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RESEARCH ARTICLE

A Direct Method to Approximate Solution of the Space-fractional Diffusion Equation

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Abstract A class of partial differential equations with fractional derivatives in the spatial variables are called space-fractional diffusion equations. They can be applied to simulate anomalous diffusion, in which the classical diffusion equation does not accurately describe how a plume of particles disperses. Analytically solving fractional diffusion equations can be problematic due to the typically complex structures of fractional derivative models. Hence, this study proposes the utilisation of a satisfier function in combination with the Ritz method to effectively address fractional diffusion equations in the Caputo sense. By employing this approach, the equations are transformed into an algebraic system, so facilitating their solution and providing a numerical result. This method can achieve a high level of accuracy in solving the Caputo fractional diffusion equations by utilising only a small number of terms from the shifted Legendre polynomials in two variables. The precision and effectiveness of our approach may be evaluated, as it yielded dependable approximations of the solutions.

Keywords: Fractional diffusion equation, Ritz method, Caputo derivative, Legendre polynomials.

Introduction

A generalisation of differential equations to any (non-integer) order is known as a fractional differential equation (FDE). Due to its capacity to represent and explain complex phenomena, FDEs have garnered a great deal of attention. FDEs have found wide-ranging applications in science and engineering, such as physics, biology, and chemistry, which has led to a major global increase in research in this field.

One of the main challenges in solving FDEs is the ‘non-locality’ of the fractional operator. This means that the fractional derivative at a specific point depends on the behaviour of the function across its entire domain. In a way, FDE is about considering the history of the function entirely, not just its behaviour at a single point.

The fractional diffusion equation is a type of FDEs. In a standard diffusion process, the rate of change of a quantity with respect to time is proportional to the second derivative of that quantity with respect to space. The space-fractional diffusion equation introduces fractional derivatives in the spatial variable. The general form of the space-fractional diffusion equation is

$$\frac{\partial u}{\partial t} = \psi \frac{\partial^\alpha u}{\partial x^\alpha},\ (x,t) \in C \quad (1)$$

where ψ is the diffusion coefficient. u is the function to be computed, and $C = [0,1] \times [0,1] \subset \mathbb{R}^2$ is a bounded domain. In fractional diffusion, the order of the derivative, α is a non-integer, allowing for more flexibility in modeling anomalous diffusion phenomena. Applications of fractional diffusion equations can be found in physics fields such as fluid flow [1] and wave solutions [2, 3]. In biology, fractional diffusion equation is demonstrated in the transmission of Lassa fever disease [4, 5], where the results may be applied to stop the disease’s spread. In finance, it is important in predicting financial and market stability [6].

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Solving fractional diffusion equations is computationally demanding because the non-local nature of fractional derivatives means that each point in time depends on a range of past times, complicating the numerical approximation. Creating precise, quick, and cost-effective numerical algorithms for solving them is necessary. For this reason, [7] applied a numerical method using a stabilized finite element formulation, finite difference method and Newton-Raphson method to find accurate approximation of convection-advection diffusion equation. The time-space fractional diffusion problem has been solved in [8] using the Fourier spectral approach and the Spectral Deferred Correction. Other numerical method is meshless method where it can be found in [9] and [10]. Gaining motivation and inspiration from the above mentioned work, in this article, a framework has been introduced to solve the space-fractional diffusion equation numerically.

We present a method for solving space-fractional diffusion equations by combining the Ritz method with a satisfier function. This method effectively transforms the equations into a set of algebraic equations system, making it easier to solve and obtain the derivation of numerical solutions. The method's efficacy lies in its ability to generate highly accurate solutions by employing a limited number of shifted Legendre polynomial terms involving two variables. Hence a complete investigation on convergence and stability analysis is presented for the proposed numerical method. The numerical example presented at the later section shows the efficiency, stability and comparability of our proposed method with existing approaches.

The structure of this article is as follows. In the following sections, we will briefly introduce a few preliminary concepts, such as the Caputo fractional derivative and the Legendre polynomials. The details of the proposed method are presented in a subsequent section. Following that, some examples are provided to demonstrate the application of the suggested method in solving fractional-order diffusion equations using the Caputo derivative. We also presented the numerical experiments that show the accuracy and validity of the proposed method. The final section emphasises the conclusion and we made some recommendations.

Preliminaries

We will concentrate on the following space-fractional order diffusion equation in the form of:

$$D_t\, u(x,t) = \psi D_x^{\alpha}\, u(x,t) + f(x,t), \quad 0 \leq x,\, t \leq 1 \tag{2}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= g_0(x), \\ u_t(x,0) &= g_1(x), \\ u(0,t) &= h_0(t), \\ u(1,t) &= h_1(t). \end{aligned}$$

The fractional derivative $D^{\alpha}u(x,t)$ is defined in the Caputo sense. The parameter α denotes the fractional order of the space derivative, with $1 < \alpha \leq 2$ and $f(x,t)$ is the source term. $g_0(x), g_1(x), h_0(t)$ and $h_1(t)$ are assumed to be sufficiently smooth functions on their domains.

Caputo Derivatives

Some definitions relating to the Caputo fractional derivative will be briefly discussed in this section.

Definition 1. The one-parameter Mittag-Leffler function is defined as [11]

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)},\ \alpha \in \mathbb{R}^+, z \in \mathbb{R} \tag{3}$$

Definition 2. The three-parameter generalization of Mittag-Leffler function is given by [11]

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\alpha k+\beta)} \frac{z^k}{k!},\ \alpha,\beta,\gamma \in \mathbb{R}^+, z \in \mathbb{R} \tag{4}$$

Definition 3. The Caputo fractional derivative D^{α} is given as [12]

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1} d\tau},\ n-1 < \alpha \leq n,\ n \in \mathbb{N} \tag{5}$$

The followings are some properties of the Caputo derivatives:

Definition 4. The Caputo derivative of power function is given as [12]

$$D^{\alpha} t^{p} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, \ p \in \mathbb{R} \tag{6}$$

Definition 5. The Caputo derivative of exponential function is given as [12]

$$D^{\alpha} e^{\lambda t} = \lambda^{m} t^{m-\alpha} E_{1,m-\alpha+1}(\lambda t), \ \lambda \geq 0 \tag{7}$$

Definition 6. The Caputo derivative of trigonometric functions is given as [12]

$$D^{\alpha} \sin \lambda t = \frac{1}{2i} (i\lambda)^{m} t^{m-\alpha} \left(E_{1,m-\alpha+1}(i\lambda t) - (-1)^{m} t^{m-\alpha} E_{1,m-\alpha+1}(-i\lambda t) \right), \ \lambda \geq 0$$
$$D^{\alpha} \cos \lambda t = \frac{1}{2i} (i\lambda)^{m} t^{m-\alpha} \left(E_{1,m-\alpha+1}(i\lambda t) + (-1)^{m} t^{m-\alpha} E_{1,m-\alpha+1}(-i\lambda t) \right), \ \lambda \geq 0 \tag{8}$$

These definitions are important, and they are essential in the calculation of the Caputo diffusion equation later. The calculation using Maple 2023 is performed by applying these definitions.

Legendre Polynomials

The Legendre polynomials serve as the fundamental basis function in the Ritz method. The Legendre polynomials, denoted as $L_n(x)$, are the coefficients of a formal expansion of the generating function in powers of t:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} L_n(x) t^n \tag{9}$$

The Legendre polynomials, $L_n(x, y)$ (or shifted Legendre polynomials, $P_n(x, y)$) can be extended in in two variables in a number of different ways. For example, in [13], the two-variable Legendre polynomials can be expressed as

$$\frac{1}{\sqrt{1-2xt+yt^2}} = \sum_{n=0}^{\infty} L_n(x, y) t^n. \tag{10}$$

The shifted Legendre polynomials in two dimensions are defined according to the reference [14]:

$$P_n(x, y) = P_n(x) P_k(y), \tag{11}$$

and the Legendre Polynomials which having two-variables is written as

$$\frac{1}{\sqrt{1-2xs+s^2-2yt+t^2}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} L_{n,k}(x, y) s^n \, t^k. \tag{12}$$

One important benefit of the two variables Legendre Polynomials that M. A. Khan and M. P. Singh derived in [14] is that the polynomials can be produced by using a generating function in the two variables form, as equation (12) illustrates. The definition of two variables shifted Legendre Polynomials, $P_{n,k}(x, y)$ are defined as follows using the definition of two variables Legendre Polynomials derived by them:

$$\frac{1}{\sqrt{1-2(2x-1)s+s^2-2(2y-1)t+t^2}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{n,k}(x, y) s^n \, t^k. \tag{13}$$

Following Section 3 in [14], we have the following analytical expression for two variables shifted Legendre Polynomials:

$$P_{n,k}(x, y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{(1/2)_{n+k-r-j} (4x-2)^{n-2r} (4y-2)^{k-2j} (-1)^{r+j}}{r! \, j! \, (n-2r)! \, (k-2j)!} \tag{14}$$

where $(1/2)_{n+k-r-j}$ denotes the falling factorial. It should be noted that the well-known shifted Legendre polynomials can be expressed as $P_{n,0}(x, y) = P_n(x)$. These two variables shifted Legendre Polynomials can also be expressed using the following hypergeometric function:

$$P_{n,k}(x,y)=\frac{2^{n+k}(1/2)_{n+k}(2x-1)^{n}(2y-1)^{k}}{n!k!}\times F^{0,1,1}_{1,0,0}\left[\begin{matrix} -; & \frac{-n}{2},\frac{1-n}{2}; & \frac{-k}{2},\frac{1-k}{2}; & \frac{1}{(2x-1)^2}\frac{1}{(2y-1)^2} \\ \frac{1-2n-2k}{2}; & -; & -; & \end{matrix}\right]. \quad (15)$$

For function $f(x,y)=x^a y^b$ with a,b are positive integers, it is clear that one can obtain it by

$$f(x,y)=x^a y^b=\frac{1}{(a+1)(b+1)}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{n,k}\, P_{n,k}(x,y) \quad (16)$$

where

$$c_{n,k}=\frac{\int_0^1\int_0^1 f(x,y)P_{n,k}(x,y)dxdy}{\int_0^1\int_0^1 P_{n,k}(x,y)P_{n,k}(x,y)dxdy} \quad (17)$$

Ritz Method

In this Ritz method approach, a pair of variables is employed, specifically shifted Legendre polynomials, $P_{n,k}(x,t)$ as the polynomial basis function for solving the space-fractional diffusion wave equation in Caputo sense given in Equation (2).

This method involves transforming fractional order FDEs with initial and boundary conditions into an optimisation problem, and then expanding the solution using polynomial basis functions with unknown coefficients. In [15], the Ritz approach was employed to solve time-delay fractional optimal control problems (TDFOCPs). Function approximations are defined and controlled using initial and boundary conditions. The unknown state or control, as well as their delayed functions, are estimated first using the Ritz technique on the Müntz-Legendre polynomials basis, and then calculated using the fractional differential equation provided. In other study, Ritz approach is used to employ the nonlinear discretized motion equations that are constructed in accordance with the Timoshenko beam theory [16]. Combining the Ritz and collocation methods allows for the solution of two types of pantograph-type problems: pantograph fractional differential equations and pantograph fractional optimum control problems. The generalised Pell wavelet basis is chosen as the trial function in the Ritz approach [17].

As per the initial and boundary conditions, we denoted the approximate solution $\tilde{u}(x,t)$ as follows:

$$\tilde{u}(x,t)=\sum_{n=0}^{M}\sum_{k=0}^{M} K_{nk}\,\omega_{nk}(x,t)+\zeta(x,t),\ \ (x,t)\in[0,L]\times[0,T], \quad (18)$$

where $\omega_{nk}(x,t)=x(x-L)t^2P_{n,k}(x,t)$. The function by $\zeta(x,t)$ is the satisfier function. The expression $P_{n,k}(x,t)$ represents pair of Legendre polynomials that have been shifted in terms of two variables, whereby the expression K_{nk} indicates the coefficient that requires computation. The primary objective of employing the satisfier function is to ensure that it satisfies both the initial and boundary conditions.

In the Ritz method, it is important to find a proper satisfier function. The satisfier function is an arbitrary equation that satisfies all conditions of the problem. The interpolation method is often used to find the satisfier function. In this article, the satisfier equation, $\zeta(x,t)$ can be constructed via the steps as in [18]. Furthermore, the coefficients K_{nk} in Equation (18) can also be achieved by utilizing the inner product depicted as follows:

$$< F(\tilde{u}),P_{n,k}(x,t) \geq \int_0^L\int_0^T F(\tilde{u})P_{n,k}(x,t)\,dt\,dx=0, \quad (19)$$

where

$$F(\tilde{u})=D_t\tilde{u}(x,t)-\psi\,{}^{C}D_x^{\alpha}\tilde{u}(x,t)-f(x,t). \quad (20)$$

$P_{n,k}(x,t)$ denote the two variables shifted Legendre polynomials. By employing Equation (19), it is possible to establish a system of linear equations. Solving this system enables us to derive the coefficients of K_{nk}, where $n=0,\ldots,M$ and $k=0,\ldots,M$. Subsequently, by substituting the determined values of K_{nk} into Equation (18), an estimated solution for the fractional diffusion equation defined in the Caputo sense which outlined in Equation (2) can be obtained.

Error Analysis

This section gives a thorough study of the error analysis with regard to the proposed method. To proceed in such a study, the following lemma is useful [18].

Lemma 1. Let $u(x,t)$ be the solution of space-fractional diffusion equation in which $u(x,t) \in C^{m+1}[0,1] \times [0,1]$. Moreover, let $Y = span\{P_{0,m}(x), P_{1,m}(x), \dots, P_{m,m}(x)\} \subset L^2[0,1]$ and $Y' = span\{P_{0,m}(t), P_{1,m}(t), \dots, P_{m,m}(t)\} \subset L^2[0,1]$. By employing the two variables shifted Legendre polynomials, we have $u_m(x,t) = Y \times Y'$ as the best approximation of $u(x,t)$. Here, the error bound may be written as [18]:

$$\| u(x,t) - u_m(x,t) \|_2 \leq \frac{M}{(m+1)!} \frac{\sqrt{2}\, 2^{m+1}}{\sqrt{(2m+3)(m+2)}}. \tag{21}$$

Proof. The Taylor series is utilised to derive

$$u(x,t) = \sum_{j=0}^{\infty} \frac{1}{j!} \left[(x-a)\frac{\partial}{\partial x} + (t-b)\frac{\partial}{\partial t} \right]^j u(x,t). \tag{22}$$

To simplify, we assume that $a = b = 0$. In practical applications, the estimation of $u(x,t)$ up to m orders is expressed as follows:

$$u_m(x,t) = \sum_{j=0}^{m} \frac{1}{j!} \left[x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \right]^j u(x,t). \tag{23}$$

Provided that $u_m(x,t)$ refers to the best approximation $u(x,t)$ out of $Y \times Y'$, we may derive the following expression

$$\begin{aligned}
\|u(x,t) - u_m(x,t)\|_2 &\leq \left\| \frac{1}{(m+1)!} \left[x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \right]^{m+1} u(\xi,\eta) \right\|_2 \\
&= \left(\int_0^1 \int_0^1 \left(\frac{1}{(m+1)!} \left[x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \right]^{m+1} u(\xi,\eta) \right)^2 dxdt \right)^{1/2} \\
&= \left(\int_0^1 \int_0^1 \left[\sum_{r=0}^{m+1} \frac{\binom{m}{r}}{(m+1)!} \frac{\partial^{m+1} u(\xi,\eta) t^r x^{m+1-r}}{\partial t^r \partial x^{m+1-r}} \right]^2 dxdt \right)^{1/2} \\
&\leq \left(\int_0^1 \int_0^1 \left(\frac{M}{(m+1)!} (x+t)^{m+1} \right)^2 dxdt \right)^{1/2} \\
&= \frac{M}{(m+1)!} \left(\int_0^1 \int_0^1 (x+t)^{2m+2}\, dxdt \right)^{1/2} \\
&\leq \frac{M}{(m+1)!} \frac{\sqrt{2}\, 2^{m+1}}{\sqrt{(2m+3)(m+2)}},
\end{aligned}$$

in which $0 \leq \xi \leq x$ as well as $0 \leq \eta \leq t$, and $M \leq \sum_{r=0}^{m+1} \binom{m}{r} \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right]^{m+1}$ in which yields the error bound given above.■

Numerical Examples

The suggested approach from the previous part will be used in this section to get the numerical solutions for specific instances of the fractional diffusion equation given by Equation (2).

Example 1

A fractional diffusion equation is considered as [19]:

$$D_t \mathrm{u}(x,t) = x^{1.8} \Gamma(1.2) D_x \mathrm{u}(x,t) + 3x^2(2x-1)\mathrm{e}^{-t}, \tag{24}$$

with initial and boundary conditions

$$\mathrm{u}(x,0) = x^2(1-x),$$
$$u(0,t) = 0, \qquad u(1,t) = 0.$$

The exact solution is given by $u(x,t) = x^2(1-x)e^{-t}$. The satisfier function is $\zeta(x,t) = x^2(1-x)(1-t)$, which we found by employing the procedure explained in [18]. The numerical result is obtained using Maple 2023 by implementing the Ritz technique procedures described in the preceding section. We display our result graphically in Figure 1.

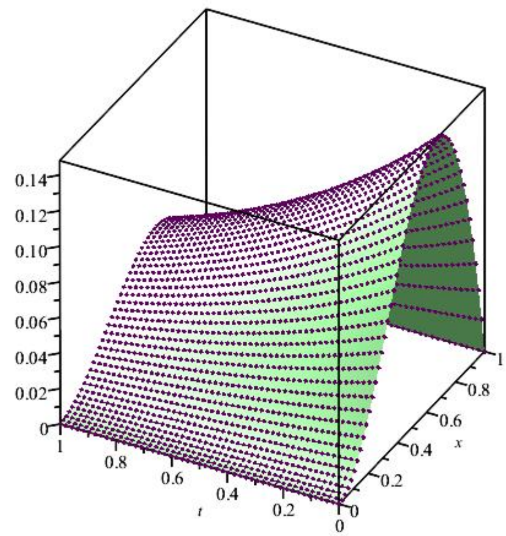

Figure 1. Diagram of the approximate and exact solution with $m = 2$ for Example 1

Figure 1 illustrate the graph for the absolute error by employing $m = 2$ for the Ritz approximation for Example 1. The green plane is the exact solution while the purple dots are our approximate numerical solution. Utilising the proposed method, we obtain the following approximation presented in Table 1. Here, we compare our method with the finite difference with Chebyshev collocation method.

Table 1. Comparison of the absolute errors obtained via the proposed method with $m = 2$ with the method in [19] with $m = 3$

(x,t)	Exact solution	Abs. error (proposed)	Abs. error [19]
(0.1,0.1)	0.008143536762	1.01295E-07	7.92E-07
(0.2,0.2)	0.026199384100	5.87510E-07	1.23E-06
(0.3,0.3)	0.046671547900	5.84220E-07	1.39E-06
(0.4,0.4)	0.064350724420	9.95960E-07	1.32E-06
(0.5,0.5)	0.075816332460	3.34620E-06	1.11E-06
(0.6,0.6)	0.079028875600	3.83660E-06	8.07E-07
(0.7,0.7)	0.072998039660	7.84620E-07	4.78E-07
(0.8,0.8)	0.057514107400	3.29450E-06	1.90E-07
(0.9,0.9)	0.032932142440	2.60854E-06	9.32E-09

According to the numerical results in Table 1, our suggested method appears to provide a more accurate approximation compared to the method which utilizes the finite difference method with shifted Chebyshev polynomials. The error for our proposed method with $m = 2$ terms falls mainly at 10^{-7} while the error for the comparison method with $m = 3$ terms is 10^{-6}.

This proves that our method is simpler in its implementation. Our method only needs a small number of two-variable shifted Legendre polynomials terms $(m = 2)$. Using numerical simulation, the Ritz method approximation significantly reduces the number of polynomial terms required to solve fractional diffusion equations. With only a few terms of two variable shifted Legendre polynomials, we successfully obtained an accurate numerical solution.

Example 2

A space-fractional diffusion equation in Caputo sense is considered as:

$$D_t u(x,t) = \frac{\Gamma(2.2)}{6} x^{2.8} D_x^{1.8} u(x,t) - (1+x)x^3 e^{-t}, \tag{25}$$

with initial and boundary conditions

$$u(x,0) = x^3,$$
$$\mathrm{u}(0,t) = 0, \;\; \mathrm{u}(1,t) = \mathrm{e}^{-t}.$$

The exact solution is given by $u(x,t) = x^3 \mathrm{e}^{-t}$, and the satisfier function is $\zeta(x,t) = x^3(1-t) + x(t-1) + x\mathrm{e}^{-t}$. The approximate solution for $m = 2$ is illustrated in the Figure 2 below.

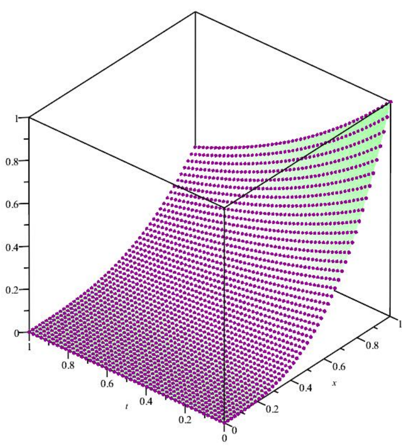

Figure 2. Diagram of the approximate and exact solution for Example 2 with $m = 2$

As we can see from Figure 2, our approximate solution is much closer to the exact solution. The data is as shown in the third column of Table 2. We extend this example by showing the approximate solution for different values of m in Table 2 below.

Table 2. Comparison of the absolute errors derived from the exact solution and the suggested method using $m = 2$ and $m = 3$

(x,t)	Exact solution	Absolute. error (M=2)	Absolute. error (M=3)
(0.1,0.1)	0.000904837418	8.89986E-07	4.25320E-08
(0.2,0.2)	0.006549846025	3.28210E-06	5.81750E-08
(0.3,0.3)	0.020002091960	2.98388E-06	1.30860E-07
(0.4,0.4)	0.042900482940	2.38925E-06	3.38640E-07
(0.5,0.5)	0.075816332460	9.53698E-06	2.37460E-07
(0.6,0.6)	0.118543313400	1.17619E-05	1.74300E-07
(0.7,0.7)	0.170328759200	5.41030E-06	4.44600E-07
(0.8,0.8)	0.230056429600	4.34210E-06	1.79000E-07
(0.9,0.9)	0.296389281900	5.68710E-06	1.84300E-07

In Table 2, we can see that with the increase in scale level from $m = 2$ to $m = 3$, the amount of absolute error decreased to a good extent. The value of m indicated the term values in the two-variables shifted Legendre polynomials. $m = 2$ means the two variables of shifted Legendre polynomials are limited up to quadratic power. As m increases to $m = 3$, the accuracy of the approximate solution improves, as shown in Table 2. Numerical simulation demonstrates that the Ritz method effectively decreases the polynomial terms needed to solve the space-fractional diffusion equations. The method is stable, since the errors decrease with an increase in m.

Discussion

We solved space-fractional diffusion equations by employing the Ritz approximation approach using shifted two-variable Legendre polynomials. We achieved an accurate numerical solution by utilising only a few of terms from shifted Legendre polynomials in two variables. The Ritz method was selected due to its high degree of flexibility, which allows for easy establishment of initial and boundary conditions.

The Ritz method's idea is that it converts the FDE into a system of algebraic equations, making it simpler to solve. Which implies that we will receive the answers for each point simultaneously. This method has advantage over other numerical methods that need iteration, linearization, or discretization.

In Table 3, we list the computational order for the numerical results. $Order = log_2 \frac{MAE_m}{MAE_{m+1}}$ having m elements of basis polynomials.

Table 3. Maximum absolute error (MAE) and computational order obtained for all examples

Example no.	m	MAE	Order
1	2	5.52794E-06	-
	3	2.1165E-07	4.706989563
	4	1.0267E-07	1.043665798
2	2	1.168307E-05	-
	3	3.899E-07	4.905171461
	4	4.829E-08	3.013307786

We may infer from all the numerical results presented in the tables that our numerical solutions agree well with the exact solutions, and that the accuracy of the approximate solutions can be increased by using more shifted Legendre polynomials terms in the basis function.

Conclusions

The Ritz method resembles the solution with respect to fractional diffusion equations as a linear combination with regard to basis functions. This allows for great flexibility in choosing the basis functions to best match the problem at hand. As far as we know, utilizing the Ritz method for the space-fractional diffusion equation is novel. Analysis and implementation led us to the conclusion that the suggested approach for approximating the solution with regard to the space-fractional diffusion equations yields promising results. In addition, this method offers a reduction in computational workload compared to classical methods while maintaining a high level of accuracy in the numerical results. Mathematicians and scientists working in the field of fractional calculus will greatly benefit from it.

Conflicts of Interest

The authors affirm that there is no conflict of interest associated with this paper's publication.

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References

[1] Rehman, A. U., Awrejcewicz, J., Riaz, M. B., & Jarad, F. (2022). Mittag-Leffler form solutions of natural convection flow of second grade fluid with exponentially variable temperature and mass diffusion using Prabhakar fractional derivative. *Case Studies in Thermal Engineering, 34*, 102018.

[2] Manikandan, K., Serikbayev, N., Aravinthan, D., & Hosseini, K. (2024). Solitary wave solutions of the conformable space–time fractional coupled diffusion equation. *Partial Differential Equations in Applied*

Mathematics, 100630.

[3] Hashemi, M. S., Mirzazadeh, M., Bayram, M., & El Din, S. M. (2023). Numerical approximation of the Cauchy non-homogeneous time-fractional diffusion-wave equation with Caputo derivative using shifted Chebyshev polynomials. *Alexandria Engineering Journal, 81*, 118–129.

[4] Alqhtani, M., Owolabi, K. M., Saad, K. M., & Pindza, E. (2022). Efficient numerical techniques for computing the Riesz fractional-order reaction-diffusion models arising in biology. *Chaos, Solitons & Fractals, 161*, 112394.

[5] Abidemi, A., Owolabi, K. M., & Pindza, E. (2022). Modelling the transmission dynamics of Lassa fever with nonlinear incidence rate and vertical transmission. *Physica A: Statistical Mechanics and Its Applications, 597*.

[6] Olayiwola, M. O., Alaje, A. I., & Yunus, A. O. (2024). A Caputo fractional order financial mathematical model analyzing the impact of an adaptive minimum interest rate and maximum investment demand. *Results in Control and Optimization, 14*, 100349.

[7] Cengizci, S., & Uğur, Natesan, S. (2024). A stabilized finite element formulation with shock-capturing for solving advection-dominated convection–diffusion equations having time-fractional derivatives. *Journal of Computational Science, 76*, 102214.

[8] Yang, Z., Chen, X., Chen, Y., & Wang, J. (2024). Accurate numerical simulations for fractional diffusion equations using spectral deferred correction methods. *Computers & Mathematics with Applications, 153*, 123–129.

[9] Xu, Y., Sun, H., Zhang, Y., Sun, H.-W., & Lin, J. (2023). A novel meshless method based on RBF for solving variable-order time fractional advection-diffusion-reaction equations in linear or nonlinear systems. *Computers & Mathematics with Applications, 142*, 107–120.

[10] Du, H., & Chen, Z. (2022). Adaptive meshless numerical method of solving 2D variable order time fractional mobile-immobile advection-diffusion equations. *Computers & Mathematics with Applications, 124*, 42–51.

[11] Salim, T. O. (2009). Some properties relating to the generalized Mittag-Leffler function. *Advances in Applied Mathematics and Analysis, 4*(1), 21–30.

[12] Shi, P. (2017). Fractional derivatives of some functions. In *Robust Adaptive Control for Fractional-Order Systems with Disturbance and Saturation* (P. Shi, Ed.).

[13] Dattoli, G., Germano, B., Martinelli, M. R., & Ricci, P. E. (2011). A novel theory of Legendre polynomials. *Mathematical and Computer Modelling, 54*(1–2), 80–87.

[14] Khan, M. A., & Singh, M. P. (2010). A study of two variables Legendre polynomials. *Pro Mathematica, 24*(47–48), 201–223.

[15] Mamehrashi, K. (2023). Ritz approximate method for solving delay fractional optimal control problems. *Journal of Computational and Applied Mathematics, 417*, 114606.

[16] Tao, Y., Chen, C., Zhou, J., & Arvin, H. (2023). Principal parametric resonance analysis of a rotating agglomerated nanocomposite beam employing the Chebyshev–Ritz method. *Engineering Analysis with Boundary Elements, 150*, 400–412.

[17] Sabermahani, S., Ordokhani, Y., & Razzaghi, M. (2023). Ritz-generalized Pell wavelet method: Application for two classes of fractional pantograph problems. *Communications in Nonlinear Science and Numerical Simulation, 119*, 107138.

[18] Md Nasrudin, F. S., Phang, C., & Kanwal, A. (2023). Fractal-fractional advection–diffusion–reaction equations by Ritz approximation approach. *Open Physics, 21*(1), 20220221.

[19] Agarwal, P., & El-Sayed, A. (2018). Non-standard finite difference and Chebyshev collocation methods for solving fractional diffusion equations. *Physica A: Statistical Mechanics and Its Applications, 500*, 40–49.