as:

$*$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

Let F be the field GF(2). Let $0.w = e$, $1.w = w$ for all $w \in V$. Then V is a vector space over F.

Define an i-v fuzzy subset $\overline{A}$ in V by
$\overline{A}(e) = [0.6, 0.9]$, $\overline{A}(x) = [0.3, 0.4] = \overline{A}(y)$, $\overline{A}(z) = [0.5, 0.8]$.
Then $\overline{A}$ is an i-v fuzzy subspace of V.

Definition 3.3. Let (i) $\overline{A}$ be an interval-valued fuzzy subspace. Then
$V^{\overline{\alpha}} = \{x \in V : \overline{A}(x) \geq [\alpha^-, \alpha^+]\}$ is called an upper level subset of $\overline{A}$.
(ii) Further $V^{\overline{\alpha}} = (\overline{A})^{-1}[\overline{\alpha}, \overline{1}]$ is known as the α-cut of $\overline{A}$.
(iii)$\overline{A}$ takes at the most $k+1$ values in $D[0,1]$, say

$$\overline{1} > \overline{\alpha_1} > ... > \overline{\alpha_k} > \overline{0}, \text{ where } \overline{\alpha_i} = [\alpha^-, \alpha^+] \quad (3)$$

$i = 1, 2, ..., k$, $\overline{0} = [0, 0]$, $\overline{1} = [1, 1]$ and

$$V \supset V^{\overline{\alpha_k}} \supset ... \supset V^{\overline{\alpha_1}} \supset \overline{0} \quad (4)$$

We can view (4) as a flag or chain of subspaces with a weight attached to each subspace.

Definition 3.4. A subset $\{x_1, x_2, ..., x_m\}$ of V is said to be interval-valued fuzzy basis if it is linearly independent and $\overline{A}(\sum_{k=1}^{m} \alpha_k x_k) = \min^i\{\overline{A}(x_k)\}$, for all $\alpha_k \in F$, $\alpha_k \neq 0$, $k = 1, ..., m$.

We now present the intersection of two interval-valued fuzzy subspaces by the operation $\cap$ and characterise its corresponding flag.

Definition 3.5. Let $(V, \overline{A})$ and $(V, \overline{B})$ be two i-v fuzzy subspaces. Define
$(\overline{A} \cap \overline{B})(x) = \min^i\{\overline{A}(x), \overline{B}(x)\}$, $x \in V$.

Theorem 3.6. Let $(V, \overline{A})$ and $(V, \overline{B})$ be two i-v fuzzy subspaces. Then $\overline{A} \cap \overline{B}$ as in Definition 3.5. is again an i-v fuzzy subspace.

Proof. Let $(V, \overline{A})$ and $(V, \overline{B})$ be two i-v fuzzy subspaces.Define
$(\overline{A} \cap \overline{B})(x) = \min^i\{\overline{A}(x), \overline{B}(x)\}$, $x \in V$. Now,

$$\begin{aligned}
&(\overline{A} \cap \overline{B})(\alpha x * \beta y)\\
&\quad = \min^i\{\overline{A}(\alpha x * \beta y), \overline{B}(\alpha x * \beta y)\}\\
&\quad \geq \min^i\{\min^i\{\overline{A}(x), \overline{A}(y)\}, \min^i\{\overline{B}(x), \overline{B}(y)\}\}\\
&\quad = \min^i\{\min^i\{\overline{A}(x), \overline{B}(x)\}, \min^i\{\overline{A}(y), \overline{B}(y)\}\}\\
&\quad = \min^i\{(\overline{A} \cap \overline{B})(x), (\overline{A} \cap \overline{B})(y)\}.
\end{aligned}$$

So,

$$(\overline{A} \cap \overline{B})(\alpha x * \beta y) \geq \min^i\{(\overline{A} \cap \overline{B})(x), (\overline{A} \cap \overline{B})(y)\}.$$

Thus $\overline{A} \cap \overline{B}$ is an i-v fuzzy subspace. □

Example 3.7. Let $V = \{e, x, y, z\}$ be the Klien 4-group given in example 3.2. Then V is a vector space over the field $GF(2)$.
Define an i-v fuzzy subset $\overline{A}$ in V by
$\overline{A}(e) = [0.6, 0.9]$, $\overline{A}(x) = [0.3, 0.4] = \overline{A}(y)$, $\overline{A}(z) = [0.5, 0.8]$
Then $\overline{A}$ is an i-v fuzzy subspace of V by Example 3.2. Also define an i-v fuzzy subset $\overline{B}$ on V by
$\overline{B}(e) = [0.5, 0.8]$, $\overline{B}(x) = [0.4, 0.7]$, $\overline{B}(y) = [0.2, 0.3] = \overline{B}(z)$.
Then $\overline{B}$ is also an i-v fuzzy subspace of V.
Now, $(\overline{A} \cap \overline{B})(e) = [0.5, 0.8]$, $(\overline{A} \cap \overline{B})(x) = [0.3, 0.4]$, $(\overline{A} \cap \overline{B})(y) = [0.2, 0.3]$, $(\overline{A} \cap \overline{B})(z) = [0.2, 0.3]$.
Thus $\overline{A} \cap \overline{B}$ is an i-v fuzzy subset of V and further it is an i-v fuzzy subspace of V.

Theorem 3.8. An interval number $\overline{\gamma} = \min\{\overline{\alpha_r}, \overline{\beta_s}\}$ is a weight for $\overline{A} \cap \overline{B}$ if and only if
(i)$\overline{\alpha_{r-1}} > \overline{\gamma} \geq \overline{\alpha_r}$ and $\overline{\beta_{s-1}} > \overline{\gamma} \geq \overline{\beta_s}$.
(ii)$V^{\overline{\alpha_r}} \cap W^{\overline{\beta_s}} \neq \overline{0}$.

Proof. Let $(V, \overline{A})$ and $(V, \overline{B})$ be two i-v fuzzy subspaces with $(\overline{A} \cap \overline{B})$ as in Definition 3.5.
Let $V \supset V^{\overline{\alpha_k}} \supset ... \supset V^{\overline{\alpha_1}} \supset \overline{0}$
and $V \supset W^{\overline{\beta_l}} \supset ... \supset W^{\overline{\beta_1}} \supset \overline{0}$ be their corresponding flags.
We need to construct a flag for $\overline{A} \cap \overline{B}$.
Clearly the values of $\overline{A} \cap \overline{B}$ are $\overline{\alpha_i}'s$ and $\overline{\beta_j}'s$.
Define $U^{\overline{\gamma}} = (\overline{A} \cap \overline{B})^{-1}([\overline{\gamma}, \overline{1}]) = V^{\overline{\gamma}} \cap W^{\overline{\gamma}}$.
Suppose $U^{\overline{\gamma}} \neq \overline{0}$. Then for some r and s
$\overline{\alpha_{r-1}} > \overline{\gamma} \geq \overline{\alpha_r}$ and $\overline{\beta_{s-1}} > \overline{\gamma} \geq \overline{\beta_s}$.
Then $V^{\overline{\gamma}} = V^{\overline{\alpha_r}}$ and $W^{\overline{\gamma}} = W^{\overline{\beta_s}}$. So $U^{\overline{\gamma}} = V^{\overline{\alpha_r}} \cap W^{\overline{\beta_s}}$.
Therefore $U^{\overline{\gamma}} = V^{\overline{\alpha_r}} \cap W^{\overline{\beta_s}}$, where $\overline{\gamma} = \min\{\overline{\alpha_r}, \overline{\beta_s}\}$.
Since the argument can be reversed, we have the required result. □

Theorem 3.9. An interval number $\overline{\gamma} = \min\{\overline{\alpha_r}, \overline{\beta_s}\}$ is a weight for $\overline{A} + \overline{B}$ if and only if $\overline{\alpha_{r-1}} > \overline{\gamma} \geq \overline{\alpha_r}$ and $\overline{\beta_{s-1}} > \overline{\gamma} \geq \overline{\beta_s}$. The subspace corresponding to γ in the flag representation of $\overline{A} + \overline{B}$ is $V^{\overline{\gamma}} + W^{\overline{\gamma}}$.

Proof. Define,
$(\overline{A} + \overline{B})(x) = \sup_{x = x_1 + x_2} \overline{A}(x_1) \cap \overline{B}(x_2)$.
Now, $x \in (\overline{A} + \overline{B})^{-1}([\overline{\gamma}, \overline{1}])$
$\Leftrightarrow \overline{A}(x_1) + \overline{B}(x_2) \geq \overline{\gamma}$
$\Leftrightarrow \overline{A}(x_1) \geq \overline{\gamma}$ and $\overline{B}(x_2) \geq \overline{\gamma}$
$\Leftrightarrow x_1 \in V^{\overline{\gamma}}$ and $x_2 \in W^{\overline{\gamma}}$
$\Leftrightarrow x_1 + x_2 \in V^{\overline{\gamma}} + W^{\overline{\gamma}}$.
So, $(\overline{A} + \overline{B})^{-1}([\overline{\gamma}, \overline{1}]) = V^{\overline{\gamma}} + W^{\overline{\gamma}} = U^{\overline{\gamma}}$, where $\overline{\gamma} = min[\overline{\alpha}_r, \overline{\beta}_s]$. □

Theorem 3.10. An interval number $\overline{\gamma} = \min\{\overline{\alpha_r}, \overline{\beta_s}\}$ is a weight for $\overline{A} \oplus \overline{B}$. The subspace corresponding to γ in the flag representation of $\overline{A} \oplus \overline{B}$ is $V^{\overline{\gamma}} \oplus W^{\overline{\gamma}}$.

Proof. Let $U = V \oplus W$. Suppose $(V, \overline{A})$ and $(W, \overline{B})$ are i-v fuzzy subspaces with their corresponding flags.
Define,
$(\overline{A} \oplus \overline{B})(x, y) = \overline{A}(x) \cap \overline{B}(y), x \in V, y \in W$.
We have, $(x, y) \in (\overline{A} \oplus \overline{B})^{-1}([\overline{\alpha}, \overline{1}])$
$\Leftrightarrow \overline{A}(x) \cap \overline{B}(y) \geq \overline{\alpha}$
$\Leftrightarrow \overline{A}(x) \geq \overline{\alpha}$ and $\overline{B}(y) \geq \overline{\alpha}$
$\Leftrightarrow (x, y) \in V^{\overline{\alpha}} \oplus W^{\overline{\alpha}}$. □

Theorem 3.11. An interval number $\overline{\gamma} = \min\{\overline{\alpha_r}, \overline{\beta_s}\}$ is a weight for $\overline{A} \otimes \overline{B}$ and the subspace corresponding to γ in the flag representation of $\overline{A} \otimes \overline{B}$ is $V^{\overline{\gamma}} \otimes W^{\overline{\gamma}}$.
Proof. Let $(V, \overline{A})$ and $(W, \overline{B})$ be i-v fuzzy subspaces with their corresponding

flags. Consider the tensor product $V \otimes W$.
Let $\{x_i\}$, $i = 1, 2, ..., n$ and $\{y_j\}$, $j = 1, 2, ..., m$ be i-v fuzzy bases of V and W respectively. Given $t \in V \otimes W$, t can be uniquely expressed as , $t = \sum_{i,j} t_{ij} x_i \otimes y_j$ where the sum is taken over all i, j for which $t_{ij} \neq 0$. Now we define
$(\overline{A} \otimes \overline{B})(t) = \vee_{i,j} \{\overline{A}(x_i) \cap \overline{B}(y_j) : t_{ij} \neq 0\}$
Now,

$$
\begin{aligned}
t \in & (\overline{A} \otimes \overline{B})^{-1}([\overline{\gamma}, \overline{1}]) \\
& \Leftrightarrow \overline{A}(x) \otimes \overline{B}(y) \geq \overline{\gamma} \\
& \Leftrightarrow \overline{A}(x_r) \cap \overline{B}(y_s) \geq \overline{\gamma} \\
& \Leftrightarrow \overline{A}(x_r) \geq \overline{\gamma} \text{ and } \overline{B}(y_s) \geq \overline{\gamma} \\
& \Leftrightarrow x_r \in V^{\overline{\gamma}} \text{ and } y_s \in W^{\overline{\gamma}}.
\end{aligned}
$$

□

4.Interval-valued fuzzy n-normed linear space

As a generalization of Definition 2.9 we have the following notion of interval-valued fuzzy n-normed linear space.

Definition 4.1. Let X be a linear space over a real field F. An interval-valued fuzzy subset $\overline{N}$ of $X^n \times R$ is called an interval-valued fuzzy n-norm if and only if :

($\overline{N}$1) For all $t \in R$ with $t \leq 0$, $\overline{N}(x_1, x_2, ..., x_n, t) = \overline{0}$.
($\overline{N}$2) For all $t \in R$ with $t > 0$, $\overline{N}(x_1, x_2, ..., x_n, t) = \overline{1}$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.
($\overline{N}$3) $\overline{N}(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
($\overline{N}$4) For all $t \in R$ with $t > 0$,
$\overline{N}(x_1, x_2, ..., cx_n, t) = \overline{N}(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0 \in F$(field).
($\overline{N}$5) For all $s, t \in R$,
$\overline{N}(x_1, x_2, ..., x_n + x_n^{'}, s + t) \geq$
$\min^i\{\overline{N}(x_1, x_2, ..., x_n, s), \overline{N}(x_1, x_2, ..., x_n^{'}, t)\}$.
($\overline{N}$6) $\overline{N}(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} \overline{N}(x_1, x_2, ..., x_n, t) = \overline{1}$.

Then $(X, \overline{N})$is called an interval-valued fuzzy n-normed linear space or briefly i-v f-n-NLS.

The following example agrees with our notion of i-v f-n-NLS.

Example 4.2. Let (X,$||\bullet, \bullet, ..., \bullet||$) be an n-normed space. Define

$$
\overline{N}(x_1, x_2, ..., x_n, t) = \begin{cases} \overline{0}, & \text{when } t \leq ||x_1, x_2, ..., x_n|| \\ \overline{1}, & \text{when } ||x_1, x_2, ..., x_n|| < t \end{cases}
$$

Then $(X, \overline{N})$ is an i-v f-n-NLS.

Proof. $(\overline{N}1)$ For all $t \in R$ with $t \leq 0$ we have by our definition,
$\overline{N}(x_1, x_2, ..., x_n, t) = \overline{0}$.
$(\overline{N}2)$ For all $t \in R$ with $t > 0$, if $x_1, x_2, ..., x_n$ are linearly dependent.
$\Rightarrow ||x_1, x_2, ..., x_n|| = 0$ by definition 2.9.
$\Rightarrow N(x_1, x_2, ..., x_n, t) = \overline{1}$ by definition.
Also $N(x_1, x_2, ..., x_n, t) = \overline{1}$.
$\Rightarrow ||x_1, x_2, ..., x_n|| < t$.
$\Rightarrow ||x_1, x_2, ..., x_n|| = 0$.
$\Rightarrow x_1, x_2, ..., x_n$ are linearly dependent.
Thus for all $t > 0$, $\overline{N}(x_1, x_2, ..., x_n, t) = \overline{1}$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.
$(\overline{N}3)$ As $||x_1, x_2, ..., x_n||$ is invariant under any permutation, it follows that $\overline{N}(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.
$(\overline{N}4)$ For all $t \in$R with $t > 0$ and $c \in F$,$c \neq 0$
$\overline{N}(x_1, x_2, ..., cx_n, t) = \overline{0}$
$\Leftrightarrow t \leq ||x_1, x_2, ..., cx_n||$
$\Leftrightarrow \frac{t}{|c|} \leq ||x_1, x_2, ..., x_n||$.
$\Leftrightarrow \overline{N}(x_1, x_2, ..., x_n, \frac{t}{|c|}) = \overline{0}$ and
$\overline{N}(x_1, x_2, ..., cx_n, t) = \overline{1}$
$\Leftrightarrow ||x_1, x_2, ..., cx_n|| < t$
$\Leftrightarrow |c| \; ||x_1, x_2, ..., x_n|| < t$
$\Leftrightarrow ||x_1, x_2, ..., x_n|| < \frac{t}{|c|}$
$\Leftrightarrow \overline{N}(x_1, x_2, ..., x_n, \frac{t}{|c|}) = \overline{1}$.
Thus $\overline{N}(x_1, x_2, ..., cx_n, t) = \overline{N}(x_1, x_2, ..., x_n, \frac{t}{|c|})$
$(\overline{N}5)$For all $s, t \in R$,
$\overline{N}(x_1, x_2, ..., x_n + x_n^{'}, s + t) = \overline{0}$
$\Leftrightarrow s + t \leq ||x_1, x_2, ..., x_n + x_n^{'}|| \leq ||x_1, x_2, ..., x_n|| + ||x_1, x_2, ..., x_n^{'}||$.
If $||x_1, x_2, ..., x_n|| < s$ then $||x_1, x_2, ..., x_n^{'}|| \not< t$.
That is, if $\overline{N}(x_1, x_2, ..., x_n, s) = \overline{1}$ then $\overline{N}(x_1, x_2, ..., x_n^{'}, t) = \overline{0}$.
Thus $\overline{N}(x_1, x_2, ..., x_n + x_n^{'}, s + t) = \overline{0}$
$\Rightarrow \min^i\{\overline{N}(x_1, x_2, ..., x_n, s), \overline{N}(x_1, x_2, ..., x_n^{'}, t)\} = \overline{0}$
Similarly, $\overline{N}(x_1, x_2, ..., x_n + x_n^{'}, s + t) = \overline{1}$
$\Rightarrow \overline{N}(x_1, x_2, ..., x_n + x_n^{'}, s + t) \geq$
$\min^i\{\overline{N}(x_1, x_2, ..., x_n, s), \overline{N}(x_1, x_2, ..., x_n^{'}, t)\}$.
Thus,
$\overline{N}(x_1, x_2, ..., x_n + x_{n'}, s + t) \geq$
$\min^i\{\overline{N}(x_1, x_2, ..., x_n, s), \overline{N}(x_1, x_2, ..., x_n^{'}, t)\}$.

$(\overline{N}6)$From the definition it is clear that $\overline{N}(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} \overline{N}(x_1, x_2, ..., x_n, t) = \overline{1}$.
Thus $(X, \overline{N})$is an i-v f-n-NLS. □

Theorem 4.3. Let $(X, \overline{N_1})$ and $(X, \overline{N_2})$ be two i-v fuzzy n-normed linear spaces. Define
$(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, t) = \min^i\{\overline{N_1}(x_1, x_2, ..., x_n, t), \overline{N_2}(x_1, x_2, ..., x_n, t)\}$
for all $(x_1, x_2, ..., x_n, t) \in X^n \times R$. Then $\overline{N_1} \cap \overline{N_2}$ is an i-v f-n-NLS.

Proof. $(\overline{N_1})$ For all $t \in R$ with $t \leq 0$ we have
$\overline{N_1}(x_1, x_2, ..., x_n, t) = \overline{0}$ and $\overline{N_2}(x_1, x_2, ..., x_n, t) = \overline{0}$
So, $(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, t) = \overline{0}$
$(\overline{N_2})$ For all $t \in R$ with $t > 0$ we have
$(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, t) = \overline{1}$
$\Leftrightarrow \min^i\{(\overline{N_1}(x_1, x_2, ..., x_n, t), \overline{N_2}(x_1, x_2, ..., x_n, t)\} = \overline{1}$
$\Leftrightarrow \overline{N_1}(x_1, x_2, ..., x_n, t) = \overline{N_2}(x_1, x_2, ..., x_n, t) = \overline{1}$.
$\Leftrightarrow x_1, x_2, ..., x_n$ are linearly dependent.
$(\overline{N}3)$ As $\overline{N_1}(x_1, x_2, ..., x_n, t)$ and $\overline{N_2}(x_1, x_2, ..., x_n, t)$ are invariant under any permutation, we have $(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$
$(\overline{N}4)$ For all $t \in R$ with $t > 0$ and $c \in F$,$c \neq 0$

$$
\begin{aligned}
&(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., cx_n, t)\\
&= \min^i\{\overline{N_1}(x_1, x_2, ..., cx_n, t), \overline{N_2}(x_1, x_2, ..., cx_n, t)\}\\
&= \min^i\{\overline{N_1}(x_1, x_2, ..., x_n, \tfrac{t}{|c|}), \overline{N_2}(x_1, x_2, ..., x_n, \tfrac{t}{|c|})\}\\
&= (\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, \tfrac{t}{|c|})
\end{aligned}
$$

$(\overline{N}5)$ For all $s, t \in R$,

$$
\begin{aligned}
&(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n + x_n^{'}, s + t)\\
&= \min^i\{\overline{N_1}(x_1, x_2, ..., x_n + x_n^{'}, s + t), \overline{N_2}(x_1, x_2, ..., x_n + x_n^{'}, s + t)\}\\
&\geq \min^i\{\min^i\{\overline{N_1}(x_1, x_2, ..., x_n, s), \overline{N_1}(x_1, x_2, ..., x_n^{'}, t)\},\\
&\qquad \min^i\{\overline{N_2}(x_1, x_2, ..., x_n, s), \overline{N_2}(x_1, x_2, ..., x_n^{'}, t)\}\}\\
&= \min^i\{\min^i\{\overline{N_1}(x_1, x_2, ..., x_n, s), \overline{N_2}(x_1, x_2, ..., x_n, s)\},\\
&\qquad \min^i\{\overline{N_1}(x_1, x_2, ..., x_n^{'}, t), \overline{N_2}(x_1, x_2, ..., x_n^{'}, t)\}\}\\
&= \min^i\{(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, s), (\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n^{'}, t)\}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n + x_n^{'}, s + t)\\
&\qquad \geq \min^i\{(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, s), (\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n^{'}, t)\}
\end{aligned}
$$

$(\overline{N}6)$ As $\overline{N_1}(x_1, x_2, ..., x_n, t)$ and $\overline{N_2}(x_1, x_2, ..., x_n, t)$ are non-decreasing functions of $t \in R$ it follows that $(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, t)$ is a non-decreasing function

of $t \in R$ and $\lim_{t\to\infty}(\overline{N_1} \cap \overline{N_2})(x_1, x_2, ..., x_n, t) = \overline{1}$.
Thus $(X, \overline{N_1} \cap \overline{N_2})$ is an i-v f-n-NLS. □

Remark 4.4. Let $(X, \overline{N_1})$ and $(X, \overline{N_2})$ be two i-v fuzzy n-normed linear spaces. Define
$(\overline{N_1} \cup \overline{N_2})(x_1, x_2, ..., x_n, t) = \max^i\{\overline{N_1}(x_1, x_2, ..., x_n, t), \overline{N_2}(x_1, x_2, ..., x_n, t)\}$
for all $(x_1, x_2, ..., x_n, t) \in X^n \times R$. Then $\overline{N_1} \cup \overline{N_2}$ is not an i-v f-n-NLS.

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