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Approximate Solution of Fuzzy Sedimentary Ocean Basin Boundary Value Problem Using Variational Iteration Method

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Abstract The problem that is considered in this paper is developed from the migration of the coastline over time in sedimentary ocean basins that may be considered as a moving boundary problem with a variable latent heat transfer equation as a governing equation if one of the boundaries is unknown, with boundary and initial conditions are given. This model discusses the shoreline movement within sedimentary basins. The main objective of the paper is to use fuzzy logic to formulate this problem when modelling sand particles migration in saltwater by utilizing fuzzy numbers whenever replacing some of the problem's parameters. In reality, the proposed model is more general than the nonfuzzy or crisp models. By employing the variational iteration method, the approximate numerical solution of the proposed problem has been found, and computer software written in Mathematica 11 have been used to obtain the results.

Keywords: Fuzzy variational iteration method, moving boundary value problem, fuzzy sediment transport, fuzzy shoreline problem.

Introduction

The shoreline problem is a widely studied phenomenon that involves the movement of the seashore within a basin due to the accumulation of sediment in an oceanic environment. This complex problem is often referred to as the "Stefan-Neumann problem" and is characterized by a variable latent heat [1,2]. It is an important area of research as it is influenced by various factors such as sea level changes, Earth's crust subsidence, and sediment flow alterations. To better understand and model this phenomenon, researchers often utilize classical diffusion transport models [3–5], which have proven to be a dependable approach for studying sediment transport in fluvial depositional systems.

The Stefan problem, in particular, is a nonlinear problem with analytical solution so difficult to be evaluated. Stefan's problem has been solved using a variety of approximation methods [5,6], such as the analytical solution of this problem, which is an extreme state of the coastline model is evaluated in 2004 by Voller *et al*. [5]. In addition, Voller *et al*. in 2006 [7] discussed a novel moving boundary problem related to shoreline movement in a sedimentary basin, which was solved using the enthalpy method. The Adomian decomposition approximation approach was considered by Rajeev *et al*. in 2013 [8] to solve movable boundary value problems in the river-deltaic sedimentary proceeding with a space and time fractional order derivatives. Furthermore, as an approach of solving Stefan problem with variable latent heat problems, Rajeev in 2014 [3] also used the homotopy perturbation method (HPM).

In 1965, Zadeh presented the concept of fuzzy set theory for the first time as a result of parameters appeared in most real life problems, which are frequently subjected to considerable uncertainty, and to those problems with parameters estimation are typically made using experimental data [9,10]. In relation to mathematical topics, Chang and Zadeh were the original developers of the concept of the fuzzy derivative in 1972. For the sake of considering fuzzy differential equations for both ordinary and partial, the Hukuhara derivative was eventually used to establish fuzzy derivatives and fuzzy differentiability [11].

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For interval-valued functions, Stefanini and Bede [12,13] extended the Hukuhara difference and differentiability, in which several definitions of differentiability are considered, as well as the relation between them and some properties. Also, there has been a significant amount of study in this topic since Buckley and Feuring [14], are the first whom proposed fuzzy-valued partial differential equations. Thus, as a consequence to the above, the analytic, numerical and approximate methods for solving fuzzy differential equations appears to be crucial, which motivate the researchers in recent times to solve some analytical and real life problems using different semi-analytical techniques, such as the "Homotopy Perturbation Method (HPM)" [3], and the "Variational Iteration Method (VIM)" [15]. Additionally, the study of heat, wave, and Poisson equations with ambiguous parameters is also covered by Awasthi *et al*. when studying the threat to the health of individuals [16]. Recently, a study comparing fuzzy solutions to the fuzzy heat equation using both the fuzzy Adomian decomposition method (ADM) and the fuzzy variational iteration method (FVIM) is presented by Osman *et al*. [17]. Melliani *et al*. [18] used the variational iteration approach to find the fuzzy solution of fuzzy heat-like equations. Allahviranloo *et al*. [19] stated and verified the existence and uniqueness theorem of the solution of fuzzy heat equation with partial derivatives based on the definition of generalized Hukuhara differentiability and obtained the analytical solutions of their considered fuzzy partial differential equations.

In today's advanced age of science and technology, numerous challenges in engineering and applied science have been effectively expressed and mathematically formulated using differential equations, accompanied with initial and/or boundary conditions known as initial-boundary value problems. A range of detailed theories and methods has been presented to tackle these initial-boundary value problems when the initial and/or boundary conditions are treated as crisp values [21,22]. However, in practice, in most instances, initial or boundary values are not clearly defined; rather, they are often vague. Consequently, fuzzy differential equations play a significant role in representing many real-world problems [20–22]. Some researchers have utilized fuzzy differential equations to model and analyze several typical initial and boundary value problems in physics and engineering [23,24].

The main theme of this article is to formulate a fluvial sedimentary problem using fuzzy logic. This process entails introducing a fuzzy parameter, represented as a fuzzy number, into the governing equation, followed by considering the resulting problem as a fuzzy moving boundary value problem. We solve the problem using the FVIM, which is based on the concept of generalized Hukuhara partial differentiability and the alpha-level set concept.

Preliminaries

In the present section, we will concentrate with the fundamental concepts and basic notations that will be utilized throughout the present work.

Let $\mathbb{R}_f$ be the set of all fuzzy subsets of $\mathbb{R}^n$, then a fuzzy set $\tilde{N}$ with membership function $\mu_{\tilde{N}}: \mathbb{R}^n \longrightarrow [0,1]$ is said to be fuzzy number if $\tilde{N}$ satisfy the following [24,25]:

(i) $\tilde{N}$ is fuzzy convex set;
(ii) $\tilde{N}$ is normal, i.e., there exists $x_0 \in \mathbb{R}$, in which $\mu_{\tilde{N}}(x_0) = 1$;
(iii) $\tilde{N}$ is upper semi-continuous;
(iv) The closure of $\{x \in \mathbb{R}^n: \mu_{\tilde{N}}(x) > 0\}$ is compact.

Commonly, $\mathbb{R}_f$ will be used to refer to the space of all fuzzy numbers.

For $0 \leq \alpha \leq 1$, the α-level set corresponding to the fuzzy number $\tilde{N}$ is defined by $[\tilde{N}]^\alpha = \{x \in \mathbb{R}^n: \mu_{\tilde{N}}(x) \geq \alpha\}$. Then from (i)-(iv), it follows that the α-level set $[\tilde{N}]^\alpha$ is the closed interval $[\underline{N}(\alpha), \overline{N}(\alpha)]$ as a subset of $\mathbb{R}^n$, for all $\alpha \in [0,1]$.

A fuzzy number in $\mathbb{R}_f$ that will be specified by an ordered triple $(a, b, c), a \leq b \leq c$ is known as a triangular fuzzy number and is denoted by $\tilde{N} = (a, b, c)$, which is also characterized using α-level set as $[\tilde{N}]^\alpha = [\underline{N}(\alpha), \overline{N}(\alpha)] = [a + (b - a)\alpha, c - (c - b)\alpha]$, for all $\alpha \in [0,1]$.

Remark 1, [26]**.** Assuming that $\tilde{M} = [\underline{M}(\alpha), \overline{M}(\alpha)]$ and $[\tilde{N}]^\alpha = [\underline{N}(\alpha), \overline{N}(\alpha)]$ as an arbitrary two fuzzy numbers, then based on interval analysis and arithmetic, the following algebraic operations are carried out between them:

1. If h is any real number, then:
   $$h\widetilde{M}=\begin{cases}[h\underline{M}(\alpha),h\overline{M}(\alpha)], \text{ if } h\geq 0\\ [h\overline{M}(\alpha),h\underline{M}(\alpha)], \text{ if } h<0\end{cases}$$
2. $\widetilde{M}-\widetilde{N}=[\underline{M}(\alpha)-\overline{N}(\alpha),\overline{M}(\alpha)-\underline{N}(\alpha)]$.
3. $\widetilde{M}\widetilde{N}=[\min d(\alpha),\max d(\alpha)]$, where:
   $d(\alpha)=\{\underline{M}(\alpha)\underline{N}(\alpha),\underline{M}(\alpha)\overline{N}(\alpha),\overline{M}(\alpha)\underline{N}(\alpha),\overline{M}(\alpha)\overline{N}(\alpha)\}$
4. $\frac{\widetilde{M}}{\widetilde{N}}=[\min d(\alpha),\max d(\alpha)]$, where:
   $d(\alpha)=\{\underline{M}(\alpha)\underline{N}(\alpha),\underline{M}(\alpha)\overline{N}\alpha),\overline{M}(\alpha)\underline{N}(\alpha),\overline{M}(\alpha)\overline{N}(\alpha)\}$

Definition 1, [27]**.** Let $\widetilde{M},\widetilde{N}\in\mathbb{R}_f$ and if there exist $\tilde{R}\in\mathbb{R}_f$, such that $\widetilde{N}=\widetilde{M}+\tilde{R}$, then $\tilde{R}$ is called the Hukuhara difference of $\widetilde{N}$ and $\widetilde{M}$, which is denoted by $\widetilde{N}\ominus\widetilde{M}$.

As in fuzzy numbers, the α-level representation of fuzzy-real valued function $\tilde{f}:[a,b]\longrightarrow\mathbb{R}_f$ is expressed also by the closed interval $\tilde{f}(x;\alpha)=[\underline{f}(x;\alpha),\overline{f}(x;\alpha)]$, for all $x\in[a,b]$, $\alpha\in[0,1]$.

As a continuation of Hukuhara basic concepts, the generalized Hukuhara differentiation is almost the generic type of fuzzy differentiation for interval valued functions. Hukuhara presented the concept of fuzzy function derivative (abbreviated as H-derivative) in 1976, which is later became a starting point for studying fuzzy differential equations. Hukuhara derivative possibly considered like a generalization of the nonfuzzy or crisp derivative, as it is seen in the next definition:

Definition 2, [28]**.** The generalized derivative of a fuzzy-real valued function $\tilde{f}:(a,b)\longrightarrow\mathbb{R}_f$ at a point x_0 is given by:

$$\tilde{f}'_{gH}(x_0)=\lim_{h\to 0}\frac{\tilde{f}(x_0+h)\ominus_{gH}\tilde{f}(x_0)}{h}$$

If $\tilde{f}'_{gH}(x_0)\in\mathbb{R}_f$, then we say that $\tilde{f}$ is generalized Hukuhara differentiable (gH-differentiable for short) at x_0, which is also written using interval α-levels as:

$$\tilde{f}'_{gH}(x_0;\alpha)=\left[\underline{f}'_{gH}(x_0;\alpha),\overline{f}'_{gH}(x_0;\alpha)\right].$$

Furthermore, $\tilde{f}$ is said to be (i)-gH-differentiable at x_0 if:

$$\tilde{f}'_{i,gH}(x_0;\alpha)=\left[\underline{f}'(x_0;\alpha),\overline{f}'(x_0;\alpha)\right],\ 0\leq\alpha\leq 1,$$

and that f is (ii)-*gH*-differentiable at x_0 if

$$\tilde{f}'_{ii.gH}(x_0;\alpha)=\left[\overline{f}'(x_0;\alpha),\underline{f}'(x_0;\alpha)\right],\ 0\leq\alpha\leq 1.$$

As in crisp calculus, ordinary derivatives of fuzzy functions using Hukuhara differentiability is generalized for partial derivatives, as it is presented in the next definitions:

Definition 3, [19]. Let $(x_0,y_0)\in\mathbb{D}\subset\mathbb{R}^n$, then the first generalized Hukuhara partial derivative (abbreviated for simplicity as p-gH-derivative) of a fuzzy-valued function $\tilde{f}(x,y):\mathbb{D}\longrightarrow\mathbb{R}_f$ at (x_0,y_0) with respect to the variable x is denoted by $\partial_{x_{gH}}\tilde{f}(x_0,y_0)$ and is given by:

$$\partial_{x_{gH}}\tilde{f}(x_0,y_0)=\lim_{h\to 0}\frac{\tilde{f}(x_0+h,y_0)\ominus_{gH}\tilde{f}(x_0,y_0)}{h}$$

While the partial derivative with respect to y at (x_0,y_0) is denoted by $\partial_{y_{gH}}\tilde{f}(x_0,y_0)$, and given by:

$$\partial_{y_{gH}}\tilde{f}(x_0,y_0)=\lim_{k\to 0}\frac{\tilde{f}(x_0,y_0+k)\ominus_{gH}\tilde{f}(x_0,y_0)}{k}$$

provided that $\partial_{x_{gH}}\tilde{f}(x_0,y_0)$ and $\partial_{y_{gH}}\tilde{f}(x_0,y_0)$ belongs to $\mathbb{R}_f$.

Similarly, the closed interval α-level sets of the p-gH-derivatives are defined as in ordinary derivatives.

Formulation of Fuzzy Fluvio-Deltaic Sedimentary Problem

This problem is the model obtained when analysing the propagation of shorelines in a sediment sea basin based on sediment flow and the earth's crust break down, as well as, the change in sea level under the suppositions of a fixed sea level at $z = 0$ and constant line flow, in addition with landslide subsidence, [3,29].

Additionally, the absence of landslide subsidence is accompanied by a smoothly sloping basement $b < a$, with the basement slope is denoted by b and the off-shore sediment wedge slope is denoted by a, [7]. Figure 1 illustrate the basin without tectonic subsidence or sea level change is presented and hence a mathematical model is required for approximating several contemporary continental edges.

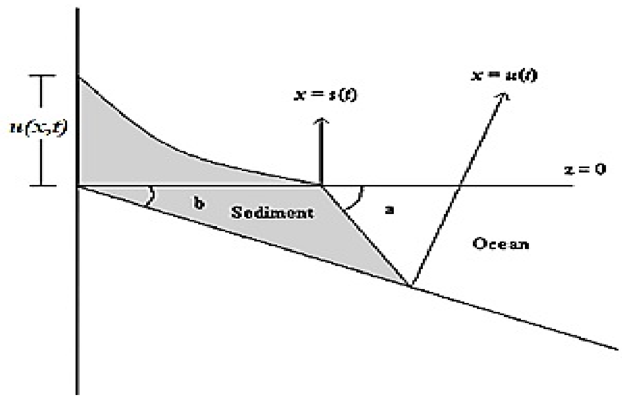

Figure 1. Schematic basin's cross sectional without changes in sea level or tectonic subsidence [7]

Because the nature of sands diffusion problem in water is similar to the nature of the heat diffusion problem, then the variable latent heat equation will be considered as the governing equation. Consequently, the sedimentation process' dynamics turn into a shifting boundary value problem, which is best represented by the following governing equation as the heat flow equation, [4,5]:

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2}, \quad s(t), 0 < x < \ t \geq 0 \tag{1}$$

with the initial and boundary conditions:

$$v\frac{\partial u}{\partial x}|_{x=0} = -q(t) \tag{2}$$
$$u(s,t) = 0 \tag{3}$$

where $q(t)$ is the sediment line flow that depends on time t, $s(t)$ is the movable boundary interface, and $u(x,t)$ is the elevation of sediment higher than the datum, and the diffusivity v is determined by the properties of the sediment grains, as well as, the average water line-discharge along the fluvial surface. Some extra conditions on the moving interface are implemented to fix this problem and make it solvable, including:

$$-v\frac{\partial u}{\partial x}|_{x=s(t)} = \gamma s\frac{ds}{dt} \tag{4}$$
$$s(0) = 0 \tag{5}$$

where γ is a constant depending on a and b.

Now, for the reason of introducing this work, it is notable that in most real-life considerations, imprecise or vague notions in the problem formulation may be highly appeared, which is due to the inaccurate formulation, or data reading or noise, etc. Thus, if we suppose the diffusivity constant v of the sands in the water to be approximately equal certain value, and hence it will be imprecise, i.e., v taken to by fuzzy number (denoted by $\tilde{v}$), then the model with this uncertainty will transform the nature of the all problem into fuzzy logic. Therefore, the appearance of the fuzzy number $\tilde{v}$, will affect on the solution and the moving interface of the deltaic sedimentary moving boundary value problem (1)-(5) also to be fuzzy functions. Thus, the formulation of the fuzzy version of the fluvio-deltaic sedimentary problem will be written as:

$$\frac{\partial \tilde{u}}{\partial t} = \tilde{v}\frac{\partial^2 \tilde{u}}{\partial x^2}, \ \ 0 < x < \tilde{s}(t), \ \ t \geq 0 \tag{6}$$

with fuzzy initial and boundary conditions:

$$\tilde{v}\frac{\partial \tilde{u}}{\partial x}|_{x=0} = -q(t) \tag{7}$$
$$\tilde{u}(\tilde{s},t) = 0 \tag{8}$$
$$-\tilde{v}\frac{\partial \tilde{u}}{\partial x}\Big|_{x=\tilde{s}(t)} = \gamma \tilde{s}\frac{d\tilde{s}}{dt} \tag{9}$$
$$\tilde{s}(0) = 0 \tag{10}$$

Analysis of the Fuzzy Fluvio-Deltaic Sedimentary Problem

Our focus in this section is on analysing the fuzzy fluvio-deltaic sedimentary problem with p-gH-differentiability using α-level sets, which will be carried based on assuming $\tilde{u}(x,t;\alpha) = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$, $\tilde{s}(t;\alpha) = [\underline{s}(t;\alpha), \overline{s}(t;\alpha)]$. Then substituting in the governing partial differential equation (6), we get:

$$\left.\begin{aligned} \min\left\{\underline{v}\frac{\partial^2}{\partial x^2}\underline{u}(x,t), \overline{v}\frac{\partial^2}{\partial x^2}\overline{u}(x,t)\right\} = \frac{\partial}{\partial t}\underline{u}(x,t) \\ \max\left\{\underline{v}\frac{\partial^2}{\partial x^2}\underline{u}(x,t), \overline{v}\frac{\partial^2}{\partial x^2}\overline{u}(x,t)\right\} = \frac{\partial}{\partial t}\overline{u}(x,t) \end{aligned}\right\} \tag{11}$$

and also substituting in the boundary condition Eq. (9), we get:

$$\left.\begin{aligned} \min\left\{\gamma\underline{s}(t)\frac{d}{dt}\underline{s}(t), \overline{\gamma}\,\overline{s}(t)\frac{d}{dt}\overline{s}(t)\right\} = -\overline{v}\frac{\partial}{\partial x}\underline{u}(\underline{s},t) \\ \max\left\{\gamma\underline{s}(t)\frac{d}{dt}\underline{s}(t), \overline{\gamma}\,\overline{s}(t)\frac{d}{dt}\overline{s}(t)\right\} = -\underline{v}\frac{\partial}{\partial x}\overline{u}(\overline{s},t) \end{aligned}\right\}. \tag{12}$$

Thus, two cases related to Eqs. (11) and two cases related to Eqs. (12) must be considered, depending on the p-gH-derivative of the lower and upper solutions. These two cases are:

Case (i): If $\frac{\partial^2}{\partial x^2}\underline{u}(x,t) \leq \frac{\partial^2}{\partial x^2}\overline{u}(x,t)$, then the possible partial differential equations resulted from system (11) are:

$$\left.\begin{aligned} \underline{v}\frac{\partial^2}{\partial x^2}\underline{u}(x,t) = \frac{\partial}{\partial t}\underline{u}(x,t) \\ \overline{v}\frac{\partial^2}{\partial x^2}\overline{u}(x,t) = \frac{\partial}{\partial t}\overline{u}(x,t) \end{aligned}\right\} \tag{13}$$

with lower and upper initial and boundary conditions related to Eqs. (7), (8) and (10).

Case (ii): If $\frac{\partial^2}{\partial x^2}\underline{u}(x,t) > \frac{\partial^2}{\partial x^2}\overline{u}(x,t)$, then the partial differential equations are reduced to:

$$\begin{aligned} \underline{v}\frac{\partial^2}{\partial x^2}\underline{u}(x,t) = \frac{\partial}{\partial t}\overline{u}(x,t) \\ \overline{v}\frac{\partial^2}{\partial x^2}\overline{u}(x,t) = \frac{\partial}{\partial t}\underline{u}(x,t) \end{aligned} \tag{14}$$

and also with lower and upper initial and boundary conditions related to Eqs. (7), (8) and (10).

Now, with the additional requirements on the moving interface of Eqs. (12), two cases for the moving boundary are also must be carried out, which are:

Case (i): If $\underline{s}'(t) \leq \bar{s}'(t)$ then the possible partial differential equations resulted from system (12) are:

$$\left.\begin{aligned} \gamma\underline{s}(t)\frac{d}{dt}\underline{s}(t) &= -\overline{v}\left(\frac{\partial}{\partial x}\underline{u}\big(\underline{s}(t),t\big)\right) \\ \gamma\overline{s}(t)\frac{d}{dt}\overline{s}(t) &= -\underline{v}\left(\frac{\partial}{\partial x}\overline{u}\big(\overline{s}(t),t\big)\right) \end{aligned}\right\} \quad (15)$$

with lower and upper initial and boundary conditions related to Eqs. (7), (8) and (10).

Case (ii): If $\underline{s}'(t) \geq \overline{s}'(t)$ then the partial differential equations are reduced to:

$$\left.\begin{aligned} \gamma\underline{s}(t)\frac{d}{dt}\underline{s}(t) &= -\underline{v}\left(\frac{\partial}{\partial x}\overline{u}\big(\overline{s}(t),t\big)\right) \\ \gamma\overline{s}(t)\frac{d}{dt}\overline{s}(t) &= -\overline{v}\left(\frac{\partial}{\partial t}\underline{u}\big(\underline{s}(t),t\big)\right) \end{aligned}\right\} \quad (16)$$

with lower and upper initial and boundary conditions related to Eqs. (7), (8) and (10).

Formulation of the Problem using Fuzzy Variational Iteration Method

According to the FVIM [15], for solving the operator form fuzzy partial differential equation $L\tilde{u}(x,t) + N\tilde{u}(x,t) = g(x,t)$, which starts by considering the correction functional in the x-direction as follows:

$$\tilde{u}_{n+1}(x,t) = \tilde{u}_n(x,t) + \int_0^x \lambda(w)\left\{L\big(\tilde{u}_n(w,t) + N\big(\tilde{u}_n^*(w,t)\big) - g(x,t)\right\}dw \quad (17)$$

where λ is the general Lagrange multiplier, which is determined optimally through employing the variational theory, the subscript n denotes the n^{th} approximate solution of $\tilde{u}$ and $\tilde{u}_n^*$ is considered as a restricted fuzzy variation, i.e., its first variational with respect to x equals zero, L and N are respectively the corresponding linear and nonlinear operators related to the governing fuzzy governing differential equation.

Remark 2. In order to find the subsequent approximate solutions using Eq. (17), we must first evaluate the value of the Lagrange multiplier λ, which will be best founded by applying integration by parts. Then the successive fuzzy approximations $\tilde{u}_n(x,t), n = 0,1, \dots;$ of the solution $[\tilde{u}(x,t)]^\alpha = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$ will be actually evaluated upon using the obtained Lagrange multiplier and starting the iterations with selective initial fuzzy function $[\tilde{u}_0(x,t)]^\alpha = [\underline{u}_0(x,t;\alpha), \overline{u}_0(x,t;\alpha)]$ as an initial guess. The zeroth approximation $\tilde{u}_0$ may be selected to be any function that just satisfies at least the fuzzy initial and boundary conditions with λ is predetermined, then several approximations $\tilde{u}_n(x,t), n = 1,2, \dots;$ follows immediately, and consequently, an approximation to the exact solution may be archived, since it may be proved in the FVIM that $\lim_{n\to\infty} \tilde{u}_n(x,t) = \tilde{u}(x,t)$.

Now, consider the linear interval problem related to Eq. (6):

$\tilde{v}\tilde{u}_{xx} = \tilde{u}_t$ or $\tilde{u}_{xx} = f(x,t,\tilde{u},\tilde{u}_t)$

which may be written equivalently in terms of lower and upper solutions after transforming throughout using the α-level sets by letting $\tilde{u} = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$, and hence:

$\underline{v}(\alpha)\,\underline{u}_{xx}(x,t;\alpha) = f(x,t,\underline{u}_t(x,t;\alpha),\overline{u}_t(x,t;\alpha))$, and.
$\overline{v}(\alpha)\overline{u}_{xx}(x,t;\alpha) = f(x,t,\underline{u}_t(x,t;\alpha),\overline{u}_t(x,t;\alpha))$

Therefore, upon using FVIM, the correction functional related to Eq. (6) for the upper and lower solutions of $\tilde{u}$ will be read for all $n = 0,1,\dots;$ as follows:

$$\underline{u}_{n+1}(x,t;\alpha) = \underline{u}_n(x,t;\alpha) + \int_0^x \lambda(w,t)\left[\underline{v}(\alpha)\frac{\partial^2}{\partial w^2}\underline{u}_n(w,t;\alpha) - f(w,t,\underline{u}_n(w,t;\alpha),\overline{u}_n(w,t;\alpha),\underline{u}_{n_t}(w,t;\alpha),\overline{u}_{n_t}(w,t;\alpha))\right]dw \quad (18)$$

$$\overline{u}_{n+1}(x,t;\alpha) = \overline{u}_n(x,t;\alpha) + \int_0^x \lambda(w,t)\left[\overline{v}(\alpha)\frac{\partial^2}{\partial w^2}\overline{u}_n(w,t;\alpha) - f(w,t,\underline{u}_n(w,t;\alpha),\overline{u}_n(w,t;\alpha),\underline{u}_{n_t}(w,t;\alpha),\overline{u}_{n_t}(w,t;\alpha))\right]dw \quad (19)$$

From the classical theory of the VIM for the second order partial differential equations, the value of the general Lagrange multiplier is found to be:

$$\lambda(w,t) = w - x \tag{20}$$

Hence, when beginning with initial approximate solutions $\underline{u}_0(x,t;\alpha) = \underline{u}_0$ and $\overline{u}_0(x,t;\alpha) = \overline{u}_0$, and by substituting the general Lagrange multiplier value given in Eq. (20) back into Eqs. (18) and (19), we get respectively the following iterated correction functionals using the VIM:

$$\underline{u}_{n+1}(x,t;\alpha) = \underline{u}_n(x,t;\alpha) + \int_0^x (w-x)\left[\underline{v}(\alpha)\frac{\partial^2}{\partial w^2}\underline{u}_n(w,t;\alpha) - f(w,t,\underline{u}_n(w,t;\alpha),\overline{u}_n(w,t;\alpha),\underline{u}_{n_t}(w,t;\alpha),\overline{u}_{n_t}(w,t;\alpha))\right]dw \tag{21}$$

$$\overline{u}_{n+1}(x,t;\alpha) = \overline{u}_n(x,t;\alpha) + \int_0^x (w-x)\left[\overline{v}(\alpha)\frac{\partial^2}{\partial w^2}\overline{u}_n(w,t;\alpha) - f(w,t,\underline{u}_n(w,t;\alpha),\overline{u}_n(w,t;\alpha),\underline{u}_{n_t}(w,t;\alpha),\overline{u}_{n_t}(w,t;\alpha))\right]dw \tag{22}$$

where $n = 0,1,\ldots$

Applying the iterations (21) and (22) for systems (13) and (14) will give approximate solutions convergent to exact solutions for increasing solution iteration [18].

In the next, we will consider the analysis of the fuzzy fluvio-deltaic sedimentary equations utilizing p-gH-differentiability with α-level sets. The FVIM will be used to answer the two situations connected to fuzzy logic.

Case (i): Based on Eq. (13), the general FVIM using Eq. (21) and Eq. (22):

$$\underline{u}_{n+1}(x,t;\alpha) = \underline{u}_n(x,t;\alpha) + \int_0^x (w-x)\left[\underline{v}\frac{\partial^2}{\partial w^2}\underline{u}_n(w,t;\alpha) - \frac{\partial}{\partial t}\underline{u}_n(x,t;\alpha)\right]dw \tag{23}$$

$$\overline{u}_{n+1}(x,t;\alpha) = \overline{u}_n(x,t;\alpha) + \int_0^x (w-x)\left[\overline{v}\frac{\partial^2}{\partial w^2}\overline{u}_n(w,t;\alpha) - \frac{\partial}{\partial t}\overline{u}_n(x,t;\alpha)\right]dw \tag{24}$$

With lower and upper initial and boundary conditions in addition to condition on the moving interface, where $n = 0,1,\ldots$, $\alpha \in [0,1]$

Case (ii): By using Eq. (14), with the same approach followed in case (i), the general FVIM arrangements will be:

$$\underline{u}_{n+1}(x,t;\alpha) = \underline{u}_n(x,t;\alpha) + \int_0^x (w-x)\left[\underline{v}\frac{\partial^2}{\partial w^2}\underline{u}_n(w,t;\alpha) - \frac{\partial}{\partial t}\overline{u}_n(x,t;\alpha)\right]dw \tag{25}$$

$$\overline{u}_{n+1}(x,t;\alpha) = \overline{u}_n(x,t;\alpha) + \int_0^x (w-x)\left[\overline{v}\frac{\partial^2}{\partial w^2}\overline{u}_n(w,t;\alpha) - \frac{\partial}{\partial t}\underline{u}_n(x,t;\alpha)\right]dw \tag{26}$$

Similar progress as presented above for cases (i) and (ii) when evaluating the approximate solution $\tilde{u}_n$, $n = 1,2,\ldots$; we can find the approximate solution of the moving boundary using the FVIM based on the following two cases:

Case (i): By using Eqs. (15), the general FVIM will be:

$$\underline{s}_{n+1}(t;\alpha) = \underline{s}_n(t;\alpha) - \int_0^t \left[\gamma\underline{s}_n(w;\alpha)\frac{d\underline{s}_n(w;\alpha)}{dw} + \overline{v}\frac{\partial\underline{u}(x,w;\alpha)}{\partial x}\Big|_{x=\underline{s}_n(w;\alpha)}\right]dw \tag{27}$$

$$\overline{s}_{n+1}(t;\alpha) = \overline{s}_n(t;\alpha) - \int_0^t \left[\gamma\overline{s}_n(w;\alpha)\frac{d\overline{s}_n(w;\alpha)}{dw} + \underline{v}\frac{\partial\overline{u}(x,w;\alpha)}{\partial x}\Big|_{x=\overline{s}_n(w;\alpha)}\right]dw \tag{28}$$

Case (ii): By using Eqs. (16), the general VIM will be:

$$\underline{s}_{n+1}(t;\alpha) = \underline{s}_n(t;\alpha) - \int_0^t \left[\gamma\underline{s}_n(w;\alpha)\frac{d\underline{s}_n(w;\alpha)}{dw} + \underline{v}\frac{\partial\overline{u}(x,w;\alpha)}{\partial x}\Big|_{x=\overline{s}_n(w;\alpha)}\right]dw \tag{29}$$

$$\overline{s}_{n+1}(t;\alpha) = \overline{s}_n(t;\alpha) - \int_0^t \left[\gamma\overline{s}_n(w;\alpha)\frac{d\overline{s}_n(w;\alpha)}{dw} + \overline{v}\frac{\partial\underline{u}(x,w;\alpha)}{\partial x}\Big|_{x=\underline{s}_n(w;\alpha)}\right]dw \tag{30}$$

Approximate Solution of the Problem

Recalling that in this problem, the fuzziness was achieved through introducing considering the parameter $\tilde{v}$ to be fuzzy number, which will be assumed in this work as a triangular fuzzy number. Then with the reference to the triangular fuzzy number given earlier in this paper, one can rewrite the fuzzy number $\tilde{v}$ in terms of its α-levels as $[\tilde{v}]^\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)]$, as follows:

$$[\tilde{v}]^\alpha = [a-(a-b)\alpha, c-(b-c)\alpha], \text{ where } \alpha \in [0,1] \tag{31}$$

and starting with case (i) related to Eqs. (13), as follows:

Case (i): By choosing the initial approximations of $\tilde{u}_0(x,t)$ and $\tilde{s}_0(t)$ [3,7] as:

$$\tilde{u}_0(x,t) = \tilde{c}(\tilde{s}_0(t)-x), \text{ where } \tilde{c} = \frac{\tilde{q}(t)}{\tilde{v}} \tag{32}$$

$$\tilde{s}_0(t) = \tilde{a}_0 t^{\frac{1}{2}}, \text{ where } \tilde{a}_0 = \left(\frac{2\tilde{q}(t)}{\tilde{\gamma}}\right)^{\frac{1}{2}} \tag{33}$$

and calculating the values of $\tilde{c}$ according to the previously specified properties for fuzzy numbers by letting $[\tilde{v}]^\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)]$; $[\tilde{q}(t)]^\alpha = [\underline{q}(t;\alpha), \overline{q}(t;\alpha)]$. Hence:

$$[\tilde{c}]^\alpha = \left[\frac{\tilde{q}(t)}{\tilde{v}}\right]^\alpha = \frac{[\underline{q},\overline{q}]}{[\underline{v},\overline{v}]} = \left[\min\left\{\frac{\underline{q}}{\underline{v}}, \frac{\underline{q}}{\overline{v}}, \frac{\overline{q}}{\underline{v}}, \frac{\overline{q}}{\overline{v}}\right\}, \max\left\{\frac{\underline{q}}{\underline{v}}, \frac{\underline{q}}{\overline{v}}, \frac{\overline{q}}{\underline{v}}, \frac{\overline{q}}{\overline{v}}\right\}\right]$$

and since $\alpha \in [0,1]$, then $[\tilde{c}]^\alpha = \left[\frac{\underline{q}}{\overline{v}}, \frac{\overline{q}}{\underline{v}}\right]$, where $\underline{c} = \frac{\underline{q}}{\overline{v}}$ and $\overline{c} = \frac{\overline{q}}{\underline{v}}$ (similarly, we can evaluate $\underline{a}_0, \overline{a}_0$).

Thus, Eqs. (32) and (33) will respectively become with the lower cases of solution as follows:

$$\begin{aligned}\underline{u}_0(x,t;\alpha) &= \underline{c}\big(\underline{s}_0(t;\alpha)-x\big),\\ &= \frac{\underline{q}\left(-x+\sqrt{t}\sqrt{\frac{2\underline{q}}{\overline{\gamma}}}\right)}{\overline{v}}\end{aligned}$$

where $\underline{s}_0(t;\alpha) = \underline{a}_0\sqrt{t}$ and $\underline{a}_0 = \sqrt{\frac{2\underline{q}}{\overline{\gamma}}}$.

The first lower approximate solution resulted after applying Eq. (23) could be found as:

$$\begin{aligned}\underline{u}_1(x,t;\alpha) &= \underline{u}_0(x,t;\alpha) + \int_0^x (w-x)\left\{\underline{v}\frac{\partial^2}{\partial w^2}\underline{u}_0(w,t;\alpha) - \frac{\partial}{\partial t}\underline{u}_0(x,t;\alpha)\right\}dw\\ &= \frac{\underline{q}\left(-x+\sqrt{t}\sqrt{\frac{2\underline{q}}{\overline{\gamma}}}\right)}{\overline{v}} + \frac{\underline{q}x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{2\sqrt{2}\sqrt{t}\overline{v}}\end{aligned}$$

while the second lower approximate solution is:

$$\begin{aligned}\underline{u}_2(x,t;\alpha) &= \underline{u}_1(x,t;\alpha) + \int_0^x (w-x)\left\{\underline{v}\frac{\partial^2}{\partial w^2}\underline{u}_1(w,t;\alpha) - \frac{\partial}{\partial t}\underline{u}_1(w,t;\alpha)\right\}dw\\ &= \frac{\underline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\right)}{\overline{v}} + \frac{\underline{q}x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{2\sqrt{2}\sqrt{t}\overline{v}} - \frac{\underline{q}x^2(24t(-1+\underline{v})+x^2)\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{48\sqrt{2}t^{3/2}\overline{v}}\end{aligned}$$

So on, the third and fourth lower approximate solutions are obtained to be:

$$\begin{aligned}\underline{u}_3(x,t;\alpha) &= \underline{u}_2(x,t;\alpha) + \int_0^x (w-x)\left\{\underline{v}\frac{\partial^2}{\partial w^2}\underline{u}_2(w,t;\alpha) - \frac{\partial}{\partial t}\underline{u}_2(w,t;\alpha)\right\}dw\\ &= \frac{\underline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\right)}{\overline{v}} + \frac{\underline{q}x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{\sqrt{2}\sqrt{t}\overline{v}} + \frac{\underline{q}\underline{v}x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{\sqrt{2}\sqrt{t}\overline{v}} - \frac{\sqrt{2}\underline{q}\underline{v}x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{\sqrt{t}\overline{v}} + \frac{\underline{q}\underline{v}^2x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{2\sqrt{2}\sqrt{t}\overline{v}} - \frac{\underline{q}\underline{v}x^4\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{24\sqrt{2}t^{3/2}\overline{v}} + \frac{\underline{q}x^6\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{960\sqrt{2}t^{5/2}\overline{v}} -\\ &\quad \frac{\underline{q}x^2(24t(-1+\underline{v})+x^2)\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{48\sqrt{2}t^{3/2}\overline{v}}\end{aligned}$$

$$\underline{u}_4(x,t;\alpha) = \underline{u}_4(x,t;\alpha) + \int_0^x (w-x)\left\{\underline{v}\frac{\partial^2}{\partial w^2}\underline{u}_3(w,t;\alpha) - \frac{\partial}{\partial t}\underline{u}_3(w,t;\alpha)\right\}dw$$
$$= \frac{\underline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\right)}{\overline{v}} + \frac{3\underline{q}x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{2\sqrt{2}\sqrt{t\overline{v}}} + \frac{3\underline{q}\underline{v}x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{2\sqrt{2}\sqrt{t\overline{v}}} - \frac{2\sqrt{2}\underline{q}\underline{v}x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{\sqrt{t\overline{v}}} + \frac{\sqrt{2}\underline{q}\underline{v}^2x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{\sqrt{t\overline{v}}} - \frac{\underline{q}\underline{v}^3x^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{2\sqrt{2}\sqrt{t\overline{v}}} - \frac{5\underline{q}x^4\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{48\sqrt{2}t^{3/2}\overline{v}} + \frac{\underline{q}\underline{v}x^4\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{6\sqrt{2}t^{3/2}\overline{v}} - \frac{\underline{q}\underline{v}^2x^4\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{16\sqrt{2}t^{3/2}\overline{v}} + \frac{\underline{q}x^6\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{240\sqrt{2}t^{5/2}\overline{v}} - \frac{\underline{q}\underline{v}x^6\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{320\sqrt{2}t^{5/2}\overline{v}} - \frac{\underline{q}x^8\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{21504\sqrt{2}t^{7/2}\overline{v}} - \frac{\underline{q}x^2(24t(-1+\underline{v})+x^2)\sqrt{\frac{\underline{q}}{\overline{\gamma}}}}{48\sqrt{2}t^{3/2}\overline{v}}$$

Following the same approach as in the above when evaluating the lower-case approximate solutions, we can find the solution for the upper case up to the fourth iteration by applying Eq. (24) starting with the initial guess approximate solution:

$$\overline{u}_0(x,t;\alpha) = \overline{c}(\overline{s}_0(t;\alpha)-x) = \frac{\overline{q}\left(-x+\sqrt{t}\sqrt{\frac{2\,\overline{q}}{\underline{\gamma}}}\right)}{\underline{v}}$$

where $\overline{s}_0(t;\alpha) = \overline{a}_0\sqrt{t}$ and $\overline{a}_0 = \sqrt{\frac{2\,\overline{q}}{\underline{\gamma}}}$. Thus, further approximations will be:

$$\overline{u}_1(x,t;\alpha) = \overline{u}_0(x,t,\alpha) + \int_0^x (w-x)\left\{\overline{v}\frac{\partial^2}{\partial w^2}\overline{u}_0(w,t;\alpha) - \frac{\partial}{\partial t}\overline{u}_0(x,t;\alpha)\right\}dw = \frac{\overline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{\underline{v}} + \frac{\overline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}}$$

$$\overline{u}_2(x,t;\alpha) = \overline{u}_1(x,t,\alpha) + \int_0^x (w-x)\left[\overline{v}\frac{\partial^2}{\partial w^2}\overline{u}_1(w,t;\alpha) - \frac{\partial}{\partial t}\overline{u}_1(w,t;\alpha)\right]dw = \frac{\overline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{\underline{v}} + \frac{\overline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}} - \frac{\overline{q}x^2(24t(-1+\overline{v})+x^2)\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\underline{v}}$$

$$\overline{u}_3(x,t;\alpha) = \frac{\overline{q}(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}})}{\underline{v}} + \frac{\overline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{2}\sqrt{t}\underline{v}} + \frac{\overline{q}\,\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{2}\sqrt{t}\underline{v}} - \frac{\sqrt{2}\overline{q}\,\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}\underline{v}} + \frac{\overline{q}\,\overline{v}^2x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}} - \frac{\overline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{24\sqrt{2}t^{3/2}\underline{v}} + \frac{\overline{q}\,\overline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{24\sqrt{2}t^{3/2}\underline{v}} + \frac{\overline{q}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{960\sqrt{2}t^{5/2}\underline{v}} - \frac{\overline{q}x^2(24t(-1+\overline{v})+x^2)\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\underline{v}}$$

$$\overline{u}_4(x,t;\alpha) = \frac{\overline{q}(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}})}{\underline{v}} + \frac{3\overline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}} + \frac{3\overline{q}\,\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}} - \frac{2\sqrt{2}\overline{q}\,\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}\underline{v}} + \frac{\sqrt{2}\overline{q}\,\overline{v}^2x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}\underline{v}} - \frac{\overline{q}\,\overline{v}^3x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}} - \frac{5\overline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\underline{v}} + \frac{\overline{q}\,\overline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{6\sqrt{2}t^{3/2}\underline{v}} - \frac{\overline{q}\,\overline{v}^2x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{16\sqrt{2}t^{3/2}\underline{v}} + \frac{\overline{q}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{240\sqrt{2}t^{5/2}\underline{v}} - \frac{\overline{q}\,\overline{v}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{320\sqrt{2}t^{5/2}\underline{v}} - \frac{\overline{q}x^8\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{21504\sqrt{2}t^{7/2}\underline{v}} - \frac{\overline{q}x^2(24t(-1+\overline{v})+x^2)\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\underline{v}}$$

Thus, we can obtain $\tilde{u}_4(x,t) = [\underline{u}_4(x,t;\alpha), \overline{u}_4(x,t;\alpha)]$ as an approximate of the height of the sediment above the datum.

The second crucial aspect of the problem is to find the moving boundary $\tilde{s}(t)$ as a fuzzy function, which must satisfy condition (9) and establish the relationship between the moving boundary $\tilde{s}(t)$ and the sedimentation height $\tilde{u}(x,t)$.

We can find the iterative functions of the lower and upper cases of the moving boundary by also applying the FVIM for Eqs. (27) and (28), and hence for case (i), we get:

$$\underline{s}_1(t;\alpha) = \underline{s}_0(t;\alpha) - \int_0^t \left\{\gamma\underline{s}_0(w;\alpha)\frac{d\underline{s}_0(w;\alpha)}{dw} + \overline{v}\frac{\partial\underline{u}_4(x,w)}{\partial x}\Big|_{x=\underline{s}_0(w;\alpha)}\right\}dw$$
$$= \sqrt{2}\sqrt{t}\sqrt{\frac{\underline{q}}{\overline{\gamma}}} + \frac{\underline{q}^5t}{336\overline{\gamma}^4} - \frac{\underline{q}^4t}{10\overline{\gamma}^3} + \frac{3\underline{q}^4t\underline{v}}{40\overline{\gamma}^3} + \frac{\underline{q}^3t}{\overline{\gamma}^2} - \frac{4\underline{q}^3t\underline{v}}{3\overline{\gamma}^2} + \frac{\underline{q}^3t\underline{v}^2}{2\overline{\gamma}^2} - \frac{4\underline{q}^2t}{\overline{\gamma}} + \frac{6\underline{q}^2t\underline{v}}{\overline{\gamma}} - \frac{4\underline{q}^2t\underline{v}^2}{\overline{\gamma}} + \frac{\underline{q}^2t\underline{v}^3}{\overline{\gamma}}$$

and

$$\overline{s}_1(t;\alpha) = \overline{s}_0(t;\alpha) - \int_0^t \left[\gamma \overline{s}_0(w;\alpha)\frac{d\overline{s}_0(w;\alpha)}{dw} + \underline{v}\frac{\partial \overline{u}_4(x,w)}{\partial x}\Big|_{x=\overline{s}_0(w;\alpha)}\right] dw$$
$$= \sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + \frac{\overline{q}^5 t}{336\underline{\gamma}^4} - \frac{\overline{q}^4 t}{10\underline{\gamma}^3} + \frac{3\overline{q}^4 t\overline{v}}{40\underline{\gamma}^3} + \frac{\overline{q}^3 t}{\underline{\gamma}^2} - \frac{4\overline{q}^3 t\overline{v}}{3\underline{\gamma}^2} + \frac{\overline{q}^3 t\overline{v}^2}{2\underline{\gamma}^2} - \frac{4\overline{q}^2 t}{\underline{\gamma}} + \frac{6\overline{q}^2 t\overline{v}}{\underline{\gamma}} - \frac{4\overline{q}^2 t\overline{v}^2}{\underline{\gamma}} + \frac{\overline{q}^2 t\overline{v}^3}{\underline{\gamma}}$$

By applying this process for the second iteration, that is evaluand using a computer program written in Mathematica 11 which is so difficult to be presented here, as an approximate solution of the movement of shoreline position.

Case (ii): Also, by choosing the initial fuzzy approximations of $\tilde{u}_0(x,t)$ and $\tilde{s}_0(x,t)$ as before when carried out in case (i), for the lower and upper cases, we can evaluate respectively the approximate solutions of case (ii) for the lower and upper solutions of Eqs. (25) and (26) up to the forth iteration as:

$$\underline{u}_1(x,t;\alpha) = \underline{u}_0(x,t;\alpha) + \int_0^x (w-x)\left\{\underline{v}\frac{\partial^2}{\partial w^2}\underline{u}_0(w,t;\alpha) - \frac{\partial}{\partial t}\overline{u}_0(x,t;\alpha)\right\} dw$$
$$= \frac{\underline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{\overline{v}} + \frac{\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}}$$

$$\overline{u}_1(x,t;\alpha) = \overline{u}_0(x,t,\alpha) + \int_0^x (w-x)\left\{\overline{v}\frac{\partial^2}{\partial w^2}\overline{u}_0(w,t;\alpha) - \frac{\partial}{\partial t}\underline{u}_0(x,t;\alpha)\right\} dw$$
$$= \frac{\underline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{\underline{v}} + \frac{\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\overline{v}}$$

$$\underline{u}_2(x,t;\alpha) = \underline{u}_1(x,t;\alpha) + \int_0^x (w-x)\left\{\underline{v}\frac{\partial^2}{\partial w^2}\underline{u}_1(w,t;\alpha) - \frac{\partial}{\partial t}\overline{u}_1(x,t;\alpha)\right\} dw$$
$$= \frac{\underline{q}(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}})}{\overline{v}} + \frac{\underline{q}(-\frac{1}{12}\underline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}} - 2t(-1+\underline{v})\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}})}{4\sqrt{2}t^{3/2}\underline{v}\overline{v}} + \frac{\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}}$$

$$\overline{u}_2(x,t;\alpha) = \overline{u}_1(x,t,\alpha) + \int_0^x (w-x)\left\{\overline{v}\frac{\partial^2}{\partial w^2}\overline{u}_1(w,t;\alpha) - \frac{\partial}{\partial t}\underline{u}_1(x,t;\alpha)\right\} dw$$
$$= \frac{\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\overline{v}} + \frac{\underline{q}(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}})}{\underline{v}} - \frac{\underline{q}x^2(24t\underline{v}(-1+\overline{v})\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + \overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}})}{48\sqrt{2}t^{3/2}\underline{v}\overline{v}}$$

$$\underline{u}_3(x,t;\alpha) = \frac{\underline{q}(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}})}{\overline{v}} + \frac{\underline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}} - \frac{\underline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{24\sqrt{2}t^{3/2}\overline{v}} + \frac{\underline{q}\underline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\overline{v}} + \frac{\underline{q}\left(-\frac{1}{12}\underline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}} - 2t(-1+\underline{v})\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{4\sqrt{2}t^{3/2}\underline{v}\overline{v}} + \frac{\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{2}\sqrt{t}} - \frac{\sqrt{2}\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}} +$$
$$\frac{\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{2}\sqrt{t}\underline{v}} + \frac{\underline{q}\underline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}} + \frac{\underline{q}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{960\sqrt{2}t^{5/2}\underline{v}}$$

$$\overline{u}_3(x,t;\alpha) = \frac{\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{2}\sqrt{t}} - \frac{\sqrt{2}\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}} + \frac{\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{2}\sqrt{t}\overline{v}} + \frac{\underline{q}\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}} + \frac{\underline{q}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{960\sqrt{2}t^{5/2}\overline{v}} + \frac{\underline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{\underline{v}} - \frac{\underline{q}x^2\left(24t\underline{v}(-1+\overline{v})\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + \overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{48\sqrt{2}t^{3/2}\underline{v}\overline{v}} +$$
$$\frac{\underline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}} - \frac{\underline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{24\sqrt{2}t^{3/2}\underline{v}} + \frac{\underline{q}\overline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\underline{v}}$$

$$\underline{u}_4(x,t;\alpha) = \frac{\underline{q}(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}})}{\overline{v}} + \frac{\underline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{12\sqrt{2}t^{3/2}} - \frac{\underline{q}\underline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}} - \frac{5\underline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\overline{v}} + \frac{\underline{q}\underline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{12\sqrt{2}t^{3/2}\overline{v}} - \frac{\underline{q}\underline{v}^2x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\overline{v}} - \frac{\underline{q}\overline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}} - \frac{\underline{q}x^8\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{21504\sqrt{2}t^{7/2}\overline{v}} +$$
$$\frac{\underline{q}\left(-\frac{1}{12}\underline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}} - 2t(-1+\underline{v})\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{4\sqrt{2}t^{3/2}\underline{v}\overline{v}} + \frac{3\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}} - \frac{2\sqrt{2}\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}} + \frac{3\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\underline{v}} + \frac{\sqrt{2}\underline{q}\underline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}} - \frac{\underline{q}\underline{v}^2x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}} - \frac{\underline{q}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{480\sqrt{2}t^{5/2}} +$$
$$\frac{\underline{q}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{240\sqrt{2}t^{5/2}\underline{v}} - \frac{\underline{q}\overline{v}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{960\sqrt{2}t^{5/2}\underline{v}}$$

$$\overline{u}_4(x,t;\alpha) = \frac{3\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}} - \frac{2\sqrt{2}\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}} + \frac{3\underline{q}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}\overline{v}} + \frac{\sqrt{2}\underline{q}\overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\sqrt{t}} - \frac{\underline{q}\overline{v}^2x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{2\sqrt{2}\sqrt{t}} - \frac{\underline{q}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{480\sqrt{2}t^{5/2}} + \frac{\underline{q}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{240\sqrt{2}t^{5/2}\overline{v}} - \frac{\underline{q}\underline{v}x^6\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{960\sqrt{2}t^{5/2}\overline{v}} +$$
$$\frac{\underline{q}\left(-x+\sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{\underline{v}} - \frac{\underline{q}x^2\left(24t\underline{v}(-1+\overline{v})\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + \overline{v}x^2\sqrt{\frac{\overline{q}}{\underline{\gamma}}}\right)}{48\sqrt{2}t^{3/2}\underline{v}\overline{v}} + \frac{\underline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{12\sqrt{2}t^{3/2}} - \frac{5\underline{q}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\underline{v}} - \frac{\underline{q}\underline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}} - \frac{\underline{q}\overline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}} + \frac{\underline{q}\overline{v}x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{12\sqrt{2}t^{3/2}\underline{v}} -$$
$$\frac{\underline{q}\overline{v}^2x^4\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{48\sqrt{2}t^{3/2}\underline{v}} - \frac{\underline{q}x^8\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{21504\sqrt{2}t^{7/2}\underline{v}}$$

The case (ii) first iterative solution of the moving boundary based on using the FVIM depending on Eqs. (29) and (30), as follows:

$$\underline{s}_1(t;\alpha) = \underline{s}_0(t;\alpha) - \int_0^t \left\{\gamma \underline{s}_0(w;\alpha)\frac{d\underline{s}_0(w;\alpha)}{dw} + \underline{v}\frac{\partial \overline{u}_4(x,w;\alpha)}{\partial x}\bigg|_{x=\overline{s}_0(w;\alpha)}\right\} dw$$

$$= \underline{q}t + \sqrt{2}\sqrt{t}\sqrt{\frac{\underline{q}}{\overline{\gamma}}} - \frac{\underline{q}t\gamma}{\underline{\gamma}} + 6\underline{q}t\underline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}} - \frac{4\underline{q}t\underline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\overline{v}} - 4\underline{q}t\underline{v}\overline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + \underline{q}t\underline{v}\overline{v}^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + \frac{\underline{q}\overline{q}^4 t}{336\overline{\gamma}^4} + \frac{\underline{q}\overline{q}^2 t}{\overline{\gamma}^2} -$$

$$\frac{2\underline{q}\overline{q}^2 t\underline{v}}{3\overline{\gamma}^2} + \frac{\underline{q}\overline{q}^2 t\underline{v}^2}{6\overline{\gamma}^2} - \frac{2\underline{q}\overline{q}^2 t\overline{v}}{3\overline{\gamma}^2} + \frac{\underline{q}\overline{q}^2 t\underline{v}\overline{v}}{6\overline{\gamma}^2} + \frac{\underline{q}\overline{q}^2 t\overline{v}^2}{6\overline{\gamma}^2} + \frac{\underline{q}\overline{q}^2 t\underline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{20\overline{\gamma}^2} - \frac{\underline{q}\overline{q}^2 t\underline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{10\overline{v}\,\overline{\gamma}^2} + \frac{\underline{q}\overline{q}^2 t\underline{v}^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{40\overline{v}\,\overline{\gamma}^2}$$

$$\overline{s}_1(t;\alpha) = \overline{s}_0(t;\alpha) - \int_0^t \left\{\gamma \overline{s}_0(w;\alpha)\frac{d\overline{s}_0(w;\alpha)}{dw} + \overline{v}\frac{\partial \underline{u}_4(x,w;\alpha)}{\partial x}\bigg|_{x=\underline{s}_0(w;\alpha)}\right\} dw$$

$$= \underline{q}t + \frac{\underline{q}^5 t}{336\underline{\gamma}^4} + \frac{\underline{q}^3 t}{\underline{\gamma}^2} - \frac{2\underline{q}^3 t\underline{v}}{3\underline{\gamma}^2} + \frac{\underline{q}^3 t\underline{v}^2}{6\underline{\gamma}^2} - \frac{2\underline{q}^3 t\overline{v}}{3\underline{\gamma}^2} + \frac{\underline{q}^3 t\underline{v}\overline{v}}{6\underline{\gamma}^2} + \frac{\underline{q}^3 t\overline{v}^2}{6\underline{\gamma}^2} + \sqrt{2}\sqrt{t}\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + 6\underline{q}t\overline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}} - \frac{4\underline{q}t\overline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{\underline{v}} -$$

$$4\underline{q}t\underline{v}\overline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + \underline{q}t\underline{v}^2\overline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}} + \frac{\underline{q}^3 t\overline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{20\underline{\gamma}^2} - \frac{\underline{q}^3 t\overline{v}\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{10\underline{v}\underline{\gamma}^2} + \frac{\underline{q}^3 t\overline{v}^2\sqrt{\frac{\underline{q}}{\overline{\gamma}}}\sqrt{\frac{\overline{q}}{\underline{\gamma}}}}{40\underline{v}\underline{\gamma}^2} - \frac{\overline{q}t\overline{\gamma}}{\overline{\gamma}}$$

and the second iteration, that is evaluand using a computer program written in Mathematica 11, which is so difficult to be presented here, as an approximate solution of the movement of shoreline position.

Numerical Results

This section presents an approximation of the sediment height $\tilde{u}(x,t)$ and the shoreline positions $\tilde{s}(t)$ as fuzzy functions, using the concept of α-level sets. The numerical calculations were performed using Mathematica 11 computer software. The results are approximations and are illustrated through figures and Tables. To validate the accuracy of the numerical results, a comparison is made with the exact solution obtained when $\alpha = 1$, as provided by Voller ***et al*** [7].

We consider the triangular fuzzy number $\tilde{v} = \tilde{2} = (1,2,3)$, as defined using interval α-level given by Eq. (31). Hence, in terms of α-levels, we can express $\tilde{v}$ as $\tilde{v} = [1+\alpha, 3-\alpha]$, $\alpha \in [0,1]$. This representation allows us to describe the range of values for $\tilde{v}$ based on different levels of α.

Figures 2 and 3 of case (i) indicate the lower and upper approximate solutions $\tilde{u}(x,t)$ for different values of α-level, namely 0, 0.4, 0.8 and 1 at the fuzzy triangular value of diffusivity coefficient ($\tilde{v} = \tilde{2}$), sediment flow $(q = 0.5)$ and time $t = 3$ for $\gamma = 10$ and $\gamma = 15$, respectively. where the dashed line stands for $\underline{u}$ and continuous line stands for $\overline{u}$. While, Figure 4-5 present the result of the lower and upper movable boundaries of shoreline position for the same above α-level, at the fuzzy triangular number of diffusivity coefficient $\tilde{v} = \tilde{2}$, and sediment flow $(q = 0.5)$ for $\gamma = 10$ and $\gamma = 15$, respectively, where the dashed line stands for $\underline{s}$ and continuous line stands for $\overline{s}$. In addition, Table 1 presents the absolute errors between the exact and approximate solutions with $\tilde{v} = \tilde{2}$, $q = 0.5$, $\gamma = 10$, and $\alpha = 1$. From Table 1 and Figures 2-5, it is appeared that the approximate solution obtained through applying the FVIM are closely approximates the exact or nonfuzzy solution provided by Voller ***et al***. [7].

Same discussion for case (ii) as in the last above paragraph, we get the result we get the Figure 6-9 and the Table 2.

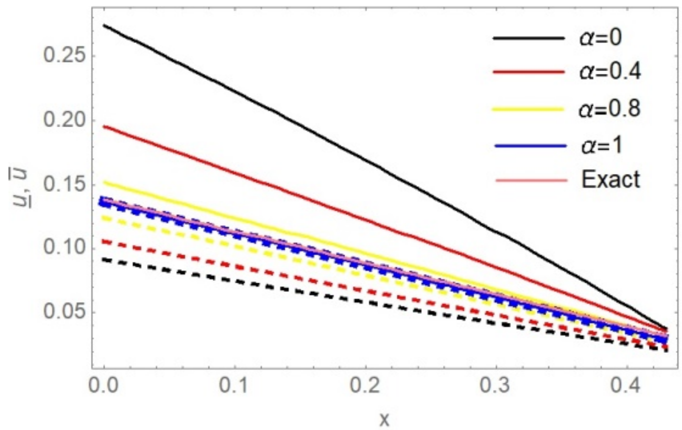

Figure 2. Plot of the 4th iterated solution with respect to x for $\gamma = 10$ in terms of $\underline{u}$ and $\overline{u}$ with different α-levels of case (i)

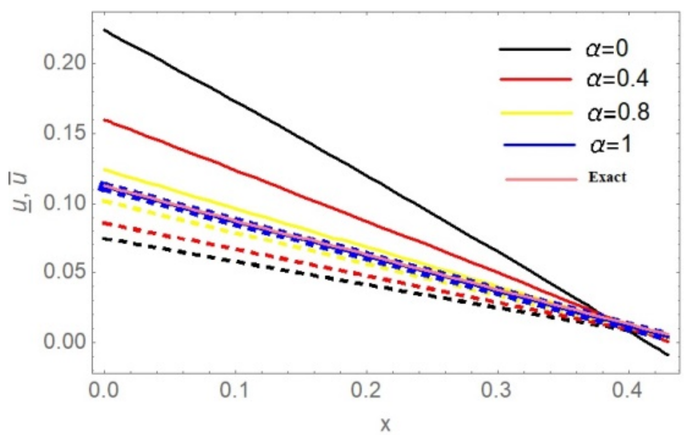

Figure 3. Plot of the 4th iterated solution with respect to x for $\gamma = 15$ in terms of $\underline{u}$ and $\overline{u}$ with different α-levels of case (i)

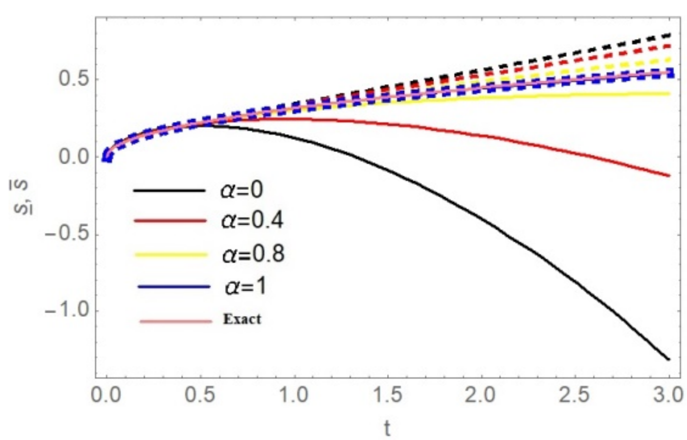

Figure 4. Plot of the 2nd iterated moving boundary with respect to x for $\gamma = 10$ in terms of $\underline{s}$ and $\overline{s}$ with different α-levels of case (i)

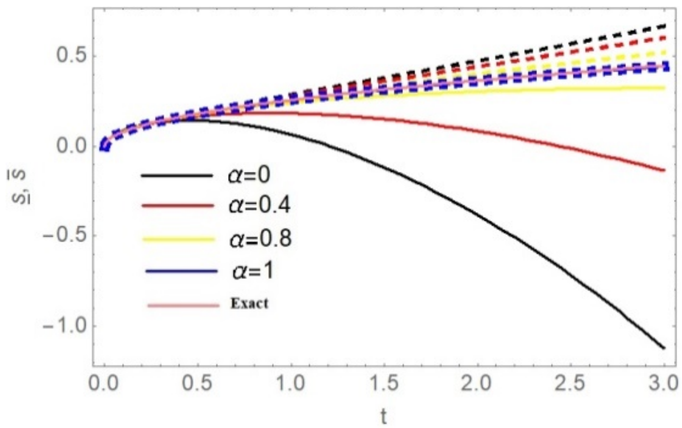

Figure 5. Plot of the 2nd iterated moving boundary with respect to x for $\gamma = 15$ in terms of $\underline{s}$ and $\overline{s}$ with different α-levels of case (i)

Table 1. The exact solution, approximate solution, absolute error of case (i), at $\tilde{v} = \tilde{2}$ and $\gamma = 10$

x	Approximate solution	Exact solution	Absolute error
0	0.13693064	0.13865301	0.00172238
0.1	0.11193061	0.11371078	0.00178018
0.2	0.08693013	0.08888404	0.00195391
0.3	0.06192807	0.06417269	0.00224457
0.4	0.03692250	0.03957634	0.00265384
0.5	0.01191075	0.01509481	0.00318407
0.6	0.01311069	−0.00927237	0.00383831
0.7	0.03814609	−0.03352575	0.00462034
0.8	−0.06320058	−0.05766592	0.00553467
0.9	−0.08828015	−0.08169359	0.00658656
1.0	−0.11339165	−0.10560956	0.00778209

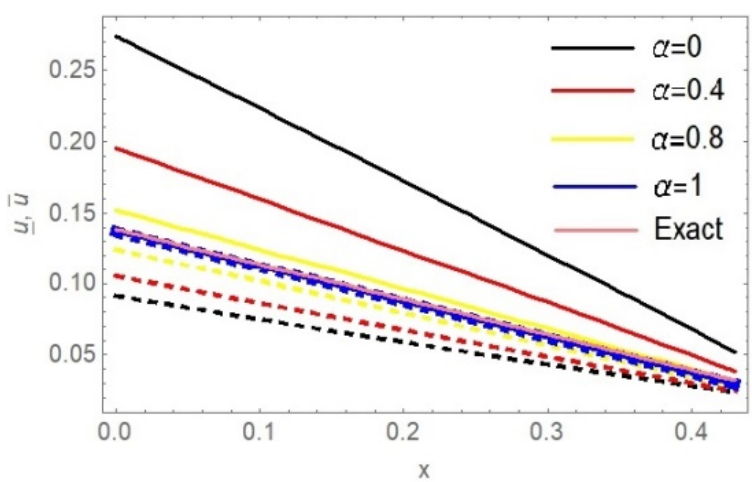

Figure 6. Plot of the 4th iterated solution with respect to x for $\gamma = 10$ in terms of $\underline{u}$ and $\overline{u}$ with different α-levels of case (ii)

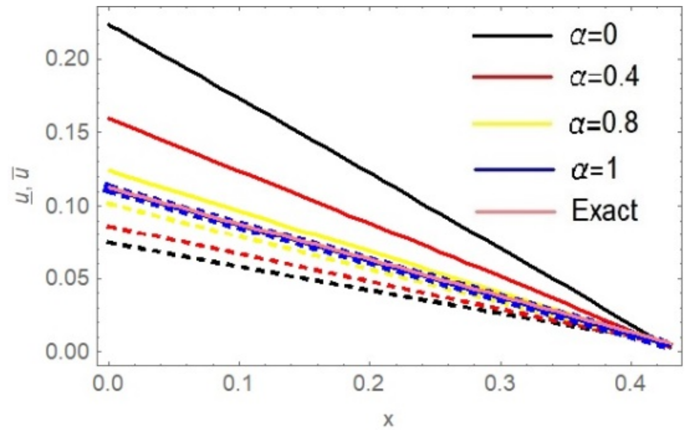

Figure 7. Plot of the 4th iterated solution with respect to x for $\gamma = 15$ in terms of $\underline{u}$ and $\overline{u}$ with different α-levels of case (ii)

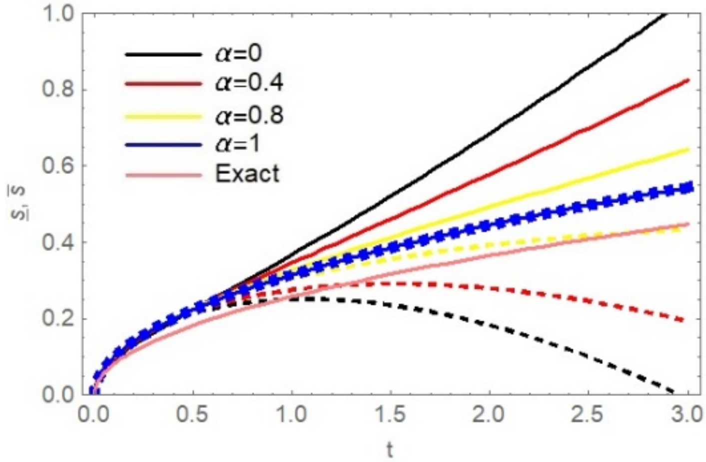

Figure 8. Plot of the 2nd iterated moving boundary with respect to x for $\gamma = 10$ in terms of $\underline{s}$ and $\overline{s}$ with different α-levels of case (ii)

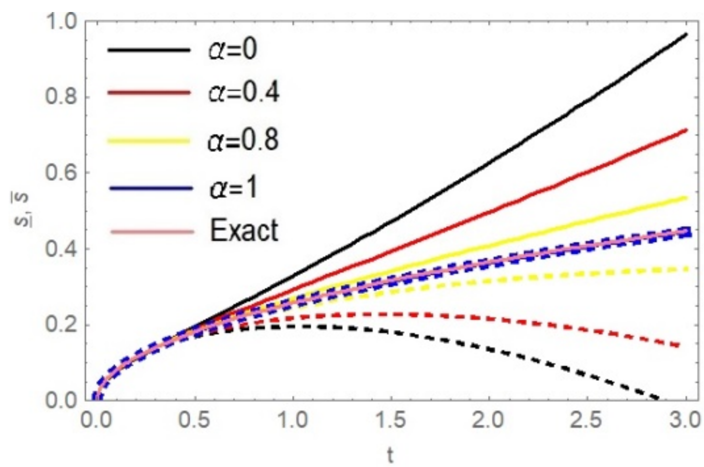

Figure 9. Plot of the 2nd iterated moving boundary with respect to x for $\gamma = 15$ in terms of $\underline{s}$ and $\overline{s}$ with different α-levels of case (ii)

Table 2. The exact solution, approximate solution, absolute error of case (i), at $\tilde{v} = \tilde{2}$ and $\gamma = 15$

x	Approximate solution	Exact solution	Absolute error
0	0.13693064	0.13865301	0.00172237
0.1	0.11193061	0.11371078	0.00178017
0.2	0.08693013	0.08888403	0.00195390
0.3	0.06192806	0.06417263	0.00224457
0.4	0.03692250	0.03957634	0.00265383
0.5	0.01191074	0.01509481	0.00318406
0.6	0.01311068	−0.00927237	0.00383831
0.7	0.03814608	−0.03352574	0.00462034
0.8	−0.06320058	−0.05766591	0.00553466
0.9	−0.08828014	−0.08169359	0.00658655
1.0	−0.11339165	−0.10560956	0.00778208

Conclusions

We investigated a mathematical model that utilizes fuzzy parameters, specifically a type of fuzzy number that transforms the fluvio deltaic sedimentary problem into a fuzzy problem. It was demonstrated that FVIM is a reliable and effective technique for addressing the fluvio-deltaic sedimentary problem, allowing for the derivation of an analytical approximate solution. The accuracy of the results was verified by comparing the approximate results for the lower and upper solutions for each α-level, as well as their convergence when $\alpha = 1$, which represents the crisp solution. We have established that sedimentation slows down as the value $\tilde{\gamma}$ increases. As the value of $\tilde{q}$ increases, the sedimentation procedure accelerates. The results are shown in Figures 2-9 and Tables 1-2, which shows the effect of fuzzy phenomena on the behaviour of the fuzzy solution in the considered numerical results for the problem. Further, the proposed procedure outlined in this paper provides an analytical approximate solution that could prove useful for future real-world applications. In such methods, analytical function solutions are considered more dependable than numerical results got through alternative methods. The Adomian decomposition method could be explored for handling the problem.

In future studies, we suggest that we use new parameters in order to transform the problem into a fuzzy formulation, use other types of fuzzy numbers, and use the residual error to analyse the fuzzy approximate solution.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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