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On the Planar Property of an Ideal-Based Weakly Zero-Divisor Graph

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Abstract Let R be a commutative ring with a nonzero identity and $Z(R)$ be the set of zero-divisors of R. The weakly zero-divisor graph of R, denoted by $W\Gamma(R)$, is the graph with the vertex set $Z(R)^* = Z(R)\backslash\{0\}$, where two distinct vertices a and b form an edge if $ar = bs = rs = 0$ for $r, s \in R\backslash\{0\}$. For an ideal I of R, the ideal-based zero-divisor graph of R, denoted by $\Gamma_I(R)$, has vertices $\{a \in R\backslash I: ab \in I$ for some $b \in R\backslash I\}$ and edges $\{(a,b): ab \in I, a, b \in R\backslash I, a \neq b\}$. In this article, an ideal-based weakly zero-divisor graph of R, denoted by $W\Gamma_I(R)$, is introduced which contains $\Gamma_I(R)$ as a subgraph and is identical to the graph $W\Gamma(R)$ when $I = \{0\}$. The relationship between the graphs $W\Gamma_I(R)$ and $W\Gamma(R/I)$ is investigated and the planar property of $W\Gamma_I(R)$ is studied. The results show that $W\Gamma(R/I)$ is isomorphic to a subgraph of $W\Gamma_I(R)$. For $W\Gamma_I(R)$ to be planar, some restraints are provided on the size of the ideal I and girth of $W\Gamma_I(R)$. In conclusion, the results suggest that $W\Gamma_I(R)$ and $W\Gamma(R/I)$ are strongly related and establish necessary and sufficient conditions for $W\Gamma_I(R)$ to be planar. In addition, rings R with planar $W\Gamma_I(R)$ are classified.

Keywords: Zero-divisor graph, commutative ring, girth, planar graph, graph theory.

Introduction

Let R be a commutative ring with identity and $Z(R)$ be the set of zero-divisors of R. The zero divisor graph of R is the graph $\Gamma(R)$ with the vertex set $Z(R)^* = Z(R) - \{0\}$ and edges set $\{(a,b): ab = 0, a, b \in Z(R)^*\}$. In 1988, Beck [1] presented the graph in the work pertaining to coloring of rings. In 1993, the graph was further studied by Anderson and Naseer [2]. These earlier works included zero in the set of vertices of $\Gamma(R)$. In 1999, Anderson and Livingston [3] gave the definition of $\Gamma(R)$ which did not include zero in the vertex set. This work produced fundamental results on $\Gamma(R)$. In [3], it was shown that the diameter of $\Gamma(R)$ is at most three and if $\Gamma(R)$ is not acyclic and R is Artinian, then the girth of $\Gamma(R)$ is at most four. The authors also determined when the graph $\Gamma(R)$ is complete or star and studied the group $\mathrm{Aut}(\Gamma(\mathrm{R}))$. Numerous authors have investigated the graph in great detail, and new results and generalizations have been provided. Generalizations of $\Gamma(\mathrm{R})$ are given in [4] and [5]. Results on structural properties of $\Gamma(R)$ for finite rings R are discussed in [6], [7], [8] and [9]. Planarity of $\Gamma(R)$ is studied in [10], [11] and [12]. Realizable zero-divisor graphs are studied in [13]. For $a, b \in R\backslash\{0\}$, where R is finite, the probability that $ab = 0$ is determined in [14].

In 2003, Redmond [15] presented a generalization of $\Gamma(R)$ which is based on an ideal I of R, where the vertex set is $V = \{a \in R\backslash I: ab \in I$ for some $b \in R\backslash I\}$ and the edge set is $E = \{(a,b): ab \in I, a, b \in R\backslash I, a \neq b\}$. The generalized graph was denoted by $\Gamma_I(R)$. Redmond [15] found the diameter,

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connectivity, clique number and girth of $\Gamma_I(R)$. Then, Redmond [15] determined graphs on n vertices which can be presented as $\Gamma_I(R)$ for specific values of n. In [2], a list of rings R is given for which $\Gamma(R)$ has one, two, three or four vertices. Redmond [15] determined all rings R with $|V(\Gamma(R))| = 5$ and also determined when $\Gamma_I(R)$ is planar. Later on, the graph has been studied by various authors such as Miamani ***et al***. [16], Atani ***et al***. [17], Smith [18], Mallika ***et al***. [19] followed by Ansari-Toroghy ***et al***. [20].

In 2021, Nikmehr ***et al.*** [21] defined weakly zero-divisor graph of a ring R as the graph $W\Gamma(R)$ with vertex set $Z(R)^*$ and distinct vertices a and b are adjacent if $\exists r \in ann(a)$ and $s \in ann(b)$ such that $rs = 0$, where $ann(a) = \{m \in R : am = 0\}$. This graph contains $\Gamma(R)$ as a subgraph. The authors studied this graph's connectedness, diameter, girth and determined when $W\Gamma(R)$ is a star. They showed that the diameter of $W\Gamma(R)$ is ≤ 2 and if $W\Gamma(R)$ is not acyclic, then the girth of $W\Gamma(R)$ is ≤ 4. They also showed when $\Gamma(R) = W\Gamma(R)$ and studied the coloring of $W\Gamma(R)$. Later on, the graph $W\Gamma(R)$ has been studied by various authors such as shariq ***et al.*** [22], Nazim ***et al.*** [23] and Rehman ***et al.*** [24].

In this article, an ideal-based weakly zero-divisor graph of a commutative ring R, denoted by $W\Gamma_I(R)$, is introduced which contains $\Gamma_I(R)$ as a subgraph and is identical to $W\Gamma(R)$ when $I = \{0\}$. The graph $W\Gamma_I(R)$ is shown to have a certain relationship with $W\Gamma(R/I)$. It is shown that $W\Gamma_I(R)$ contains $|I|$ disjoint subgraphs isomorphic to $W\Gamma(R/I)$. For $\Gamma_I(R)$ to be planar, some restrictions on the size of I and girth of $\Gamma(R/I)$ are discussed in [15] and later improved in [18]. As in [18], the graph $\Gamma_I(R)$ is planar iff girth of $\Gamma(R/I)$ is infinite and either (a) $|I| = 2$ or (b) $|V(\Gamma(R/I))| \leq 1$ and $|I| \in \{2, 3, 4\}$. Since $\Gamma_I(R) \subseteq W\Gamma_I(R)$, it is evident that if $W\Gamma_I(R)$ is planar then $\Gamma_I(R)$ is planar. It is discussed when $\Gamma_I(R)$ is planar but $W\Gamma_I(R)$ is not planar. It is shown that if R/I is non-reduced and $|V(\Gamma(R/I))| > 2$, then $W\Gamma_I(R)$ is not planar. For $W\Gamma_I(R)$ to be planar, some restrictions on the girth of $W\Gamma(R/I)$ are also discussed. The results show that if the girth of $W\Gamma(R/I)$ is 3 or 4, then $W\Gamma_I(R)$ is not planar. Necessary and sufficient conditions for $W\Gamma_I(R)$ to be planar are provided and rings R with non-trivial planar $W\Gamma_I(R)$ are classified. In addition to introducing an ideal based generalization of $W\Gamma(R)$, this study gives an understanding of the relationship between $W\Gamma(R/I)$ and the generalized graph. Specifically, in regard to planar property of graphs, the study shows that the girth of $W\Gamma(R/I)$ has to do with the generalized graph being planar and classifies rings R for which $W\Gamma_I(R)$ is planar.

The ring R in this article is commutative with $0 \neq 1$ and I is a proper ideal of R. The radical of I, denoted by $\sqrt{I}$, is the set $\{a \in R : a^n \in I$ for some $n \geq 1\}$. For $a \in R$, define $ann(a) = \{m \in R : am = 0\}$. Define *nil*$(R) = \{a \in R \mid a^n = 0$ for some $n \in \mathbb{N}\}$. The ring R is reduced if *nil*$(R) = \{0\}$. For a graph G, let $d(a, b)$ be the length of the shortest path from a to b. The diameter of G, denoted by *diam*(G), is $\sup\{d(a, b) : a, b \in V(G)\}$. The girth of G, denoted by *gr*(G), is the length of the shortest cycle in G (*gr*$(G) = \infty$ if G is acyclic). The graph G is connected if there is a path between any two vertices. A planar graph is one which can be drawn in the plane in such a way that no two edges intersect. For any undefined graph theoretic terms, see Chartrand [25] and Bollobas [26].

Preliminaries

In an earlier study by Redmond [15], the following result was presented regarding the diameter and girth of $\Gamma_I(R)$.

Theorem 1 [15]. Let I be an ideal of R. Then:

(1) $\Gamma_I(R)$ is connected with *diam*$(\Gamma_I(R)) \leq 3$.

(2) *gr*$(\Gamma_I(R)) \leq$ *gr*$(\Gamma(R/I))$. In particular, if *gr*$(\Gamma(R/I)) \neq \infty$, then *gr*$(\Gamma_I(R)) \neq \infty$, and therefore *gr*$(\Gamma_I(R)) \leq$ *gr*$(\Gamma(R/I)) \leq 4$.

It is shown that $\Gamma_I(R) \subseteq W\Gamma_I(R)$ and this fact is used together with Theorem 1 to find the diameter and girth of $W\Gamma_I(R)$. In [15], it was shown how $\Gamma_I(R)$ and $\Gamma(R/I)$ are related. An analogous relationship between $W\Gamma_I(R)$ and $W\Gamma(R/I)$ is established by defining, for each $i \in I$, a subgraph of $W\Gamma_I(R)$ isomorphic to $W\Gamma(R/I)$.

Since $V(W\Gamma_I(R)) = V(\Gamma_I(R))$, the following theorem from [15] gives the cardinality of $W\Gamma_I(R)$.

Proposition 2 [15]. Let I be an ideal of R. Then $|V(\Gamma_I(R))| < \infty$ iff either $|R| < \infty$ or I is a prime. Moreover, if $|V(\Gamma(R/I))| = n$, then $|V(\Gamma_I(R))| = n \cdot |I|$.

While studying the planarity of $W\Gamma_I(R)$, some restrictions were given on the size of I and girth of $W\Gamma(R/I)$. The following result from [18] was used together with the fact that $\Gamma_I(R) \subseteq W\Gamma_I(R)$ to give restraints on $|I|$ when $W\Gamma_I(R)$ is planar.

Proposition 3 [18]. Let I be an ideal of R. If $\Gamma_I(R)$ is planar, then $|I| \leq 2$ or $|V(\Gamma(R/I))| \leq 1$.

For a graph G, the following result from [13] shows when $G \ncong \Gamma(R)$ for a ring R.

Theorem 4 [13]. Let G be a graph with $|G| > 2$. If G contains a looped endpoint, then $G \ncong \Gamma(R)$ for a ring R.

The following is the definition of the subdivision of a graph G.

Definition 5 [15]. A graph H obtained from a graph G by subdividing some edges of G is called a subdivision of G.

For $a, b \in V(W\Gamma_I(R)), \overline{a} = \overline{b}$, it was determined when $a, b \in E(W\Gamma_I(R))$. This result was used together with Theorem 4 and the following well known Kuratowski's theorem for planar graphs to give restraint on the cardinality of $\Gamma(R/I)$ when $W\Gamma_I(R)$ is planar and *nil*$(R/I) \neq \{\overline{0}\}$.

Theorem 6 [26]. A graph G is planar iff G does not contain a subdivision of K^5 or $K^{3,3}$.

The following theorem from [18] determines when $\Gamma_I(R)$ is planar.

Theorem 7 [18]. Let $|R| < \infty, \{0\} \neq I \subset R$ and I is a nonprime ideal of R. Then $\Gamma_I(R)$ is planar iff *gr*$(\Gamma(R/I)) = \infty$ and either (a) $|I| = 2$ or (b) $|V(\Gamma(R/I))| = 1$ and $|I| \leq 4$.

The following theorem from [7] was used to investigate the planarity of $W\Gamma_I(R)$ when $|R| < \infty$, *gr*$(\Gamma(R/I)) = \infty$ and *nil*$(R/I) \neq \{\overline{0}\}$.

Theorem 8 [7]. Let *nil*$(R) \neq \{0\}$. Then the following three statements are equivalent.

(1) *gr*$(\Gamma(R)) = \infty$.

(2) $R \cong B$ or $R \cong \mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, or $\Gamma(R)$ is a star graph.

(3) $\Gamma(R)$ is a singleton, a $\overline{K}^{1,3}$, or a $K^{1,n}$, $n \geq 1$.

Remark 9 [7]. If R is finite, *nil*$(R) \neq \{0\}$ and $\Gamma(R)$ is a star graph, then $|V(\Gamma(R))| \leq 3$. If $|V(\Gamma(R))| = \infty$ and $\Gamma(R)$ is star graph, then either $R \cong Z_2 \times D$ for an integral domain D, or *nil*$(R) = \{0, a\}$, where $Z(R) = ann(a)$.

By Theorem 8, one of the cases when *gr*$(\Gamma(R/I)) = \infty$ and *nil*$(R) \neq \{0\}$ is that $\Gamma(R/I)$ is a star graph. Redmond [15] defined the connected columns in $\Gamma_I(R)$ as follows.

Definition 10 [15]. Let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\Gamma(R/I)$; that is, $\cup_{\lambda \in \Lambda}\{a_\lambda + I\} = Z(R/I) - \{0 + I\}$. If ${a_\lambda}^2 \in I$, then $\overline{a_\lambda} = a_\lambda + I$ is called a connected column of $\Gamma_I(R)$.

Given $\Gamma(R/I)$ is a star graph, the connected columns in $\Gamma_I(R)$ are discussed in the following lemma from [8].

Lemma 11 [8]. Let I be an ideal of R. Then:

(1) If $\Gamma(R/I) \cong K^n$, $n = 1$ or $n \geq 3$, then $a^2 \in I \ \forall a \in V(\Gamma_I(R))$. Moreover, $\Gamma_I(R) \cong K^{n \cdot |I|}$.

(2) If $|V(\Gamma(R/I))| = 2$, then either (a) $a^2 \notin I \ \forall a \in V(\Gamma_I(R))$ or (b) $a^2 \in I \ \forall a \in V(\Gamma_I(R))$.

(3) If $\Gamma(R/I) \cong K^{1,2}$, then either (a) $a^2 \notin I \ \forall a \in V(\Gamma_I(R))$ or (b) $c^2 \in I$, where $\overline{c}$ is the center of $\Gamma(R/I)$, and $a^2 \notin I \ \forall a \in V(\Gamma_I(R))$, $\overline{a} \neq \overline{c}$.

By using Lemma 11, Remark 9 and Theorem 7 together with the condition for $a, b \in V(W\Gamma_I(R)), \overline{a} = \overline{b}$, to be adjacent in $W\Gamma_I(R)$, the structure and planarity of $W\Gamma_I(R)$ were investigated in the case when $\Gamma(R/I)$ is a star graph and *nil*$(R/I) \neq \{\overline{0}\}$.

If $\Gamma(R/I)$ is not star, *gr*$(\Gamma(R/I)) = \infty$ and *nil*$(R) \neq \{\overline{0}\}$, then $R/I \cong B$ or $\mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, by Theorem 8. If $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, then, by using the following theorem from [18] together with Lemma 11 and Theorem 6, the cardinality of I was determined so that $W\Gamma_I(R)$ is planar.

Theorem 12 [18]. $|Z(R)| = 2$ iff $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Moreover, $|V(\Gamma(R/I))| = 1$ iff $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.

Some useful properties of $W\Gamma(R)$ are given in the following lemma from [21] which were used to investigate the structure of $W\Gamma_I(R)$ when $R/I \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$.

Lemma 13 [21]. The following three statements hold:
(1) If $(a, b) \in E(\Gamma(R))$, then $(a, b) \in E(W\Gamma(R))$.

(2) If $a \in$ *nil*$(R)\backslash\{0\}$, then $(a, b) \in E(W\Gamma(R)) \forall b \in V(W\Gamma(R))$.

(3) *nil*$(R)\backslash\{0\}$ is a complete subgraph of $W\Gamma(R)$.
The following theorem from [21] gives the girth of $W\Gamma(R)$.

Theorem 14 [21]. The graph $W\Gamma(R)$ is connected and *diam*$(W\Gamma(R)) \leq 2$. Moreover, if *gr*$(W\Gamma(R)) \neq \infty$, then *gr*$(W\Gamma(R)) \leq 4$.

In order to give restraints on the girth of $W\Gamma(R/I)$, the case *gr*$(W\Gamma(R/I)) \neq \infty$ was investigated. First, the case *gr*$(W\Gamma(R/I)) = 4$ was considered and the planarity of $W\Gamma_I(R)$ was investigated by using the following result from [21] which is part of the proof of [21, Theorem 2.3] together with Theorem 7.

Theorem 15 [21]**.** Let *gr*$(W\Gamma(R)) \neq \infty$. If *gr*$(W\Gamma(R)) = 4$, then *nil*$(R) = \{0\}$ and $W\Gamma(R) = \Gamma(R)$.
Then the planarity of $W\Gamma_I(R)$ was investigated in the case when *gr*$(W\Gamma(R/I)) = 3$ by using the following theorem from [7] together with Theorem 7.

Theorem 16 [7]. Let *nil*$(R) = \{0\}$. Then the following three statements are equivalent for a reduced ring R.

(1) $\Gamma(R)$ is nonempty with *gr*$(\Gamma(R)) = \infty$.

(2) $T(R) = \mathbb{Z}_2 \times K$, where K is a field.

(3) $\Gamma(R) = K^{1,n}$, $n \geq 1$.

Theorem 14 was also used to investigate the planarity of $W\Gamma_I(R)$ when *gr*$(W\Gamma(R/I)) = \infty$ and *nil*$(R/I) = \{\overline{0}\}$. These results were then summarized in the form of a theorem to give necessary and sufficient conditions for $W\Gamma_I(R)$ to be planar for finite ring R. Finally, rings R with nontrivial planar $W\Gamma_I(R)$ were classified by using the classification of rings R with $\Gamma_I(R) \cong K^2, K^3$ or K^4 as well as rings R with planar non-trivial $\Gamma_I(R)$ and *nil*$(R/I) = \{\overline{0}\}$ presented in [18] and finite planar non-trivial graphs $W\Gamma_I(R)$ were also given.

Results and Discussion

First, the definition of the weakly zero-divisor graph of a ring R with respect to an ideal I of R and some basic results on the structure of this graph are given. Then the results on the planar property of this graph are presented and rings are classified whose ideal-based weakly zero-divisor graphs are planar.

Some Definitions and Basic Structures

Definition 17. The ideal-based weakly zero-divisor graph, denoted by $W\Gamma_I(R)$, is an undirected graph with vertices $\{a \in R - I \mid ab \in I$ for some $b \in R - I\}$, where distinct vertices a and b are adjacent if and only if there exist $r \in (I: a)$ and $s \in (I: b)$ such that $rs \in I$, where $(I: a) = \{r \in R \mid ra \in I\}$.

Example 18.
(i) Let $R=\mathbb{Z}_2\times\mathbb{Z}_{12}$, $I=\mathbb{Z}_2\times\{0\}$. The vertices of the graph $W\Gamma_I(R)$, listed in order from 1 to 14, are $(0,2),(1,2),(0,4),(1,4),(0,6),(1,6),(0,8),(1,8),(0,10),(1,10),(0,3),(1,3),(0,9)$ and $(1,9)$. Figure 1 depicts the graph of $W\Gamma_I(R)$. We can see that $W\Gamma_I(R)\neq\Gamma_I(R)$ (e.g., $((0,2),(0,3))\in E(W\Gamma_I(R))$ but $((0,2),(0,3))\notin E(\Gamma_I(R))$).

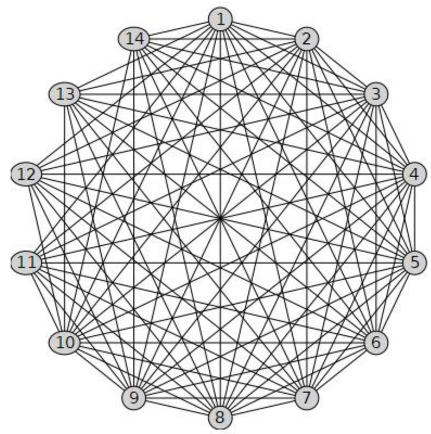

Figure 1. The graph of $W\Gamma_I(R)$

(ii) Let $R=\mathbb{Z}_{24}$ and $I=\langle 8\rangle$. The graph $W\Gamma_I(R)$ is complete while $\Gamma_I(R)$ is not complete, as shown in Figure 2.

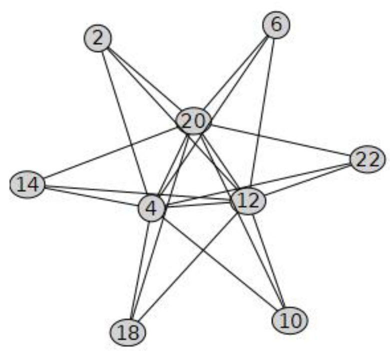

Figure 2. The graph of $\Gamma_I(R)$

The next result shows when $W\Gamma_I(R)$ is empty and when it is identical to $W\Gamma(R)$.

Proposition 19. Let I be an ideal of R.

(1) If $I=(0)$, then $W\Gamma_I(R)=W\Gamma(R)$.
(2) If $I\neq(0)$, then $W\Gamma_I(R)=\emptyset$ iff I is a prime ideal of R.
Proof. (1) If $I=(0)$, then $V(W\Gamma_I(R))=V(W\Gamma(R))=Z(R)^*$ and $(I:a)=ann(a)$. Thus $(a,b)\in E(W\Gamma_I(R))$ iff $(a,b)\in E(W\Gamma(R))$.
(2) Assume that I is prime. Then $ab\in I$ implies $a\in I$ or $b\in I$. Hence $V(W\Gamma_I(R))=\emptyset$.

Conversely, assume that $W\Gamma_I(R)=\emptyset$. Therefore, if $a\in R-I$ and $ab\in I$ for some $b\in R$, then $b\in I$ (otherwise, $a\in V(W\Gamma_I(R))$). Hence I is prime.

Since I is prime iff R/I is an integral domain, it follows from Proposition 19 that $W\Gamma_I(R)=\emptyset$ iff $W\Gamma(R/I)=\emptyset$. We will investigate how $W\Gamma_I(R)$ and $W\Gamma(R/I)$ are related.

Some useful properties of $W\Gamma_I(R)$ are given in the next lemma which is needed in the proof of the theorem following it.

Lemma 20. Let I be an ideal of R.

(1) If $(a,b) \in E(\Gamma_I(R))$, then $(a,b) \in E(W\Gamma_I(R))$.

(2) If $a \in \sqrt{I}\backslash I$, then $(a,b) \in E(W\Gamma_I(R)) \forall b \in V(W\Gamma_I(R))$.

(3) $\sqrt{I}\backslash I$ is complete.

Proof. (1) Let $(a,b) \in E(\Gamma_I(R))$, for distinct $a,b \in V(\Gamma_I(R))$. Then $ab \in I$ and clearly $b \in (I{:}\,a)$ and $a \in (I{:}\,b)$. Hence $(a,b) \in E(W\Gamma_I(R))$.

(2) Let $a \in \sqrt{I}\backslash I$. Let $b \in V(W\Gamma_I(R))$ and $r \in (I{:}\,b)$. Since $a \in \sqrt{I}$, $\exists n \in \mathbb{N}$ such that $a^n \in I$ and $a^i \notin I$, $\forall i, 1 \leq i \leq n-1$. Clearly $a^{n-1} \in (I{:}\,a)$. If $a^{n-1}r \in I$, then $(a,b) \in E(W\Gamma_I(R))$. If $a^{n-1}r \notin I$, then $a^{n-1}r \in (I{:}\,a) \cap (I{:}\,b)$ and $a^{n-1}ra^{n-1}r \in I$. Thus, $(a,b) \in E(W\Gamma_I(R))$.

(3) It is clear, by part (ii).

Next, the diameter and girth of $W\Gamma_I(R)$ are found by using Lemma 20 and Theorem 1.

Theorem 21. Let I be an ideal of R. Then $W\Gamma_I(R)$ is connected with *diam*$(W\Gamma_I(R)) \leq 2$. If *gr*$(W\Gamma_I(R)) \neq \infty$, then *gr*$(W\Gamma_I(R)) \leq 4$.

Proof. By Lemma 20, if $(a,b) \in E(\Gamma_I(R))$, then $(a,b) \in E(W\Gamma_I(R))$. Thus, it follows from Theorem 1 that $W\Gamma_I(R)$ is connected and *gr*$(W\Gamma_I(R)) \leq 4$. Suppose that $(a,b) \notin E(W\Gamma_I(R))$, for distinct $a,b \in V(W\Gamma_I(R))$. Then $rs \notin I$, for every $r \in (I{:}\,a)$ and $s \in (I{:}\,b)$. Since $rsa \in I$ and $rsb \in I$, we have a path $a - rs - b$ in $W\Gamma_I(R)$. Thus, *diam*$(W\Gamma_I(R)) \leq 2$.

In the next few results, it is investigated how $W\Gamma_I(R)$ and $W\Gamma(R/I)$ are related.

Theorem 22. Let $a,b \in R\backslash I$. Then:

(1) If $(\overline{a},\overline{b}) \in E(W\Gamma(R/I))$, then $(a,b) \in E(W\Gamma_I(R))$.
(2) If $(a,b) \in E(W\Gamma_I(R))$ and $\overline{a} \neq \overline{b}$, then $(\overline{a},\overline{b}) \in E(W\Gamma(R/I))$.
(3) If $(a,b) \in E(W\Gamma_I(R))$ and $\overline{a} = \overline{b}$, then $\exists r,\ s \in (I{:}\ a)\ (= (I{:}\ b))$ such that $rs \in I$.

Proof. (1) and (2) are clear. We prove (3).

Let $\overline{a} = \overline{b}$ and $p \in (I{:}\,a)$. Then $\overline{bp} = \overline{b}.\overline{p} = \overline{a}.\overline{p} = \overline{ap} = I$ and so $bp \in I$. Thus $(I{:}\,a) \subseteq (I{:}\,b)$. Similarly, $(I{:}\,b) \subseteq (I{:}\,a)$. Therefore, $(I{:}\,a) = (I{:}\,b)$. Since $(a,b) \in E(W\Gamma_I(R))$, $\exists r,s \in (I{:}\,a)$ such that $rs \in I$.

The following corollary is a direct consequence of Theorem 22.

Corollary 23. If $(a,b) \in E(W\Gamma_I(R))$, $\overline{a} \neq \overline{b}$ then $(a+i, b+j) \in E(W\Gamma_I(R)) \forall i,j \in I$. If $\exists r,\ s \in (I{:}\,a)$ such that $rs \in I$, then $(a+i, a+j) \in E(W\Gamma_I(R)) \forall i,j \in I, i \neq j$.
For a graph G, $\{G_\delta\}_{\delta \in \Delta}$ is a collection of disjoint subgraphs of G if, for each G_δ, $V(G_\delta) \subset G$, $E(G_\delta) \subset G$ and there is no common vertex between any two $G_\delta{}'s$.

Proposition 24. Let I be an ideal of R. The graph $W\Gamma_I(R)$ has $|I|$ disjoint subgraphs isomorphic to $W\Gamma(R/I)$.

Proof. Consider $\{a_\lambda \in R :\ a_\lambda + I \in V(W\Gamma(R/I)), \lambda \in \Lambda\}$ and if $\lambda \neq \beta$, then $\overline{a_\lambda} \neq \overline{a_\beta}$. For $i \in I$, a graph G_i can be defined whose vertices are $\{a_\lambda + i{:}\, \lambda \in \Lambda\}$ and $(a_\lambda + i, a_\beta + i) \in E(G_i)$ whenever $(\overline{a_\lambda}, \overline{a_\beta}) \in E(W\Gamma(R/I))$; i.e., whenever $\exists r \in (I{:}\,a_\lambda)$, $s \in (I{:}\,a_\beta)$ such that $rs \in I$. By Theorem 22, $G_i \subseteq W\Gamma_I(R)$. Also, $G_i \cong W\Gamma(R/I)\ \forall i \in I$, $V(G_i) \cap V(G_j) = \emptyset$ if $i \neq j$.

There is a significant relation between $W\Gamma(R/I)$ and $W\Gamma_I(R)$. The graph $W\Gamma_I(R)$ can be constructed by the method that follows. Let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq R$ and G_i $(i \in I)$ be as in the proof of Proposition 24. Define the graph G with $V(G) = \cup_{i \in I} G_i$. $E(G)$ is defined to be: (1) $\cup_{i \in I} E(G_i)$, (2) for distinct $\lambda, \beta \in \Lambda$ and for any $i,j \in I$, $(a_\lambda + i, a_\beta + j) \in E(G)$ iff $(\overline{a_\lambda}, \overline{a_\beta}) \in E(W\Gamma(R/I))$ (i.e., $\exists r \in (I{:}\,a_\lambda)$, $s \in (I{:}\,a_\beta)$ such that $sr \in I$), (3) for $\lambda \in \Lambda$ and distinct $i,j \in I$, $(a_\lambda + i, a_\lambda + j) \in E(G)$ iff $\exists r,s \in (I{:}\,a_\lambda)$ such that $rs \in I$.

Remark 25. By Proposition 2, $|V(\Gamma_I(R))| < \infty$ iff $|R| < \infty$ or I is prime. Since $V(W\Gamma_I(R)) = V(\Gamma_I(R))$, it follows that $|V(W\Gamma_I(R))| < \infty$ iff $|R| < \infty$ or I is prime. Moreover, the above construction shows that if $|V(W\Gamma(R/I))| = n$, then $|V(W\Gamma_I(R))| = n \cdot |I|$.

Planar Property

Now, the planar property of the graph $W\Gamma_I(R)$ is discussed. Some restraints on the size of the ideal I and the girth of $W\Gamma(R/I)$ will be provided.

Restraints on $|I|$

First, some restraints on the size of the ideal I of R are discussed.

Proposition 26. Let I be an ideal of R. If $W\Gamma_I(R)$ is planar, then $|I| \leq 2$ or $|V(\Gamma(R/I))| \leq 1$.

Proof. Suppose that $W\Gamma_I(R)$ is planar. Then $\Gamma_I(R)$ is planar since $\Gamma_I(R) \subseteq W\Gamma_I(R)$. By Proposition 3, $|I| \leq 2$ or $|V(\Gamma(R/I))| \leq 1$. Since $V(W\Gamma(R/I)) = V(\Gamma(R/I))$, the result follows.

Proposition 27. Let I be an ideal of R. If $|V(W\Gamma(R/I))| = 1$, then $W\Gamma_I(R)$ is planar iff $1 \leq |I| \leq 4$.

Proof. Suppose $|V(W\Gamma(R/I))| = 1$. Then $|V(\Gamma(R/I))| = 1$ since $V(\Gamma(R/I)) = V(W\Gamma(R/I))$. By Lemma 11, $\Gamma_I(R) = K^{|I|}$. Since $V(W\Gamma_I(R)) = V(\Gamma_I(R))$, it follows from Lemma 20 that $W\Gamma_I(R) = K^{|I|}$. By Theorem 15, $W\Gamma_I(R)$ is planar iff $1 \leq |I| \leq 4$.

The following theorem gives a restriction on $|V(W\Gamma(R/I))|$ when R/I is non-reduced.

Proposition 28. Let I be an ideal of R and let R/I be non-reduced. If $I \neq \{0\}$ and $|V(W\Gamma(R/I))| > 2$, then $W\Gamma_I(R)$ is not planar.

Proof. Since R/I is non-reduced, $\exists a \in R\backslash I$ such that $a^2 \in I$. Since $\overline{a}$ is a looped vertex of $\Gamma(R/I)$ and $|V(\Gamma(R/I))| = |V(W\Gamma(R/I))| > 2$, $\overline{a}$ is not an end vertex of $\Gamma(R/I)$ by Theorem 4. Thus, $\exists b, c \in R\backslash I$ such that $\overline{a} \neq \overline{b}, \overline{a} \neq \overline{c}, \overline{b} \neq \overline{c}$ and $ab, ac \in I$. Since I is nonzero, choose $0 \neq l \in I$. Then $ab, ac \in I$ implies that $a(b+l), (a+l)b, (a+l)(b+l)$ and $(a+l)c$ are all in I. Thus, $(a,b), (a,b+l), (a,c), (a+l,b), (a+l,b+l)$ and $(a+l,c)$ are all in $E(\Gamma_I(R))$ and hence in $E(W\Gamma_I(R))$. Since $ab, a(b+l), ac$ and a^2 are all in I, we have that $(b,c), (b+l,c)$ and $(b,b+l)$ are all in $E(W\Gamma_I(R))$. Also, $(a,a+l) \in E(W\Gamma_I(R))$ since $(a,a+l) \in E(\Gamma_I(R))$. Thus, $\{a, a+l, b, b+l, c\}$ is a subgraph of $W\Gamma_I(R)$ isomorphic to K^5. Hence $W\Gamma_I(R)$ is non-planar.

Remark 29. (1) Suppose that $|R| < \infty$, $I \neq \{0\}$ and I is a nonprime ideal of R. If R/I is non-reduced and *gr*$(\Gamma(R/I)) = \infty$, then $R/I \cong B$ or $R/I \cong \mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, or $\Gamma(R/I)$ is star by Theorem 8. If $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, then $|V(W\Gamma(R/I))| = |V(\Gamma(R/I))| = 1$ by Theorem 12. By Proposition 27, $W\Gamma_I(R)$ is planar iff $1 \leq |I| \leq 4$.

If $R/I \cong Z_2 \times Z_4$ or $Z_2 \times Z_2[X]/(X^2)$, then $\Gamma(R/I) \cong \underline{K}^{1,3}$ (see Figure 3). Since $|V(W\Gamma(R/I))| = |V(\Gamma(R/I))| > 2$, the graph $W\Gamma_I(R)$ is not planar by Proposition 28. However, if $|I| = 2$, then $\Gamma_I(R)$ is planar by Theorem 7. Note that the vertex of degree three in Figure 3 is the only vertex of $\Gamma(R/I)$ whose square is zero. Let $\underline{a}$ be the vertex of degree three and let $b, c, d \in R\backslash I$ such that $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ are all distinct and $(\underline{a}, \underline{b}), (\underline{a}, \underline{c}), (\underline{a}, \underline{d}) \in E(\Gamma(R/I))$. By Figure 3, one of $\underline{b}, \underline{c}$ and $\underline{d}$ has degree 2 in $\Gamma(R/I)$. Let *deg*$(\underline{c}) = 2$. Let $e \in R\backslash I$ such that $\underline{a} \neq \underline{e}, \underline{c} \neq \underline{e}$ and $(\underline{c}, \underline{e}) \in E(\Gamma(R/I))$ (by Figure 3, *deg*$(\underline{e}) = 1$ in $\Gamma(R/I)$). By Lemma 13, we have that $(\underline{a}, \underline{b}), (\underline{a}, \underline{c}), (\underline{a}, \underline{d})$ and $(\underline{a}, \underline{e})$ are all in $E(W\Gamma(R/I))$. Also, $(\underline{b}, \underline{c}), (\underline{b}, \underline{d})$ and $(\underline{c}, \underline{d})$ are all in $E(W\Gamma(R/I))$ since ab, ac, ad and a^2 are all in I. Since ce, ab, ac and ad are all in I, we have that $(\underline{b}, \underline{e}), (\underline{d}, \underline{e}) \in E(W\Gamma(R/I))$. By Lemma 13, $(\underline{c}, \underline{e}) \in E(W\Gamma(R/I))$ since $(\underline{c}, \underline{e}) \in E(\Gamma(R/I))$. Thus, $W\Gamma(R/I)$ is complete (and so *gr*$(W\Gamma(R/I)) = 3$). Then $(a+l, b+m)$, $(a+l, c+m)$, $(a+l, d+m)$, $(a+l, e+m)$, $(b+l, c+m)$, $(b+l, d+m)$, $(b+l, e+m)$, $(c+l, d+m)$, $(c+l, e+m)$ and $(d+l, e+m)$ are all in $E(W\Gamma_I(R))$, $\forall l, m \in I$ by Corollary 23. Also, $ab \in I$ implies that $a(b+l) \in I \; \forall l \in I$. Since $a^2 \in I$, we have that $(b+l, b+m) \in E(W\Gamma_I(R))$ $\forall l, m \in I, l \neq m$. Similarly, $(c+l, c+m), (d+l, d+m) \in E(W\Gamma_I(R))$, $\forall l, m \in I, l \neq m$, since ac, ad and $a^2 \in I$. By Lemma 20, $(a+l, a+m) \in E(W\Gamma_I(R))$ since $(a+l, a+m) \in E(\Gamma_I(R))$ $\forall l, m \in I, l \neq m$. Thus, $\underline{a} \cup \underline{b} \cup \underline{c} \cup \underline{d} \cup \{e\}$ is a complete subgraph of $W\Gamma_I(R)$. Since $\underline{e}$ is not adjacent to any vertex other than $\underline{c}$ in $\Gamma(R/I)$, it follows that $(I : e+l) = \underline{c} \; \forall l \in I$. If $(c+l)(c+m) \in I$ for some $l, m \in I, l \neq m$, then $c^2 \in I$, a contradiction. Since $(I : e+l) = \underline{c} \forall l \in I$, it follows that $rs \notin I$, for every $r \in (I : e+l)$ and $s \in (I : e+m)$ and so $(e+l, e+m) \notin E(W\Gamma_I(R))$ $\forall l, m \in I, l \neq m$.

Suppose that $\Gamma(R/I)$ is a star graph. Since R/I is non-reduced, $|V(\Gamma(R/I))| \leq 3$ by Remark 9. If $\Gamma(R/I) = K^{1,1}$ with vertices $\overline{a}$ and $\overline{b}$, then $a^2, b^2 \in I$ by Lemma 11 since R/I is non-reduced. Then $(a+l, a+m), (b+l, b+m) \in E(\Gamma_I(R)) \forall l, m \in I, l \neq m$. Also, $ab \in I$ implies that $(a+l)(b+m) \in I$ and so $(a+l)(b+m) \forall l, m \in I$. Thus, $\Gamma_I(R)$ is complete and so $W\Gamma_I(R) = \Gamma_I(R)$. If $|I| = 2$, then $W\Gamma_I(R)$ is planar by Theorem 7. Suppose that $\Gamma(R/I) = K^{1,2}$. By Proposition 28, $W\Gamma_I(R)$ is not planar. However, if $|I| = 2$, then $\Gamma_I(R)$ is planar by Theorem 7. Since R/I is non-reduced, $\exists a \in R \backslash I$ such that $a^2 \in I$. By Lemma 11, $\overline{a}$ is the center of $\Gamma(R/I)$. If $\overline{b}$ and $\overline{c}$ are distinct non-central vertices of $\Gamma(R/I)$, then $(\overline{a}, \overline{b}), (\overline{a}, \overline{c}) \in E(W\Gamma(R/I))$ since $(\overline{a}, \overline{b}), (\overline{a}, \overline{c}) \in E(\Gamma(R/I))$. Also $(\overline{b}, \overline{c}) \in E(W\Gamma(R/I))$ since ab, ac and $a^2 \in I$. Thus, $W\Gamma(R/I) = K^3$ (and so $gr(W\Gamma(R/I)) = 3$). By Corollary 23, $(a+l, b+m), (a+l, c+m)$ and $(b+l, c+m)$ are all in $E(W\Gamma_I(R)) \forall l, m \in I$. Since $ab, a^2 \in I$, we have that $(b+l, b+m) \in E(W\Gamma_I(R)) \forall l, m \in I, l \neq m$ by Corollary 23. Also, $(c+l, c+m) \in E(W\Gamma_I(R)) \forall l, m \in I, l \neq m$, since $ac, a^2 \in I$. Finally, since $a^2 \in I$, we have that $(a+l, a+m) \in E(\Gamma_I(R)) \forall l, m \in I, l \neq m$. By Lemma 20, $(a+l, a+m) \in E(W\Gamma_I(R))\ \forall l, m \in I, l \neq m$. Thus, $W\Gamma_I(R)$ is complete.

(2) Suppose that $|R| < \infty$ and $\{0\} \neq I$ is a nonprime ideal of R. Since $\Gamma(R/I) \subseteq W\Gamma(R/I)$, we deduce that $gr(W\Gamma(R/I)) \leq gr(\Gamma(R/I))$. Thus, if $nil(R/I) \neq \{\overline{0}\}$ and $gr(W\Gamma(R/I)) = \infty$, then $W\Gamma(R/I)$ has at most two vertices by (1) (otherwise, $gr(W\Gamma(R/I)) = 3$ by (1)).

(3) Assume that $I \neq \{0\}$ and R/I is non-reduced. If $\Gamma(R/I)$ is an infinite star graph, then, by Remark 9, $nil(R/I) \neq \{\overline{0}, \overline{a}\}$, where $Z(R/I) = ann(\overline{a})$. Then $\overline{a}$ is the center of $\Gamma(R/I)$ and $a^2 \in I$. By the reasoning in (1), it can be shown that $W\Gamma(R/I)$ and $W\Gamma_I(R)$ are complete graphs. Note that $W\Gamma_I(R)$ is not planar by Proposition 28.

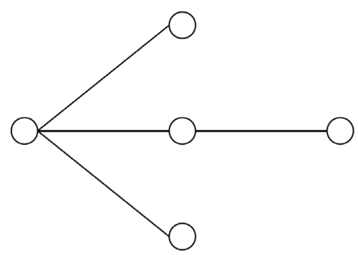

Figure 3. $\overline{K}^{1,3}$

Restraints on $gr(W\Gamma(R/I))$

Now, some restraints on $gr(W\Gamma(R/I))$ are given.

Proposition 30. Let I be an ideal of R. If $gr(W\Gamma(R/I)) = 4$, then $W\Gamma_I(R)$ is non-planar.

Proof. Assume that $gr(W\Gamma(R/I)) = 4$. By Theorem 15, $W\Gamma(R/I) = \Gamma(R/I)$, and so $gr(\Gamma(R/I)) = 4$. By Theorem 7, $\Gamma_I(R)$ is non-planar. Since $\Gamma_I(R) \subseteq W\Gamma_I(R)$ and $\Gamma_I(R)$ is not planar, $W\Gamma_I(R)$ is non-planar.

In the next result it is shown that if $gr(W\Gamma(R/I)) = 4$, then $W\Gamma_I(R) = \Gamma_I(R)$.

Proposition 31. Let I be an ideal of R. If $gr(W\Gamma(R/I)) = 4$, then $W\Gamma_I(R) = \Gamma_I(R)$.

Proof. Suppose that $gr(W\Gamma(R/I)) = 4$. Then, by Theorem 15, R/I is reduced and $W\Gamma(R/I) = \Gamma(R/I)$. Assume that $W\Gamma_I(R) \neq \Gamma_I(R)$. Then $\exists a, b \in V(W\Gamma_I(R)), a \neq b$, such that $(a, b) \in E(W\Gamma_I(R))$ and $(a, b) \notin E(\Gamma_I(R))$. Then $ab \notin I$ and there exist $r \in (I : a)$ and $s \in (I : b)$ such that $rs \in I$. Assume that $\overline{a} = \overline{r}$. Then $r = a + l$ for some $l \in I$. Then $ar \in I$ implies that $a^2 \in I$ and so $nil(R/I) \neq \{\overline{0}\}$. This is a contradiction as R/I is reduced. Thus, $\overline{a} \neq \overline{r}$. Similarly, $\overline{r} \neq \overline{s}$ and $\overline{s} \neq \overline{b}$. If $\overline{a} = \overline{s}$, then, since $bs \in I$, we have $\overline{ab} = \overline{a}.\overline{b} = \overline{s}.\overline{b} = \overline{bs} = I$ and so $ab \in I$. This is a contradiction. Thus $\overline{s} \neq \overline{a}$. If $\overline{a} \neq \overline{b}$, then $(\overline{a}, \overline{b}) \in E(W\Gamma(R/I))$ by Theorem 22 since $(a, b) \in E(W\Gamma_I(R))$. Then $(\overline{a}, \overline{b}) \in E(\Gamma(R/I))$ since $W\Gamma(R/I) = \Gamma(R/I)$. Thus, $ab \in I$, a contradiction. Therefore $\overline{a} = \overline{b}$. Then $\overline{as} = \overline{a}.\overline{s} = \overline{b}.\overline{s} = \overline{bs} = I$ and so $as \in I$. Thus, $\overline{a}, \overline{r}$ and $\overline{s}$ is a triangle in $\Gamma(R/I)$, a contradiction as $gr(\Gamma(R/I)) = gr(W\Gamma(R/I)) = 4$. Thus, $W\Gamma_I(R) = \Gamma_I(R)$.

Proposition 32. Let I be an ideal of R. If $gr(W\Gamma(R/I)) = 3$, then $W\Gamma_I(R)$ is not planar.

Proof. Suppose that *gr*$(W\Gamma(R/I)) = 3$. If *gr*$(\Gamma(R/I))$ is finite, then $\Gamma_I(R)$ is non-planar by Theorem 7 and so $W\Gamma_I(R)$ is non-planar. Suppose *gr*$(\Gamma(R/I)) = \infty$. We claim that *nil*$(R/I) \neq \{\overline{0}\}$. Suppose *nil*$(R/I) = \{\overline{0}\}$. Then, by Theorem 16, $\Gamma(R/I) \cong K^{1,n}$ for some $n \geq 2$. Let $\overline{c}$ be the center of $\Gamma(R/I)$ and let $\overline{a}, \overline{b} \in V(\Gamma(R/I))\backslash\{\overline{c}\}, \overline{a} \neq \overline{b}$. Then $ann\{\overline{a}\} = ann\{\overline{b}\} = \{\overline{0}, \overline{c}\}$. Since $c^2 \notin I, (\overline{a}, \overline{b}) \notin E(W\Gamma(R/I))$. Therefore $W\Gamma(R/I) = \Gamma(R/I)$, a contradiction. Thus, R/I is non-reduced. By Proposition 28, $W\Gamma_I(R)$ is not planar.

Proposition 33. Let I be an ideal of R. If *gr*$(W\Gamma(R/I)) = \infty$ and $|I| = 2$, then $W\Gamma_I(R)$ is planar.

Proof. If $I = R$ or I is prime, then both $W\Gamma(R/I)$ and $W\Gamma_I(R)$ are empty, and so they are both planar. Suppose that $I \subsetneq R$ and I is non-prime. Then $W\Gamma(R/I)$ is nonempty. The case $|V(W\Gamma(R/I))| = 1$ has been handled in Proposition 27. Suppose that $|V(W\Gamma(R/I))| \geq 2$. If *nil*$(R/I) \neq \{\overline{0}\}$, then, by (2) of Remark 29, $|V(W\Gamma(R/I))| = 2$. By (1) of Remark 29, $W\Gamma_I(R)$ is planar. If *nil*$(R/I) = \{\overline{0}\}$, then $\Gamma(R/I) \cong K^{1,n}$ for some $n \geq 1$ by Theorem 16 since *gr*$(\Gamma(R/I)) = \infty$. We claim that $W\Gamma_I(R) = W\Gamma_I(R)$. Assume that $W\Gamma_I(R) \neq W\Gamma_I(R)$. Then $\exists a, b \in V(W\Gamma_I(R))$ such that $(a, b) \in E(W\Gamma_I(R))$ and $(a, b) \notin E(\Gamma_I(R))$. Since $(a, b) \in E(W\Gamma_I(R)), \exists r \in (I:a)$ and $s \in (I:b)$ such that $rs \in I$. Suppose that $\overline{a} \neq \overline{b}$. Then $(\overline{a}, \overline{b}) \notin E(\Gamma(R/I))$ since $ab \notin I$. Thus, neither $\overline{a}$ nor $\overline{b}$ is the central vertex of $\Gamma(R/I)$. If $\overline{r} = \overline{a}$, then $r = a + l$ for $0 \neq l \in I$. Then $ar \in I$ implies that $a^2 \in I$. This is a contradiction since R/I is reduced and $a \notin I$. Therefore $\overline{a} \neq \overline{r}$. Similarly, $\overline{b} \neq \overline{s}$ and $\overline{r} \neq \overline{s}$. Also, $(\overline{r}, \overline{a}), (\overline{s}, \overline{b}) \in E(\Gamma(R/I))$ since $ra \in I$ and s$b \in I$. Since $\overline{a}$ and $\overline{b}$ are non-central vertices of $\Gamma(R/I)$, we must have $\overline{r} = \overline{s} = \overline{c}$, where $\overline{c}$ is the center of $\Gamma(R/I)$. This is a contradiction. Assume that $\overline{a} = \overline{b}$. Then $(\overline{s}, \overline{a}) \in E(\Gamma(R/I))$ since $(\overline{s}, \overline{b}) \in E(\Gamma(R/I))$. Since $(\overline{r}, \overline{a}), (\overline{s}, \overline{a}) \in E(\Gamma(R/I))$, it follows that $\overline{r}$ and $\overline{s}$ are non-central vertices of $\Gamma(R/I)$, a contradiction as $rs \in I$. Thus, $W\Gamma_I(R) = \Gamma_I(R)$. By Theorem 7, $W\Gamma_I(R)$ is planar.

By combining all the previous propositions, the following result is obtained:

Theorem 34. Suppose that $|R| < \infty$ and I is non-zero, non-prime ideal of R. Then $W\Gamma_I(R)$ is planar iff *gr*$(W\Gamma(R/I)) = \infty$ and either (a) $|I| = 2$ or (b) $|V(W\Gamma(R/I))| = 1$ and $|I| \in \{2, 3, 4\}$.

Proof. Suppose that $W\Gamma_I(R)$ is planar. By Proposition 30, *gr*$(W\Gamma(R/I)) \neq 4$ and, by Proposition 32, *gr*$(W\Gamma(R/I)) \neq 3$. Thus, *gr*$(W\Gamma(R/I)) = \infty$ by Theorem 14. Suppose that $|I| \neq 2$. Then, by Proposition 26, $|V(W\Gamma(R/I))| \leq 1$. Since I is non-prime, $W\Gamma_I(R) \neq \emptyset$ by Proposition 19. Thus $W\Gamma(R/I) \neq \emptyset$ and so $|V(W\Gamma(R/I))| = 1$. Since $I \neq \{0\}$ and $W\Gamma_I(R)$ is planar, $|I| \in \{2, 3, 4\}$ by Proposition 27. For the converse, see Proposition 27 and Proposition 33.

Classification

In this section, finite rings R with planar $W\Gamma_I(R)$ are classified. First, the case $\sqrt{I} = I$ is considered.

Proposition 35. Suppose that $|R| < \infty$ and I is a non-zero, non-prime ideal of R such that $\sqrt{I} = I$. Then *gr*$(W\Gamma(R/I)) = \infty$ and $|I| = 2$ iff R is isomorphic to one of the rings from Table 1 (with appropriately chosen ideal I).

Table 1. Rings for Proposition 34

Ring	Ideal
$\mathbb{Z}_2 \times K$	$\langle 2 \rangle \times \{0\}$
$\mathbb{Z}_2[X]/(X^2) \times K$	$\langle x \rangle \times \{0\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times K$	$\mathbb{Z}_2 \times \{0\} \times \{0\}$, $\{0\} \times \mathbb{Z}_2 \times \{0\}$ or $\{0\} \times \{0\} \times K$

Proof. If *gr*$(W\Gamma(R/I)) = \infty$, then *gr*$(\Gamma(R/I)) = \infty$ since $\Gamma(R/I) \subseteq W\Gamma(R/I)$. Since $\sqrt{I} = I$, the ring R/I is reduced. By [18, Proposition 3.9], $R \cong S$ for some ring S from Table 1 with corresponding I. Conversely suppose $R \cong S$ for some ring S from Table 1 with corresponding ideal. Then, by [18, Proposition 3.9], $gr(\Gamma(R/I)) = \infty$ and $|I| = 2$. By the proof of Proposition 33, $W\Gamma_I(R) = \Gamma_I(R)$. If $W\Gamma(R/I) \neq \Gamma(R/I)$, then $\exists a, b \in R\backslash I$ such that $\overline{a} \neq \overline{b}$ and $(\overline{a}, \overline{b}) \in E(\Gamma(R/I))$ but $(\overline{a}, \overline{b}) \notin E(\Gamma(R/I))$. Then $(a, b) \in E(W\Gamma_I(R))$ by Theorem 22. Since $W\Gamma_I(R) = \Gamma_I(R)$, we have $(a, b) \in E(\Gamma_I(R))$. This is a contradiction as $ab \notin I$. Thus, $W\Gamma(R/I) = \Gamma(R/I)$ and so *gr*$(W\Gamma(R/I)) = \infty$.

The next result determines rings R and ideals I of R with planar $W\Gamma_I(R)$ in the case when $\sqrt{I} \neq I$.

Proposition 36. Suppose that $|R| < \infty$ and I is a non-zero, non-prime ideal of R such that $\sqrt{I} \neq I$. Then $gr(W\Gamma(R/I)) = \infty$ and $|I| = 2$ iff R is isomorphic to one of the rings from Table 2 (with appropriately chosen ideal I).

Proof. $nil(R/I) \neq \{\overline{0}\}$ since $\sqrt{I} \neq I$. Suppose that $gr(W\Gamma(R/I)) = \infty$ and $|I| = 2$. Then $W\Gamma(R/I)$ has at most two vertices by (2) of Remark 29. If $|V(W\Gamma(R/I)| = 1$, then $W\Gamma_I(R) = \Gamma_I(R) \cong K^2$ by the proof of Proposition 27. By [18, Proposition 2.4], $R \cong S$, where S is one of the first seven rings in Table 2. Suppose that $|V(W\Gamma(R/I))| = 2$. Then $|V(\Gamma(R/I))| = 2$. By (1) of Remark 29, $\Gamma_I(R)$ is complete. By Proposition 2, $|V(\Gamma_I(R)| = |I||V(\Gamma(R/I))| = 2.2 = 4$ and so $\Gamma_I(R) \cong K^4$. Thus, $W\Gamma_I(R) = \Gamma_I(R) \cong K^4$. Rings R with $\Gamma_I(R) \cong K^4, \{0\} \neq I < R$, are classified in section 2.4 of [18]. In the case 2 of section 2.4 in [18], it has been shown that if $|I| = 2$ and $|V(\Gamma(R/I))| = 2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_9$ or $\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$.
The converse is clear.

Table 2. Non-radical case: $gr(W\Gamma(R/I)) = \infty$ and $|I| = 2$

Ring	Ideal
$\mathbb{Z}_8$	$\langle 4\rangle = \{0,4\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\{0\} \times \mathbb{Z}_2$
$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2$	$\{0\} \times \mathbb{Z}_2$
$\mathbb{Z}_4[X]/(X^2, 2X)$	$\langle x\rangle, \langle 2\rangle$ or $\langle x+2\rangle$
$\mathbb{Z}_4[X]/(2X, X^2-2)$	$\langle 2\rangle$
$\mathbb{Z}_2[X]/(X^3)$	$\langle x^2\rangle$
$\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$	$\langle x\rangle, \langle y\rangle$ or $\langle x+y\rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_9$	$\mathbb{Z}_2 \times \{0\}$
$\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$	$\mathbb{Z}_2 \times \{0\}$

Finite Planar Non-Trivial $W\Gamma_I(R)$

Finite planar graphs corresponding to non-empty $W\Gamma_I(R)$ are now presented. If $\sqrt{I} \neq I$, $gr(W\Gamma(R/I)) = \infty$ and $|I| = 2$, then $W\Gamma_I(R) = \Gamma_I(R) \cong K^2$ or K^4 by the proof of Proposition 36. Also, if $|V(W\Gamma(R/I))| = 1$ and $|I| \in \{3,4\}$, then $W\Gamma_I(R) = \Gamma_I(R) \cong K^3$ or K^4 by the proof of Proposition 27. Commutative rings R and ideal I of R with $\Gamma_I(R) \cong K^3$ and $\Gamma_I(R) \cong K^4$ have been determined in [18, Proposition 2.6] and [18, Proposition 2.7] respectively.

Now consider the case $\sqrt{I} = I$ (that is, R/I is reduced). If $W\Gamma_I(R)$ is a finite planar graph, then $gr(W\Gamma(R/I)) = \infty$ and $|I| = 2$. Since $nil(R/I) = \{\overline{0}\}$ and $gr(\Gamma(R/I)) = \infty$, $\Gamma(R/I) \cong K^{1,n}, n \geq 1$, by Theorem 16. By the proof of Proposition 33, $W\Gamma_I(R) = \Gamma_I(R)$. The graph $\Gamma_I(R)$ is given in [18, Figure 3.7] which we present in Figure 4.

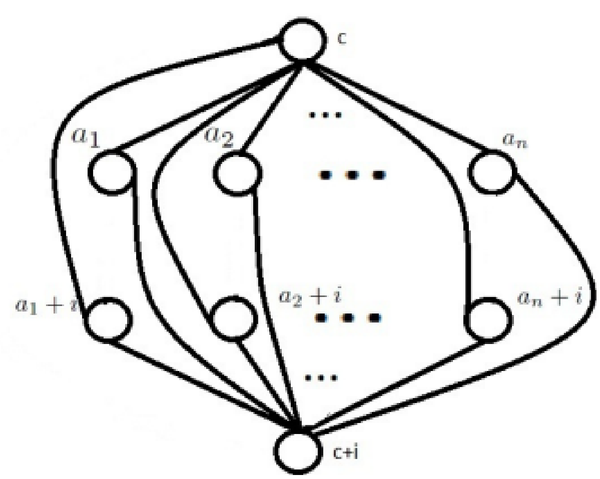

Figure 4. Finite planar $W\Gamma_I(R), \sqrt{I} = I$, $I \neq \{0\}$

Conclusions

In conclusion, an ideal-based weakly zero-divisor graph of R, denoted by $W\Gamma_I(R)$, is defined and it is shown that $\Gamma_I(R) \subseteq W\Gamma_I(R)$ and $W\Gamma_I(R) = W\Gamma(R)$ when $I = \{0\}$. The results suggested that the graphs $W\Gamma_I(R)$ and $W\Gamma(R/I)$ are strongly related. Specifically, it is shown that $W\Gamma_I(R)$ has $|I|$ disjoint subgraphs isomorphic to $W\Gamma(R/I)$. A similar relationship had already been established between $\Gamma_I(R)$ and $\Gamma(R/I)$ in previous research. A method for constructing the graph $W\Gamma_I(R)$ is also presented. In previous research, it was established that $\Gamma(R)$ does not have a looped end point. This result was used to show that if the ideal I is non-radical ($\sqrt{I} \neq I$) and $|V(\Gamma(R/I))| > 2$, then $W\Gamma_I(R)$ is not planar. However, it was established in previous research that if $|I| = 2$ and *gr*$(\Gamma(R/I)) = \infty$ (where R is finite), then the subgraph $\Gamma_I(R)$ of $W\Gamma_I(R)$ is planar (regardless of whether or not $\sqrt{I} \neq I$). A result given in an earlier research state that if $\sqrt{I} \neq I$ and *gr*$(\Gamma(R/I)) = \infty$, then $\Gamma(R/I)$ is a singleton, a $\bar{K}^{1,3}$, or a $K^{1,n}$, $n \geq 1$. This research shows that under these assumptions that $W\Gamma(R/I)$ is complete. It is also shown that under these assumptions that $W\Gamma_I(R)$ is complete except when $\Gamma(R/I) = \bar{K}^{1,3}$. Further, the results showed that if *gr*$(W\Gamma(R/I)) = 3$ or 4, then $W\Gamma_I(R)$ is not planar. However, if R is finite, *gr*$(W\Gamma(R/I)) = \infty$ and $|I| = 2$, then $W\Gamma_I(R)$ is planar. The main result states that if $\{0\} \neq I$, R is finite and I is non-prime, then $W\Gamma_I(R)$ is planar iff *gr*$(W\Gamma(R/I)) = \infty$ and either (a) $|I| = 2$ or (b) $|V(W\Gamma(R/I))| = 1$ and $|I| \in \{2,3,4\}$. Also, finite rings R and ideals I of R are determined such that $W\Gamma_I(R)$ is planar. The results showed that if $I \neq \{0\}$ and $\sqrt{I} \neq I$, then $W\Gamma_I(R)$ is planar iff $W\Gamma_I(R) \cong K^2, K^3$ or K^4. If $W\Gamma_I(R)$ is planar and $\sqrt{I} = I$, then the results revealed that $W\Gamma_I(R) = \Gamma_I(R)$ and $W\Gamma_I(R)$ is not complete. As suggested by these results, the graph $W\Gamma_I(R)$ can be studied in relation to some of the existing graphs, namely $\Gamma(R/I)$, $W\Gamma(R/I)$ and $\Gamma_I(R)$. In some previous researches it was investigated how the properties of $\Gamma_I(R)$ and $\Gamma(R/I)$ are related (for example, girth, clique number, connectivity, independence number and domination number). The results presented in this research motivate one to further study the graphical properties of $W\Gamma_I(R)$ in relation to the properties of $W\Gamma(R/I)$. This will enhance the understanding of the structure of some of the existing graphs as well as the structure of the ring.

Conflicts of Interest

The authors declare that they have no competing interests with respect to the publication of this work.

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