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Picone Identities of a Certain Class of Conformable Half Linear Anisotropic Biharmonic Equations

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Abstract Main aim of this article, we derive sufficient conditions of new results for Picone identities for a certain class of conformable half-linear anisotropic biharmonic equations. We derive a Strumian comparison theorem and oscillation results. Furthermore, the oscillation results are different from the most known ones in the sense that they are based on the information for radial solutions. This paper's results expand upon and broaden a few of the previously established results for conformable half-linear anisotropic biharmonic equations. If $p_i = 2, \gamma = 1$and $\alpha = 1,$ then conformable half-linear anisotropic biharmonic equations (4) become the classical biharmonic equation. These novel outcomes add to the body of knowledge already available in the classical example. To illustrate the usefulness of our new results, we give an example.

Keywords: Picone identities, anisotropic biharmonic equations, anisotropic hardy type inequality, oscillation.

Introduction

To establish oscillation theory and the Strumian comparison theorem, a Picone identity is fundamental [2], [3]. For fourth-order ordinary equations, there are two varieties of Picone identities that are recognized [12], [13], [17] – [20], [22]. There has been a lot of interest lately in researching the anisotropic Laplacian has important mathematical theoretical merit but is not widely used in natural science applications. The applications are in science and engineering. When the media's conductivity varies in each direction, the anisotropic physical characteristics of some reinforced materials describe the fluid dynamics in the anisotropic media and image processing [4], [6], [7], [14], [21]. Khalil et. al.[9] developed a limit-based derivative with good behavior termed the conformable fractional derivative. Day to day, there has been significant growth in their direction. The reference lists a few of the studies [1], [5], [8] – [11], [15], [16].

Motivated by Picone's identity [19], we study the j and J ordinary differential operators by,

$$\mathrm{j}[\mathrm{u}] = (\mathrm{a}_1\mathrm{u}'')'' - (\mathrm{a}_2\mathrm{u}')' + \mathrm{a}_3\mathrm{u}, \quad \text{and} \tag{1}$$

$$\mathrm{J}[\mathrm{v}] = (\mathrm{A}_1\mathrm{v}'')'' - (\mathrm{A}_2\mathrm{v}')' + \mathrm{A}_3\mathrm{v}. \tag{2}$$

Prove the identity that is

$$\frac{d}{dt}\left[\frac{u}{v}\{u(A_1v'')' - v(a_1u'')'\} + \frac{u'}{v'}\{v'(a_1u'') - u'(A_1v'')\} + \frac{u}{v}\{v(a_2u') - u(A_2v')\}\right]$$
$$= (a_1 - A_1)(u'')^2 + (a_2 - A_2)(u')^2 + (a_3 - A_3)(u)^2 + A_1\left(u'' - \frac{u'}{v'}v''\right)^2$$
$$+(-v'(A_1v'')' + A_2(v')^2)\left(\frac{u'}{v'} - \frac{u}{v}\right)^2 + \frac{u}{v}(uJ[v] - vj[u]). \tag{3}$$

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then, the oscillation theory of solutions and the Sturmian comparison principle was obtained.

This paper aims to identify a Picone identity for conformable half-linear anisotropic biharmonic equations. As an application of the form, we derived the Strumian comparison theorem,

$$p_{\alpha,\gamma}[u] = \sum_{i=1}^{n} D_{x_i}^{2\alpha}\left(|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}u_{x_ix_i}\right) + c(x)|u|^{p_i+\gamma-3}u, \qquad p_i > 1, \tag{4}$$

where $\alpha \in (0,1]$, $\gamma > 0$. Note that if $\gamma = 1$, the equation (4) becomes a conformable anisotropic equations. If $p_i = 2$, $\gamma = 1$, the equation (4) is called a conformable biharmonic equations. If $p_i = 2$, $\gamma = 1$ and $\alpha = 1$, then equation (4) is called the classical biharmonic equations. $u(x) \in \mathbb{C}^{4\alpha}(\mathbb{G},\mathbb{R}) \cap \mathbb{C}(\overline{\mathbb{G}},\mathbb{R})$ which is absolutely continuous with four times $\alpha -$ fractional derivative in every compact subset of $\mathbb{G}$ and almost everywhere on $\mathbb{G}$. This article is structured as follows: In section 2, a few necessary definitions and a lemma are provided. In Section 3, we examine the Strumian comparison theorem, the Picone identity. Section 4 discusses a few oscillation results. We provide an example of our results in section 5.

Preliminaries

The Basic definitions of conformable derivatives and discuss analogue of conformable Picone identities.

Definition 1.1 [9] Given $u: \mathbb{R}_+ \to \mathbb{R}$. Conformable fractional derivative of u of α order is defined by $D^{\alpha}(u)(y) = \lim_{\epsilon\to 0}\frac{u(y+\epsilon y^{1-\alpha})-u(y)}{\epsilon}$.
$\forall$ y> 0, $\alpha \in (0,1]$. If u is α-differentiable in some $(0,a), a > 0$ and $\lim_{y\to 0^+} u^{\alpha}(y)$exists, then given by $u^{\alpha}(0) = \lim_{y\to 0^+} u^{\alpha}(y)$.

Definition 1. 2 [9] $I_a^{\alpha}(u)(y) = I_1^{\alpha}(y^{\alpha-1})(u) = \int_a^y \frac{u(x)}{x^{1-\alpha}}dx$.
The integral is this case is the regular Riemann improper integral.

Proposition 1.1 [9] Assume $\alpha \in (0,1]$, at some point $y > 0$, u, v be α-differentiable. The

1. $D^{\alpha}(a_1u + a_2v) = a_1D^{\alpha}(u) + a_2D^{\alpha}(v)$,
2. $D^{\alpha}(y^q)= qy^{q-\alpha}$,
3. $D^{\alpha}(a) =0$,
4. $D^{\alpha}(uv) = uD^{\alpha}(v) + vD^{\alpha}(u)$,
5. $D^{\alpha}\left(\frac{u}{v}\right) = \frac{vD^{\alpha}(u)-uD^{\alpha}(v)}{v^2}$,
6. If u is differentiable, then $D^{\alpha}(u(y)) = y^{1-\alpha}\frac{du(y)}{dy}$.

Definition 1.3 [5] Consider u to a function with m variables $y_1, \ldots\ldots, y_m$, then conformable partial derivative of u in order $0 < \alpha \leq 1$ in y_i is given

$$D^{\alpha}u_{y_i} = \frac{\partial^{\alpha}}{\partial {y_i}^{\alpha}}u(y_1, \ldots\ldots, y_m) = \lim_{\epsilon\to 0}\frac{u(y_1, \ldots y_{i-1}, y_i + \epsilon\, {y_i}^{1-\alpha}, \ldots, y_m) - u(y_1, \ldots\ldots, y_m)}{\epsilon}.$$

For conformable half linear biharmonic operators $p_{\alpha,\gamma}$ and $P_{\alpha,\gamma}$ defined by, we prove (1) and (2) to an analogue of Picone identity,

$$p_{\alpha,\gamma}[u] = D^{2\alpha}(a_1(x)|D^{2\alpha}u|^{\gamma-1}D^{2\alpha}u) + a_2(x)|u|^{\gamma-1}u\,, \tag{5}$$

$$P_{\alpha,\gamma}[v] = D^{2\alpha}(A_1(x)|D^{2\alpha}v|^{\gamma-1}D^{2\alpha}v) + A_2(x)|v|^{\gamma-1}v. \tag{6}$$

Assuming G to be a bounded domain in $\mathbb{R}^n$with a piecewise smooth boundary $\partial\mathbb{G}$, we can say that $A, a \in \mathbb{C}^{2\alpha}(\mathbb{G},\mathbb{R}_+)$, and $C, c \in \mathbb{C}(\overline{\mathbb{G}},\mathbb{R})$.

The $u \in \mathbb{C}^{4\alpha}(\mathbb{G},\mathbb{R}) \cap \mathbb{C}(\overline{\mathbb{G}},\mathbb{R})$ is defined to the domain $\mathbb{D}_{p_{\alpha,\gamma}}(\mathbb{G})$ of $p_{\alpha,\gamma}$. A similar definition applies to the domain $\mathbb{D}_{P_{\alpha,\gamma}}(\mathbb{G})$.

Lemma 2.1 If $u \in \mathbb{D}_{p_{\alpha,\gamma}}(\mathbb{G})$, $v \in \mathbb{D}_{P_{\alpha,\gamma}}(\mathbb{G})$ and v is not equal to 0 in $\mathbb{G}$, then,

$$D^{2\alpha}\left[\frac{u}{v}\{uD^{\alpha}(A_1(x)|D^{2\alpha}v|^{\gamma-1}D^{2\alpha}v) - vD^{\alpha}(a_1(x)|D^{2\alpha}u|^{\gamma-1}D^{2\alpha}u)\} + \frac{D^{\alpha}u}{D^{\alpha}v}\{D^{\alpha}v(a_1(x)|D^{2\alpha}u|^{\gamma-1}D^{2\alpha}u) - D^{\alpha}u(A_1(x)|D^{2\alpha}v|^{\gamma-1}D^{2\alpha}v)\}\right]$$

$$= \frac{u}{v}\left(uP_{\alpha,\gamma}[v] - vp_{\alpha,\gamma}[u]\right) + \left(a_1(x) - A_1(x)\right)|D^{2\alpha}u|^{\gamma-1} + \left(a_2(x)|u|^{\gamma-1} - A_2(x)|v|^{\gamma-1}\right)u^2$$
$$+A_1(x)\left(D^{2\alpha}u - \frac{D^{\alpha}u}{D^{\alpha}v}\Delta^{\alpha}v\right)^2 - D^{\alpha}v\left(D^{\alpha}\left(A_1(x)|D^{2\alpha}v|^{\gamma-1}D^{2\alpha}v\right)\right). \quad (7)$$

We generalized (7) to an analogue of conformable biharmonic equations $v > 0$ and $u \geq 0$,

$$\left(D^{2\alpha}u - \frac{D^{\alpha}u}{D^{\alpha}v}D^{2\alpha}v\right)^2 = |D^{2\alpha}u|^2 + \left(\frac{D^{\alpha}u}{D^{\alpha}v}\right)^2 - 2\left(\frac{D^{\alpha}u}{D^{\alpha}v}\right)D^{2\alpha}uD^{2\alpha}v,$$
$$= |D^{2\alpha}u|^2 - D^{\alpha}\left(\frac{(D^{\alpha}u)^2}{D^{\alpha}v}\right)D^{2\alpha}v. \quad (8)$$

We extended (8) to a conformable p-biharmonic equations $v > 0$ and $u \geq 0$,

$$\left(D^{2\alpha}u - \frac{D^{\alpha}u}{D^{\alpha}v}D^{2\alpha}v\right)^p = |D^{2\alpha}u|^p + (p-1)\left(\frac{D^{\alpha}u}{D^{\alpha}v}\right)^p - p\left(\frac{D^{\alpha}u}{D^{\alpha}v}\right)^{p-1}D^{2\alpha}u|D^{2\alpha}v|^{p-1}D^{2\alpha}v,$$
$$= |D^{2\alpha}u|^p - D^{\alpha}\left(\frac{(D^{\alpha}u)^p}{(D^{\alpha}v)^{p-1}}\right)|D^{2\alpha}v|^{p-1}D^{2\alpha}v. \quad (9)$$

Half Linear Anisotropic Biharmonic Equations

In this section, we see the conformable half linear anisotropic biharmonic equations and we derive a Strumian comparison theorem.

$$p_{\alpha,\gamma}[u] = \sum_{i=1}^{n} D_{x_i}^{2\alpha}\left(a_i(x)|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}u_{x_ix_i}\right) + c(x)|u|^{p_i+\gamma-3}u, \quad x \in \mathbb{G}, \quad (10)$$
$$P_{\alpha,\gamma}[v] = \sum_{i=1}^{n} D_{x_i}^{2\alpha}\left(A_i(x)|D^{2\alpha}v_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}\right) + C(x)|v|^{p_i+\gamma-3}v, \quad x \in \mathbb{G}, \quad (11)$$

Assuming G to be a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\mathbb{G}$, we can say that $A_i, a_i \in \mathbb{C}^{2\alpha}(\mathbb{G}, \mathbb{R}_+)$ $(i = 1,2,..,n), C, c \in \mathbb{C}(\overline{\mathbb{G}}, \mathbb{R})$.

The $u \in \mathbb{C}^{4\alpha}(\mathbb{G}, \mathbb{R}) \cap \mathbb{C}(\overline{\mathbb{G}}, \mathbb{R})$ is defined by the domain $\mathbb{D}_{p_{\alpha,\gamma}}(\mathbb{G})$ of $p_{\alpha,\gamma}$. According to definition, the domain $\mathbb{D}_{P_{\alpha,\gamma}}(\mathbb{G})$ is the same as $P_{\alpha,\gamma}$.

Theorem 3.1 If two continuously conformable differentiable sets u and v in $\mathbb{G} \in \mathbb{R}^n$, they are,

$$L_{\alpha}(u,v) = \sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1} + \sum_{i=1}^{n}(p_i+\gamma-2)$$
$$-\sum_{i=1}^{n}(p_i+\gamma-1)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right)^{p_i+\gamma-2}D^{2\alpha}u_{x_ix_i}|D^{2\alpha}v_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i} \quad (12)$$

$$R_{\alpha}(u,v) = \sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1} - \sum_{i=1}^{n}D_{x_i}^{\alpha}\left(\frac{\left(D^{\alpha}u_{x_i}\right)^{p_i+\gamma-1}}{\left(D^{\alpha}v_{x_i}\right)^{p_i+\gamma-2}}\right)|D^{2\alpha}v_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}. \quad (13)$$

i) $L_{\alpha}(u,v) = R_{\alpha}(u,v)$,
ii) $L_{\alpha}(u,v) \geq 0$ and
iii) $L_{\alpha}(u,v) = 0$ iff $D_{x_i}^{2\alpha}\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right) = 0$ in Ω.

Proof. i) $L_{\alpha}(u,v) = R_{\alpha}(u,v)$ (14)

$$R_{\alpha}(u,v) = \sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1} - \sum_{i=1}^{n}D_{x_i}^{\alpha}\left(\frac{\left(D^{\alpha}u_{x_i}\right)^{p_i+\gamma-1}}{\left(D^{\alpha}v_{x_i}\right)^{p_i+\gamma-2}}\right)|D^{2\alpha}v_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}$$
$$= \sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1} + \sum_{i=1}^{n}(p_i+\gamma-2)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}|D^{2\alpha}v_{x_ix_i}|\right)^{p_i+\gamma-1}$$
$$-\sum_{i=1}^{n}(p_i+\gamma-1)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right)^{p_i+\gamma-2}D^{2\alpha}u_{x_ix_i}|D^{2\alpha}v_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}$$
$$= L_{\alpha}(u,v).$$

ii) To prove $L_{\alpha}(u,v) \geq 0$, we rewrite by

$$L_{\alpha}(u,v) = \sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1} + \sum_{i=1}^{n}(p_i+\gamma-2)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}|D^{2\alpha}v_{x_ix_i}|\right)^{p_i+\gamma-1}$$
$$+\sum_{i=1}^{n}(p_i+\gamma-1)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right)^{p_i+\gamma-2}\left|D^{2\alpha}v_{x_ix_i}\right|^{p_i+\gamma-3}\left(|D^{2\alpha}v_{x_ix_i}|\left|D^{2\alpha}u_{x_ix_i}\right| - D^{2\alpha}u_{x_ix_i}D^{2\alpha}v_{x_ix_i}\right)$$
$$-\sum_{i=1}^{n}(p_i+\gamma-1)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right)^{p_i+\gamma-2}\left|D^{2\alpha}v_{x_ix_i}\right|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}|D^{2\alpha}u_{x_ix_i}|$$
$$= I + II, \quad (15)$$

Where

$$I=\sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1}+\sum_{i=1}^{n}(p_i+\gamma-2)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}|D^{2\alpha}v_{x_ix_i}|\right)^{p_i+\gamma-1}$$

$$-\sum_{i=1}^{n}(p_i+\gamma-1)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right)^{p_i+\gamma-2}\left|D^{2\alpha}v_{x_ix_i}\right|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}|D^{2\alpha}u_{x_ix_i}| \quad (16)$$

$$II=\sum_{i=1}^{n}(p_i+\gamma-1)\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right)^{p_i+\gamma-2}\left|D^{2\alpha}v_{x_ix_i}\right|^{p_i+\gamma-3}\left(|D^{2\alpha}v_{x_ix_i}|\left|D^{2\alpha}u_{x_ix_i}\right|-D^{2\alpha}u_{x_ix_i}D^{2\alpha}v_{x_ix_i}\right) \quad (17)$$

Recall, the Young's inequality $m\geq 0$ and $n\geq 0$. We have

$mn\leq \frac{m^p}{p}+\frac{n^q}{q}$, where $\frac{1}{p}+\frac{1}{q}=1$.

Equality holds if $m^p=n^q$, where $p>1, q>1$. We take

$m=\sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|$ and $n=\sum_{i=1}^{n}\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}|D^{2\alpha}v_{x_ix_i}|\right)^{p_i+\gamma-2}$

$$\sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}|D^{2\alpha}v_{x_ix_i}|\right)^{p_i+\gamma-2}\leq \sum_{i=1}^{n}\frac{1}{(p_i+\gamma-1)}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1}$$

$$+\sum_{i=1}^{n}\left(\frac{p_i+\gamma-1}{p_i+\gamma-2}\right)\left(\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}|D^{2\alpha}v_{x_ix_i}|\right)^{p_i+\gamma-2}\right)^{\frac{p_i+\gamma-1}{p_i+\gamma-2}}. \quad (18)$$

Then, using then equation from (16), we obtain $I\geq 0$.

Clearly, $II\geq 0$ in virtue of $\sum_{i=1}^{n}\left(|D^{2\alpha}v_{x_ix_i}||D^{2\alpha}u_{x_ix_i}|-D^{2\alpha}u_{x_ix_i}D^{2\alpha}v_{x_ix_i}\right)\geq 0$ holds simultaneously.

Hence $L_{\alpha}(u,v)\geq 0$.

Let us consider that iii) $L_{\alpha}(u,v)=0$. In fact, if $L_{\alpha}(u,v)(x_0)=0$ and $(x_0)\neq 0$, $(x_0)\in\Omega$.

$\sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|\left|D^{2\alpha}v_{x_ix_i}\right|=\sum_{i=1}^{n}D^{2\alpha}u_{x_ix_i}D^{2\alpha}v_{x_ix_i}$ yields

$\sum_{i=1}^{n}D_{x_i}^{2\alpha}\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right)(x_0)=0.$

If $u(x_0)=0$, then $\sum_{i=1}^{n}D^{\alpha}u_{x_i}=0$ a.e on Ω and $\sum_{i=1}^{n}D_{x_i}^{2\alpha}\left(\frac{D^{\alpha}u_{x_i}}{D^{\alpha}v_{x_i}}\right)=0$. Hence the proof.

Theorem 3.3 Suppose $a_i=A_i$, $c<C$ in the bounded domain $\mathbb{G}$. Let $u\in\mathbb{C}^{4\alpha}(\mathbb{G},\mathbb{R})\ni\mathbb{D}_{p_{\alpha,\gamma}}[u]$ is zero,

$\sum_{i=1}^{n}D_{x_i}^{2\alpha}\left(a_i(x)\left|D^{2\alpha}u_{x_ix_i}\right|^{p_i+\gamma-3}D^{2\alpha}u_{x_ix_i}\right)+c(x)|u|^{p_i+\gamma-3}u.$

Next, the following conformable half linear anisotropic biharmonic equations can be solved non trivially by v,

$\sum_{i=1}^{n}D_{x_i}^{2\alpha}\left(A_i(x)\left|D^{2\alpha}v_{x_ix_i}\right|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}\right)+C(x)|v|^{p_i+\gamma-3}v$, must change sign.

Proof. Assume to the contrary, that $v\neq 0$ in $\mathbb{G}$. Let $v>0$ we observe that,

$$0\leq\int_{\mathbb{G}}L_{\alpha}(u,v)\,d_{\alpha}x=\int_{\mathbb{G}}R_{\alpha}(u,v)\,d_{\alpha}x$$

$$=\int_{\mathbb{G}}\left[\sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1}-\sum_{i=1}^{n}D_{x_i}^{\alpha}\left(\frac{(D^{\alpha}u_{x_i})^{p_i+\gamma-1}}{(D^{\alpha}v_{x_i})^{p_i+\gamma-2}}\right)|D^{2\alpha}v_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}\right]d_{\alpha}x$$

$$=\int_{\mathbb{G}}\sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1}\,d_{\alpha}x-\int_{\mathbb{G}}\sum_{i=1}^{n}D_{x_i}^{\alpha}\left(\frac{(D^{\alpha}u_{x_i})^{p_i+\gamma-1}}{(D^{\alpha}v_{x_i})^{p_i+\gamma-2}}\right)|D^{2\alpha}v_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}d_{\alpha}x$$

$$=\int_{\mathbb{G}}\sum_{i=1}^{n}|D^{2\alpha}u_{x_ix_i}|^{p_i+\gamma-1}\,d_{\alpha}x-\int_{\mathbb{G}}\sum_{i=1}^{n}D_{x_i}^{\alpha}\left(|D^{2\alpha}v_{x_ix_i}|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}\right)\frac{(D^{\alpha}u_{x_i})^{p_i+\gamma-1}}{(D^{\alpha}v_{x_i})^{p_i+\gamma-2}}d_{\alpha}x$$

$$=\sum_{i=1}^{n}\int_{\mathbb{G}}(c(x)-C(x))\left(D^{\alpha}u_{x_i}\right)^{p_i+\gamma-1}d_{\alpha}x\leq 0.$$

Which is contradiction. Hence the proof.

Oscillation Results

Now we investigate the oscillation solutions of

$$P_{\alpha,\gamma}[v]=\sum_{i=1}^{n}D_{x_i}^{2\alpha}\left(A_i(x)\left|D^{2\alpha}v_{x_ix_i}\right|^{p_i+\gamma-3}D^{2\alpha}v_{x_ix_i}\right)+C(x)|v|^{p_i+\gamma-3}v=0. \quad (19)$$

Where $\gamma>0,\alpha\in(0,1]$ and a bounded domain $\mathbb{G}\subset\Omega$, that Ω is an exterior domain in $\mathbb{R}^n$, assume that $A_i\in\mathbb{C}^{2\alpha}(\mathbb{G},\mathbb{R}_+)$ $(i=1,2,..,n)$ and $C\in\mathbb{C}(\overline{\mathbb{G}},\mathbb{R})$.

A solution if, for every $r>0$, $\Omega_r=\Omega\cap\{x\in\mathbb{R}^n:|x|>r\}, v\in\mathbb{D}_{P_{\alpha,\gamma}}(\Omega)$ has a zero, it becomes oscillatory in Ω.

$A(x)=n\max_{1\leq i\leq n}A_i(x).$

$\bar{A}(r)$ and $\bar{C}(r)$ are the spherical means of $A(x)$ and $C(x)$ to the sphere $S_r = \{x \in \mathbb{R}^n : |x| = r\}$, respectively.

$\bar{A}(r) = \frac{1}{w_n r^{n-1}} \int_{S_r} A(x)\, d_\alpha S,$

$\bar{C}(r) = \frac{1}{w_n r^{n-1}} \int_{S_r} C(x)\, d_\alpha S.$

Here w_n is the sphere's surface area. We observe that the conformable half linear anisotropic biharmonic equations have a solution $y(r)$.

$$D^\alpha \left(r^{n-1} D^\alpha \left(\bar{A}(r) r^{1-n} \left((|D^\alpha(r^{n-1}D^\alpha y)|)^{p_i+\gamma-3} D^\alpha(r^{n-1}D^\alpha y) \right) \right) \right) + r^{n-1}\bar{C}(r)|y|^{p_i+\gamma-3} y = 0,$$

then a conformable, radially symmetric solution of is $v(x) = y(r)$ of

$$P_{\alpha,\gamma}[v] = \sum_{i=1}^{n} D_{x_i}^{2\alpha}(\bar{A}(|x|)\left|D^{2\alpha} v_{x_i x_i}\right|^{p_i+\gamma-3} D^{2\alpha} v_{x_i x_i}) + \bar{C}(|x|)|v|^{p_i+\gamma-3} v = 0.$$

Theorem 4.1 Consider the conformable half linear anisotropic biharmonic equations

$$D^\alpha \left(r^{n-1} D^\alpha \left(\bar{A}(r) r^{1-n} \left((|D^\alpha(r^{n-1}D^\alpha y)|)^{p_i+\gamma-3} D^\alpha(r^{n-1}D^\alpha y) \right) \right) \right) + r^{n-1}\bar{C}(r)|y|^{p_i+\gamma-3} y = 0. \quad (20)$$

If each solution $v \in \mathbb{D}_{P_{\alpha,\gamma}}(\Omega)$ of the conformable half linear anisotropic biharmonic equations (19) is oscillatory at $r = \infty$, then Ω is oscillation.

Proof. If $y(r)$ of (20) has a nontrivial solution, let $\{r_k\}_{k=1}^{\infty}$ be the zero sequence such that $r_0 \le r_1 < r_2 \ldots,$ $\lim_{k\to\infty} r_k = \infty$. Get,

$\mathbb{G}_k = \{x \in \mathbb{R}^n ; r_k < |x| < r_{k+1}\} \quad (k = 1,2,\ldots),$

$v(x) = y(r)$, define that

$$\begin{aligned} M_{G_k}[v] &= \int_{\mathbb{G}_k} \left[\sum_{i=1}^{n} A_i(x)\left(D^{2\alpha} v_{x_i x_i}\right)^2 \left|D^{2\alpha} v_{x_i x_i}\right|^{p_i+\gamma-3} + C(x)|v|^{p_i+\gamma-3} v^2 \right] d_\alpha x \\ &\le \int_{\mathbb{G}_k} \left[\sum_{i=1}^{n} \max_{1\le i\le n} A_i(x) \left(D^{2\alpha} v_{x_i x_i}\right)^2 \left|D^{2\alpha} v_{x_i x_i}\right|^{p_i+\gamma-3} + C(x)|v|^{p_i+\gamma-3} v^2 \right] d_\alpha x \\ &= w_n \int_{r_k}^{r_{k+1}} \left[r^{n-1} \left(\bar{A}(r) r^{1-n} \left((|D^\alpha(r^{n-1}D^\alpha y)|)^{p_i+\gamma-3} D^\alpha(r^{n-1}D^\alpha y) \right) \right) (D^\alpha D^\alpha y) + r^{n-1}\bar{C}(r)|y|^{p_i+\gamma-3} y^2 \right] d_\alpha r \\ &\le 0, \qquad (p_i > 1) \end{aligned}$$

that v oscillates in Ω since it contains a zero on every $\overline{\mathbb{G}_k}(k = 1,2,\ldots)$. Hence the proof.

Remark 4.1. In the instance, here $\bar{A}(r) = 1, \alpha = 1, p_i = 2, \gamma = 1$, equation (20) was derived by Yoshida [21], (see Lemma 1.6.1, P. No. 27).

Example

We consider the conformable half linear anisotropic biharmonic equations

$$\sum_{i=1}^{n} D_{x_i}^{2\alpha}(A_0\left|D^{2\alpha} v_{x_i x_i}\right|^{p_i+\gamma-3} D^{2\alpha} v_{x_i x_i}) + C_0|v|^{p_i+\gamma-3} v = 0, \quad (p_i > 1) \quad (21)$$

in $\mathbb{G}_{k_1} = \{x \in \mathbb{R}^n : |x| \ge r_0\}(r_0 > 0)$, where $\gamma > 0, A_0 > 0$ and $C_0 > 0$ are constants and $\alpha \in (0,1]$. Then the conformable radially symmetric solution for (21) is the following

$$D^\alpha \left(r^{n-1} D^\alpha \left(A_0 r^{1-n} \left((|D^\alpha(r^{n-1}D^\alpha y)|)^{p_i+\gamma-3} D^\alpha(r^{n-1}D^\alpha y) \right) \right) \right) + r^{n-1} C_0 |y|^{p_i+\gamma-3} y = 0, \quad r \ge r_0 \quad (22)$$

This, with $\bar{A}(r) = A_0$ and $\bar{C}(r) = C_0$, is a special case of (20).

i) Assume that $n \le p_i + \gamma - 1$. Then we obtain

$$\int_{r_0}^{\infty} \left(\frac{1}{r^{n-1}\,\bar{A}(r)} \right)^{\frac{1}{p_i+\gamma-2}} d_\alpha r = \int_{r_0}^{\infty} A_0^{-\frac{1}{p_i+\gamma-2}} r^{\frac{(n-1)}{p_i+\gamma-2}} d_\alpha r = \infty.$$

Since $\int_{r_0}^{\infty} r^{n-1}\bar{C}(r) d_\alpha r = C_0 \int_{r_0}^{\infty} r^{n-1} d_\alpha r = \infty,$

Since equation (22) oscillates at $r = \infty$, Theorem (4.1) states that every solution $v \in \mathbb{D}_{P_{\alpha,\gamma}}(\mathbb{G}_{k_1})$ of equation (21) oscillates in $\mathbb{G}_{k_1}$.

ii) Suppose that $n > p_i + \gamma - 1$. We get

$$\int_{r_0}^{\infty} \left(\frac{1}{r^{1-n}\,\bar{A}(r)} \right)^{\frac{1}{p_i+\gamma-2}} d_\alpha r = \int_{r_0}^{\infty} A_0^{-\frac{1}{p_i+\gamma-2}} r^{\frac{(n-1)}{p_i+\gamma-2}} d_\alpha r < \infty.$$

It is simple to observe that

$$\begin{aligned} \bar{\pi}(r) &= \int_{r_0}^{\infty} \left(\frac{1}{s^{n-1}\,\bar{A}(s)} \right)^{\frac{1}{p_i+\gamma-2}} d_\alpha s \\ &= A_0^{-\frac{1}{p_i+\gamma-2}} \left(\frac{p_i+\gamma-2}{1-n-\alpha(p_i+\gamma-2)} \right) r^{\frac{n-1+\alpha(p_i+\gamma-2)}{p_i+\gamma-2}}, \end{aligned}$$

And therefore

$$\left(\bar{\pi}(r)\right)^{p_i+\gamma-1} \bar{C}(r) = A_0^{-\frac{(p_i+\gamma-1)}{p_i+\gamma-2}} C_0 \left(\frac{p_i + \gamma - 2}{1 - n - \alpha(p_i + \gamma - 2)} \right)^{p_i+\gamma-1} r^{\frac{(n-1+\alpha(p_i+\gamma-2))(p_i+\gamma-1)}{p_i+\gamma-2}}$$

If $n \le (p_i + \gamma - 2)^2$, then $\int_{r_0}^{\infty} \left(\bar{\pi}(r)\right)^{p_i+\gamma-1} r^{n-1}\bar{C}(r)d_\alpha r = \infty$, and if $n > (p_i + \gamma - 2)^2$, then $\frac{1}{\bar{\pi}(r)}\int_{r_0}^{\infty} \left(\bar{\pi}(s)\right)^{p_i+\gamma-1} s^{n-1}\bar{C}(s)d_\alpha s < \infty$,

which tends to ∞ as $r \to \infty$. As a result, according to Theorem (4.1), the solution $v \in \mathbb{D}_{P_{\alpha,\gamma}}(\mathbb{G}_{k_1})$ of equation (21) is oscillatory in $\mathbb{G}_{k_1}$ since equation (22) is oscillatory at $r = \infty$.

We conclude that for each $n \in N$, and for non negative constants $A_0 > 0, C_0 > 0$, every solution v of (21) is oscillatory in $\mathbb{G}_{k_1}$.

Conclusions

In this article, we obtain some new results for Picone identities for a certain class of conformable half linear anisotropic biharmonic equations, derived a Strumian comparison theorem and oscillation results. We have provided an example to show how our new results are useful.

Conflicts of Interest

The author state that there is no conflict of interest.

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