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RESEARCH ARTICLE

Szeged and Padmakar-Ivan Energies of Non-Commuting Graph for Dihedral Groups

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Abstract The graph can represent the molecule structure and the π-electron energy derives the concept of graph energy. The graph also can be related to the groups or rings as its vertex set. The non-commutative graph is a type of graph whose construction is determined by the structure of a group. This paper focuses on the energy of the non-commuting graph for dihedral groups using the Szeged and Padmakar-Ivan matrices. Both matrices are constructed based on the distance between two vertices in the graph. The eigenvalues of these matrices lead to the formulation of the graph's energy values. Interestingly, the energies obtained are equal to twice the spectral radius and are hyperenergetic.

Keywords: Szeged matrix, Padmakar-Ivan matrix, non-commuting graph, dihedral group.

Introduction

The non-commuting graph on a finite group G, denoted by Γ_G, with the set of non-central elements of G as the vertex set of Γ_G. Two vertices $v_p \neq v_q$ of Γ_G are adjacent whenever $v_p v_q \neq v_q v_p$ [1]. Alimon, *et al*. [3] have discussed the Szeged index of this graph for the dihedral group, D_{2n}. The Szeged index definition was originally defined by Gutman & Dobrynin [8] in 1998. Fath-Tabar, *et al*. [6] introduced the Szeged matrix of a graph in 2010. Later, Habibi and Ashrafi [9] continued the lines to define the revised Szeged matrix of a graph. Furthermore, the graph matrix was extended to the Padmakar-Ivan matrix [12].

Initially, the connection between the graph and algebra is represented by the adjacency matrix. Gutman [7] introduced the energy of a graph based on the eigenvalues of this matrix in 1978. The interesting results on the energy value were given by [5] and [13] who described the values as neither an odd number nor the square root of an odd number. Such authors investigated their research on the energy of Γ_G for the dihedral group, D_{2n}, such as the Seidel Laplacian and Seidel signless Laplacian energy [19], and Sombor energy [21]. In addition, D_{2n} group is also a research topic in building power graphs as has been done by Rana *et al*. [15] and another research by Romdhini *et al*. [18] who discuss the spectral perspectives. Likewise, Sehgal *et al*. [23] discussed the coprime graph of D_{2n}.

The applications of graph energy can be found in Sun *et al* [24] to analyze and compare the protein sequences, satellite communication [2], and decision-making theory [20,22]. Moreover, other applications are reported for recognizing patterns and faces [4], identifying objects [25], analyzing images [26], and international transfer of cancer patients [16].

Motivated by this, the authors investigate the Szeged and Padmakar-Ivan matrices of Γ_G for D_{2n} and observe the characteristic polynomial, spectrum, and energy. The methodology consists of constructing the Szeged and Padmakar-Ivan matrices of Γ_G, analyzing the spectrum and spectral radius of Γ_G, and computing calculating the Szeged and Padmakar-Ivan energies.

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Preliminaries

This research focuses on the dihedral group of order $2n$, where $n \geq 3$, $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$. For further discussion, we denote $\Gamma_{D_{2n}}$ as the non-commuting graph for D_{2n}. The following result gives the isomorphism of $\Gamma_{D_{2n}}$ with some common types of graphs.

Theorem 2.1. [14] Let K_n is a complete graph on n vertices, then

$$\Gamma_{D_{2n}} \cong \begin{cases} K_n, & \text{for odd } n \\ K_n - \frac{n}{2}K_2, & \text{for even } n. \end{cases}$$

The graph $\Gamma_{D_{2n}}$ is associated with the Szeged and Padmakar-Ivan matrices. We first consider Definition 2.1 for defining the Szeged and Padmakar-Ivan matrices definition in the next two consecutive definitions.

Definition 2.1. [8] For $e = (v_p v_q) \in \Gamma_{D_{2n}}$,

$$n_1(e|\Gamma_{D_{2n}}) = \left|\{x|x \in \Gamma_{D_{2n}}, d(x, v_p|\Gamma_{D_{2n}}) < d(x, v_q|\Gamma_{D_{2n}})\}\right|,$$
$$n_2(e|\Gamma_{D_{2n}}) = \left|\{v|x \in \Gamma_{D_{2n}}, d(x, v_p|\Gamma_{D_{2n}}) > d(x, v_q|\Gamma_{D_{2n}})\}\right|.$$

Definition 2.2. [6] The Szeged (S) matrix of $\Gamma_{D_{2n}}$, $S(\Gamma_{D_{2n}}) = [s_{pq}]$ in which (p, q) –th entry is

$$s_{pq} = \begin{cases} n_1(e|\Gamma_{D_{2n}}) \cdot n_2(e|\Gamma_{D_{2n}}), & \text{if } e = (v_p v_q) \in \Gamma_{D_{2n}} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3. [12] The Padmakar-Ivan (PI) matrix of $\Gamma_{D_{2n}}$, $PI(\Gamma_{D_{2n}}) = [pi_{pq}]$ in which (p, q) –th entry is

$$pi_{pq} = \begin{cases} n_1(e|\Gamma_{D_{2n}}) + n_2(e|\Gamma_{D_{2n}}), & \text{if } e = (v_p v_q) \in \Gamma_{D_{2n}} \\ 0, & \text{otherwise.} \end{cases}$$

The formula of Szeged and Padmakar-Ivan energies of $\Gamma_{D_{2n}}$ are $E_S(\Gamma_{D_{2n}}) = \sum_{i=1}^{m}|\lambda_i|$ and $E_{PI}(\Gamma_{D_{2n}}) = \sum_{i=1}^{m}|\mu_i|$ [7], where $\lambda_1, \lambda_2, \dots, \lambda_m$ and $\mu_1, \mu_2, \cdots, \mu_m$ are the eigenvalues of $S(\Gamma_{D_{2n}})$ and $PI(\Gamma_{D_{2n}})$, respectively. In this case, $m = 2n - 1$ for odd n and $m = 2n - 2$ for even n. The spectrum of $\Gamma_{D_{2n}}$, $Spec(\Gamma_{D_{2n}})$, is $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_m^{k_m}\}$ or $\{\mu_1^{k_1}, \mu_2^{k_2}, \dots, \mu_m^{k_m}\}$ associated with $S(\Gamma_{D_{2n}})$ or $PI(\Gamma_{D_{2n}})$, respectively, where $k_1, k_2, \dots, k_m$ are their respective multiplicities. The spectral radius of $\Gamma_{D_{2n}}$ is $\rho_S(\Gamma_{D_{2n}}) = max\{|\lambda|: \lambda \in Spec(\Gamma_{D_{2n}})\}$ or $\rho_{PI}(\Gamma_{D_{2n}}) = max\{|\mu|: \mu \in Spec(\Gamma_{D_{2n}})\}$ [10]. Moreover, $\Gamma_{D_{2n}}$ can be stated as hyperenergetic if the energy is more than $4(n-1)$ for odd n or more than $4(n-1) - 2$ for even n [11].

The following lemma and proposition are from previous results that are used in formulating the characteristic polynomial of $\Gamma_{D_{2n}}$.

Lemma 2.1. [14] For real numbers a, b, c and d, and an $n \times n$ matrix J_n in which all entries are 1, then

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}_{(n_1+n_2)\times(n_1+n_2)}$$

can be simplified as

$$(\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}\left((\lambda - (n_1 - 1)a)(\lambda - (n_2 - 1)b) - n_1 n_2 cd\right),$$

where $1 \le n_1, n_2 \le n$ and $n_1 + n_2 = n$.

Theorem 2.2. [17] If r, s, t are real numbers, and even number n, then the characteristic polynomial of an $(2n-2) \times (2n-2)$ matrix

$$M = \begin{bmatrix} 0 & \cdots & 0 & s & \cdots & s & s & \cdots & s \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & s & \cdots & s & s & \cdots & s \\ s & \cdots & s & 0 & \cdots & t & 0 & \cdots & t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s & \cdots & s & t & \cdots & 0 & t & \cdots & 0 \\ s & \cdots & s & 0 & \cdots & t & 0 & \cdots & t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s & \cdots & s & t & \cdots & 0 & t & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0_{n-2} & sJ_{(n-2)\times\frac{n}{2}} & sJ_{(n-2)\times\frac{n}{2}} \\ sJ_{\frac{n}{2}\times(n-2)} & t(J-I)J_{\frac{n}{2}} & t(J-I)J_{\frac{n}{2}} \\ sJ_{\frac{n}{2}\times(n-2)} & t(J-I)J_{\frac{n}{2}} & t(J-I)J_{\frac{n}{2}} \end{bmatrix}$$

can be simplified in an expression as

$$P_M(\lambda) = (\lambda)^{\frac{3n-6}{2}}(\lambda + 2t)^{\frac{n}{2}-1}(\lambda^2 - (n-2)t\lambda - n(n-2)s^2).$$

Szeged Energy of $\Gamma_{D_{2n}}$

Let us start with the characteristic polynomial of $\Gamma_{D_{2n}}$ corresponds to the Szeged matrix.

Theorem 3.1. In $\Gamma_{D_{2n}}$, then

(1) for odd n, $P_{S(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{n-2}(\lambda+1)^{n-1}(\lambda^2 + (1-n)\lambda + n(1-n)^3)$,
(2) for even n, $P_{S(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{\frac{3n-6}{2}}(\lambda+8)^{\frac{n}{2}-1}(\lambda^2 - 4(n-2)\lambda - 4n(n-2)^3)$.

Proof.

(1) Let n is odd and suppose $G_1 = \{a, a^2, \cdots, a^{n-1}\}$ and $G_2 = \{a^i b: 1 \le i \le n\}$. If $e = uv$ is an edge of $\Gamma_{D_{2n}}$, then the entries of the Szeged matrix of $\Gamma_{D_{2n}}$, $S(\Gamma_{D_{2n}})$, are described as follows:

(i) From Theorem 2.1, there is no edge between $a, a^2, \cdots, a^{n-1}$ in $\Gamma_{D_{2n}}$ which means for $u, v \in G_1$, the entries of $S(\Gamma_{D_{2n}})$ are zero.
(ii) For $u \in G_1$ and $v \in G_2$, or vice versa, then $n_1(e|\Gamma_{D_{2n}}) = 1$ and $n_2(e|\Gamma_{D_{2n}}) = n-1$. Consequently, the entries of $S(\Gamma_{D_{2n}})$ are $1 \cdot (n-1) = n-1$.
(iii) For $u, v \in G_2$, then $n_1(e|\Gamma_{D_{2n}}) = 1$ and $n_2(e|\Gamma_{D_{2n}}) = 1$. Hence the entries of $S(\Gamma_{D_{2n}})$ are $1 \cdot 1 = 1$.

Therefore, $S(\Gamma_{D_{2n}})$ is a $(2n-1) \times (2n-1)$ matrix as follows:

$$S(\Gamma_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & n-1 & n-1 & \cdots & n-1 \\ 0 & 0 & \cdots & 0 & n-1 & n-1 & \cdots & n-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & n-1 & n-1 & \cdots & n-1 \\ n-1 & n-1 & \cdots & n-1 & 0 & 1 & \cdots & 1 \\ n-1 & n-1 & \cdots & n-1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & \cdots & n-1 & 1 & 1 & \cdots & 0 \end{bmatrix} \end{array}.$$

Then, the Szeged matrix of $\Gamma_{D_{2n}}$ can be expressed as

$$S(\Gamma_{D_{2n}}) = \begin{bmatrix} 0_{n-1} & (n-1)J_{(n-1)\times n} \\ (n-1)J_{n\times(n-1)} & (J-I)_n \end{bmatrix},$$

and the determinant below is the characteristic polynomial for $S(\Gamma_{D_{2n}})$,

$$P_{S(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda I_{n-1} & (1-n)J_{(n-1)\times n} \\ (1-n)J_{n\times(n-1)} & (\lambda+1)I_n - J_n \end{vmatrix}.$$

Based on Lemma 2.1, with $a = 0$, $b = 1$, $c = d = 1-n$, $n_1 = n-1$, and $n_2 = n$, therefore

$$P_{S(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{n-2}(\lambda+1)^{n-1}(\lambda^2 + (1-n)\lambda + n(1-n)^3).$$

(2) Let n is odd and suppose $G_1 = \left\{a, a^2, \cdots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \cdots, a^n\right\}$ and $G_2 = \{a^i b: 1 \le i \le n\}$. If $e = uv$ is an edge of $\Gamma_{D_{2n}}$, then the entries of the Szeged matrix of $\Gamma_{D_{2n}}$, $S(\Gamma_{D_{2n}})$, are described as follows:

(i) From Theorem 2.1, there is no edge between $a, a^2, \cdots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \cdots, a^n$ in $\Gamma_{D_{2n}}$ which means for $u, v \in G_1$, the entries of $S(\Gamma_{D_{2n}})$ are zero.

(ii) For $u \in G_1$ and $v \in G_2$, or vice versa, then $n_1(e|\Gamma_{D_{2n}}) = 2$ and $n_2(e|\Gamma_{D_{2n}}) = n-2$. Consequently, the entries of $S(\Gamma_{D_{2n}})$ are $2 \cdot (n-2) = 2(n-2)$.

(iii) For $u, v \in G_2$, then $n_1(e|\Gamma_{D_{2n}}) = 2$ and $n_2(e|\Gamma_{D_{2n}}) = 2$. Hence the entries of $S(\Gamma_{D_{2n}})$ are $2 \cdot 2 = 4$, except for $a^i b, a^{\frac{n}{2}+i}b \in G_2$ which are zero since there is no edge between those two vertices.

Therefore, $S(\Gamma_{D_{2n}})$ is a $(2n-2) \times (2n-2)$ matrix as follows

$$
\begin{array}{c|ccccccccccccc|}
 & a & \cdots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}+1} & \cdots & a^{n-1} & b & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & \cdots & a^{n-1}b \\
a & 0 & \cdots & 0 & 0 & \cdots & 0 & 2(n-2) & \cdots & 2(n-2) & 2(n-2) & \cdots & 2(n-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{\frac{n}{2}-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 2(n-2) & \cdots & 2(n-2) & 2(n-2) & \cdots & 2(n-2) \\
a^{\frac{n}{2}+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 2(n-2) & \cdots & 2(n-2) & 2(n-2) & \cdots & 2(n-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 2(n-2) & \cdots & 2(n-2) & 2(n-2) & \cdots & 2(n-2) \\
b & 2(n-2) & \cdots & 2(n-2) & 2(n-2) & \cdots & 2(n-2) & 0 & \cdots & 4 & 0 & \cdots & 4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{\frac{n}{2}-1}b & 2(n-2) & \cdots & 2(n-2) & 2(n-2) & \cdots & 2(n-2) & 4 & \cdots & 0 & 4 & \cdots & 0 \\
a^{\frac{n}{2}}b & 2(n-2) & \cdots & 2(n-2) & 2(n-2) & \cdots & 2(n-2) & 0 & \cdots & 4 & 0 & \cdots & 4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{n-1}b & 2(n-2) & \cdots & 2(n-2) & 2(n-2) & \cdots & 2(n-2) & 4 & \cdots & 0 & 4 & \cdots & 0
\end{array}.
$$

In other words, S-matrix of $\Gamma_{D_{2n}}$can be written as

$$
S(\Gamma_{D_{2n}}) = \begin{bmatrix} 0_{n-2} & 2(n-2)J_{(n-2)\times\frac{n}{2}} & 2(n-2)J_{(n-2)\times\frac{n}{2}} \\ 2(n-2)J_{\frac{n}{2}\times(n-2)} & 4(J-I)_{\frac{n}{2}} & 4(J-I)_{\frac{n}{2}} \\ 2(n-2)J_{\frac{n}{2}\times(n-2)} & 4(J-I)_{\frac{n}{2}} & 4(J-I)_{\frac{n}{2}} \end{bmatrix}.
$$

Based on Theorem 2.2 with $s = 2(n-2)$ and $t = 4$, we get

$$
P_{S(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{\frac{3n-6}{2}}(\lambda+8)^{\frac{n}{2}-1}(\lambda^2 - 4(n-2)\lambda - 4n(n-2)^3).
$$

□

In the next results, we prove the S-spectral radius, $\rho_S(\Gamma_{D_{2n}})$, and S-energy of $\Gamma_{D_{2n}}$.

Theorem 3.2. In $\Gamma_{D_{2n}}$

(1) $\rho_S(\Gamma_{D_{2n}}) = n(n-1)$, for odd n, and
(2) $\rho_S(\Gamma_{D_{2n}}) = 2n(n-2)$ for even n.

Proof.

(1) Based on Theorem 3.1 (1) when n is odd, we have $P_{S(\Gamma_{D_{2n}})}(\lambda)$ which implies four eigenvalues of $\Gamma_{D_{2n}}$. Then we get $\lambda_1 = 0$ of multiplicity $n-2$, $\lambda_2 = -1$ of multiplicity $n-1$, $\lambda_3 = n(n-1)$ and $\lambda_4 = -(n-1)^2$, each of multiplicity 1. Thus, the S –spectrum of $\Gamma_{D_{2n}}$ is

$$
Spec(\Gamma_{D_{2n}}) = \{(n(n-1))^1, (0)^{n-2}, (-1)^{n-1}\ (-(n-1)^2)^1\}.
$$

Now for $i = 1,2,3,4$, based on $Spec(\Gamma_{D_{2n}})$, the maximum of absolute eigenvalues $|\lambda_i|$ is the S-spectral radius of $\Gamma_{D_{2n}}$,

$$
\rho_S(\Gamma_{D_{2n}}) = n(n-1).
$$

(2) Performing $P_{S(\Gamma_{D_{2n}})}(\lambda) = 0$ from Theorem 3.1 (2) for even n, we get the eigenvalues of $\Gamma_{D_{2n}}$, which are $\lambda_1 = 0$ of multiplicity $\frac{3n-6}{2}$, $\lambda_2 = -8$ of multiplicity $\frac{n}{2} - 1$ and the other two eigenvalues are $\lambda_3 = 2n(n-2)$ and $\lambda_4 = -2(n-2)^2$ each of multiplicity 1. So that

$$Spec(\Gamma_{D_{2n}}) = \left\{(2n(n-2))^1, (0)^{\frac{3n-6}{2}}, (-8)^{\frac{n}{2}-1}, (-2(n-2)^2)^1\right\}.$$

From the spectrum mentioned above, we finally arrive at

$$\rho_S(\Gamma_{D_{2n}}) = 2n(n-2).$$

□

The result of the Szeged energy of $\Gamma_{D_{2n}}$ is presented below.

Theorem 3.3. In $\Gamma_{D_{2n}}$,

(1) $E_S(\Gamma_{D_{2n}}) = 2n(n-1)$, for odd n,
(2) $E_S(\Gamma_{D_{2n}}) = 4n(n-2)$, for even n.

Proof.

1. From the eigenvalues of $Spec(\Gamma_G)$ in Theorem 3.2 (1) for odd n, we can obtain the S-energy of $\Gamma_{D_{2n}}$. Since $n \geq 3$ for $n \in \mathbb{N}$ and n is odd, then

$$E_S(\Gamma_{D_{2n}}) = (1)|n(n-1)| + (n-2)|0| + (n-1)|-1| + |-(n-1)^2| = 2n(n-1).$$

2. For even n, it follows from Theorem 3.2 (2), the S-energy is presented below

$$E_S(\Gamma_{D_{2n}}) = (1)|2n(n-2)| + \left(\frac{3n-6}{2}\right)|0| + \left(\frac{n}{2}-1\right)|-8| + |-2(n-2)^2| = 4n(n-2).$$

□

Padmakar-Ivan Energy of $\Gamma_{D_{2n}}$

This section focuses on the Padmakar-Ivan matrix of $\Gamma_{D_{2n}}$.

Theorem 3.4. In $\Gamma_{D_{2n}}$, then

(1) for odd n, $P_{PI(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{n-2}(\lambda+2)^{n-1}(\lambda^2 + 2(1-n)\lambda + (1-n)n^3)$,
(2) for even n, $P_{PI(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{2n-4}(\lambda^2 + n(n-2)(3n^2 - 16n + 16)^2)$.

Proof.

(1) Let n is odd, for the same reason as the proofing part of Theorem 3.1 (1), then the entries of the Padmakar-Ivan matrix of $\Gamma_{D_{2n}}$ are as the following:
(2)

i. For $u, v \in G_1$, the entries of $PI(\Gamma_{D_{2n}})$ are zero;
ii. For $u \in G_1$ and $v \in G_2$, or vice versa, then the entries of $PI(\Gamma_{D_{2n}})$ are $1 + (n-1) = n$;
iii. For $u, v \in G_2$, then the entries of $PI(\Gamma_{D_{2n}})$ are $1 + 1 = 2$.

Therefore, $PI(\Gamma_{D_{2n}})$ is a $(2n-1) \times (2n-1)$ matrix as follows

$$PI(\Gamma_{D_{2n}}) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & n & n & \cdots & n \\ 0 & 0 & \cdots & 0 & n & n & \cdots & n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & n & n & \cdots & n \\ n & n & \cdots & n & 0 & 2 & \cdots & 2 \\ n & n & \cdots & n & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & \cdots & n & 2 & 2 & \cdots & 0 \end{bmatrix} \end{array}.$$

Then, the Padmakar-Ivan matrix of $\Gamma_{D_{2n}}$ can be expressed as

$$PI(\Gamma_{D_{2n}}) = \begin{bmatrix} 0_{n-1} & nJ_{(n-1)\times n} \\ nJ_{n\times(n-1)} & 2(J-I)_n \end{bmatrix},$$

and the determinant below is the characteristic polynomial for $PI(\Gamma_{D_{2n}})$,

$$P_{PI(\Gamma_{D_{2n}})}(\lambda) = \begin{vmatrix} \lambda I_{n-1} & nJ_{(n-1)\times n} \\ nJ_{n\times(n-1)} & (\lambda+2)I_n - 2J_n \end{vmatrix}.$$

Based on Lemma 2.1, with $a = 0,\ b = 2,\ c = d = n,\ n_1 = n-1$, and $n_2 = n$, therefore

$$P_{PI(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{n-2}(\lambda+2)^{n-1}(\lambda^2 + 2(1-n)\lambda + (1-n)n^3).$$

(3) For even n and by the same reason as the proofing part of Theorem 3.1 (2), then the entries of the Padmakar-Ivan matrix of $\Gamma_{D_{2n}}$ are as the following:

i. For $u, v \in G_1$, the entries of $PI(\Gamma_{D_{2n}})$ are zero;
ii. For $u \in G_1$ and $v \in G_2$, or vice versa, then the entries of $PI(\Gamma_{D_{2n}})$ are $2 + (n-2) = n$;
iii. For $u, v \in G_2$, then the entries of $PI(\Gamma_{D_{2n}})$ are $2 + 2 = 4$, except for $a^i b, a^{\frac{n}{2}+i} b \in G_2$ which are zero since there is no edge between those two vertices.

Therefore, $PI(\Gamma_{D_{2n}})$ is a $(2n-2)\times(2n-2)$ matrix as follows

$$PI(\Gamma_{D_{2n}}) = \begin{array}{c} \\ a \\ \vdots \\ a^{\frac{n}{2}-1} \\ a^{\frac{n}{2}+1} \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccccccc} a & \cdots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}+1} & \cdots & a^{n-1} & b & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & \cdots & a^{n-1}b \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & n & \cdots & n & n & \cdots & n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & n & \cdots & n & n & \cdots & n \\ 0 & \cdots & 0 & 0 & \cdots & 0 & n & \cdots & n & n & \cdots & n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & n & \cdots & n & n & \cdots & n \\ n & \cdots & n & n & \cdots & n & 0 & \cdots & 4 & 0 & \cdots & 4 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ n & \cdots & n & n & \cdots & n & 4 & \cdots & 0 & 4 & \cdots & 0 \\ n & \cdots & n & n & \cdots & n & 0 & \cdots & 4 & 0 & \cdots & 4 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ n & \cdots & n & n & \cdots & n & 4 & \cdots & 0 & 4 & \cdots & 0 \end{bmatrix} \end{array}.$$

The PI-matrix of $\Gamma_{D_{2n}}$ can be written as

$$PI(\Gamma_{D_{2n}}) = \begin{bmatrix} 0_{n-2} & nJ_{(n-2)\times\frac{n}{2}} & nJ_{(n-2)\times\frac{n}{2}} \\ nJ_{\frac{n}{2}\times(n-2)} & 4(J-I)_{\frac{n}{2}} & 4(J-I)_{\frac{n}{2}} \\ nJ_{\frac{n}{2}\times(n-2)} & 4(J-I)_{\frac{n}{2}} & 4(J-I)_{\frac{n}{2}} \end{bmatrix}.$$

Repeated application of Theorem 3.1, with $s = n$ dan $t = 4$, we get

$$P_{PI(\Gamma_{D_{2n}})}(\lambda) = (\lambda)^{\frac{3n-6}{2}}(\lambda+8)^{\frac{n}{2}-1}(\lambda^2 - 4(n-2)\lambda - n^3(n-2)).$$

□

The next theorem is the spectral radius of $\Gamma_{D_{2n}}$ associated with the Padmakar-Ivan matrix.

Theorem 3.6. In $\Gamma_{D_{2n}}$

(1) $\rho_{PI}(\Gamma_{D_{2n}}) = n - 1 + \sqrt{(n-1)^2 + n^3(n-1)}$, for odd n, and
(2) $\rho_{PI}(\Gamma_{D_{2n}}) = 2(n-2) + \sqrt{4(n-2)^2 + n^3(n-2)}$, for even n.

Proof.

(1) Based on Theorem 3.5 (1) when n is odd, we have $P_{PI(\Gamma_{D_{2n}})}(\lambda)$ which implies four eigenvalues of $\Gamma_{D_{2n}}$. Then we get $\lambda_1 = 0$ of multiplicity $n-2$, $\lambda_2 = -2$ of multiplicity $n-1$, $\lambda_{3,4} = n-1 \pm \sqrt{(n-1)^2 + n^3(n-1)}$ each of multiplicity 1. Thus, the PI –spectrum of $\Gamma_{D_{2n}}$ is

$$Spec(\Gamma_{D_{2n}}) = \left\{\left(n-1+\sqrt{(n-1)^2+n^3(n-1)}\right)^1, (0)^{n-2}, (-2)^{n-1} \left(n-1-\sqrt{(n-1)^2+n^3(n-1)}\right)^1\right\}.$$

Now for $i = 1,2,3,4$, based on $Spec(\Gamma_{D_{2n}})$, the maximum of absolute eigenvalues $|\lambda_i|$ is the PI-spectral radius of $\Gamma_{D_{2n}}$,

$$\rho_{PI}(\Gamma_{D_{2n}}) = n-1+\sqrt{(n-1)^2+n^3(n-1)}.$$

(2) Performing $P_{PI(\Gamma_{D_{2n}})}(\lambda) = 0$ from Theorem 3.5 (2) for even n, we get the eigenvalues of $\Gamma_{D_{2n}}$, which are $\lambda_1 = 0$ of multiplicity $\frac{3n-6}{2}$, $\lambda_2 = -8$ of multiplicity $\frac{n}{2} - 1$ and the other two eigenvalues are $\lambda_{3,4} = 2(n-2) \pm \sqrt{4(n-2)^2 + n^3(n-2)}$ each of multiplicity 1. So that

$$Spec(\Gamma_{D_{2n}}) = \left\{\left(2(n-2)+\sqrt{4(n-2)^2+n^3(n-2)}\right)^1, (0)^{\frac{3n-6}{2}}, (-8)^{\frac{n}{2}-1}, \left(2(n-2)-\sqrt{4(n-2)^2+n^3(n-2)}\right)^1\right\}.$$

From the spectrum mentioned above, we finally arrive at

$$\rho_{PI}(\Gamma_{D_{2n}}) = 2(n-2)+\sqrt{4(n-2)^2+n^3(n-2)}.$$

□

Theorem 3.7. In $\Gamma_{D_{2n}}$,

(1) $E_{PI}(\Gamma_{D_{2n}}) = 2\left(n-1+\sqrt{(n-1)^2+n^3(n-1)}\right)$, for odd n,
(2) $E_{PI}(\Gamma_{D_{2n}}) = 2\left(2(n-2)+\sqrt{4(n-2)^2+n^3(n-2)}\right)$, for even n.

Proof.

(1) From the eigenvalues of $Spec(\Gamma_G)$ in Theorem 3.6 (1) for odd n, we can obtain the PI-energy of $\Gamma_{D_{2n}}$. Since $n \geq 3$ for $n \in \mathbb{N}$ and n is odd, then

$$\begin{aligned} E_{PI}(\Gamma_{D_{2n}}) &= (n-2)|0| + (n-1)|-2| + \left|n-1 \pm \sqrt{(n-1)^2+n^3(n-1)}\right| \\ &= 2\left(n-1+\sqrt{(n-1)^2+n^3(n-1)}\right). \end{aligned}$$

(2) For even n, it follows from Theorem 3.6 (2), the PI-energy is presented below

$$\begin{aligned} E_{PI}(\Gamma_{D_{2n}}) &= (1)\left|2(n-2) \pm \sqrt{4(n-2)^2+n^3(n-2)}\right| + \left(\frac{3n-6}{2}\right)|0| + \left(\frac{n}{2}-1\right)|-8| \\ &= 2\left(2(n-2)+\sqrt{4(n-2)^2+n^3(n-2)}\right). \end{aligned}$$

□

Discussions

By examining the results of Theorems 3.2 and 3.3 and Theorems 3.5 and 3.6 we obtain the explicit fact that the obtained energies are twice their spectral radius of $\Gamma_{D_{2n}}$.

Corollary 4.1. In $\Gamma_{D_{2n}}$, $E_S(\Gamma_{D_{2n}}) = 2 \cdot \rho_S(\Gamma_{D_{2n}})$ and $E_{PI}(\Gamma_{D_{2n}}) = 2 \cdot \rho_{PI}(\Gamma_{D_{2n}})$.

The classification of $\Gamma_{D_{2n}}$ is presented below.

Corollary 4.2. $\Gamma_{D_{2n}}$ is hyperenergetic corresponds to the Szeged and Padmakar-Ivan matrices.

Based on Theorems 3.3 and 3.6, we derive the following fact.

Corollary 4.3. In $\Gamma_{D_{2n}}$, the Szeged energy is always an even integer.

Corollary 4.4. In $\Gamma_{D_{2n}}$, the Padmakar-Ivan energy is never an odd integer.

As the illustration, we provide one example for D_8 as presented in Example 4.1.

Example 4.1. Let $D_8 = \{e,\ a,\ a^2,\ a^3, b,\ ab,\ a^2b,\ a^3b\}$. By excluding the center of D_8, we have 6 vertices in Γ_{D_8}. The non-commuting graph for D_8 is as in Figure 1.

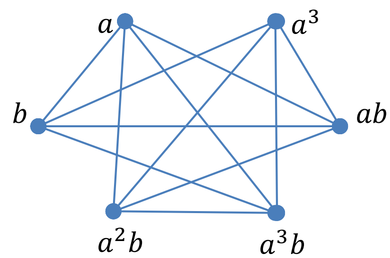

Figure 1. Non-Commuting graph for D_8

The construction of the Szeged matrix of Γ_{D_8} is as the following

$$S(\Gamma_{D_8}) = PI(\Gamma_{D_8}) = \left[\begin{array}{cc|cc|cc} 0 & 0 & 4 & 4 & 4 & 4 \\ 0 & 0 & 4 & 4 & 4 & 4 \\ \hline 4 & 4 & 0 & 4 & 0 & 4 \\ 4 & 4 & 4 & 0 & 4 & 0 \\ \hline 4 & 4 & 0 & 4 & 0 & 4 \\ 4 & 4 & 4 & 0 & 4 & 0 \end{array}\right] = \begin{bmatrix} 0_2 & 4J_2 & 4J_2 \\ 4J_2 & -4I_2 + 4J_2 & -4I_2 + 4J_2 \\ 4J_2 & -4I_2 + 4J_2 & -4I_2 + 4J_2 \end{bmatrix}.$$

The characteristic formula of $S(\Gamma_{D_8})$ and $PI(\Gamma_{D_8})$ are

$$P_{S(\Gamma_{D_8})}(\lambda) = P_{PI(\Gamma_{D_8})}(\lambda) = (\lambda)^3(\lambda + 8)^2(\lambda - 16).$$

By using Maple, we have confirmed that

$$Spec(\Gamma_{D_8}) = \{16^{(1)},\ 0^{(3)},\ -8^{(2)}\} \text{ and } \rho_S(\Gamma_{D_8}) = \rho_{PI}(\Gamma_{D_8}) = 16.$$

Therefore, the Szeged and Padmakar-Ivan energies of Γ_{D_8} are as follows:

$$E_S(\Gamma_{D_8}) = E_{PI}(\Gamma_{D_8}) = (1)|16| + (3)|0| + (2)|-8| = 32 = 2 \cdot \rho_S(\Gamma_{D_8}) = 2 \cdot \rho_{PI}(\Gamma_{D_8}).$$

Conclusion

Graphs can be defined in groups as demonstrated in this paper. They can also be associated with matrices based on certain definitions such as the Szeged and Padmakar-Ivan matrices. The eigenvalues of these matrices provide the concept of spectral graph theory. In this article, the formula of the spectral radius and energy of $\Gamma_{D_{2n}}$ associated with both matrices are determined. We also presented the obtained energies are always twice their spectral radius and are hyperenergetic. For further research, the discussion can be extended to the graph defined on rings including the prime ideal graph.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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