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RESEARCH ARTICLE

Exploring The Boundaries of Uncertainty: Interval Valued Pythagorean Neutrosophic Set and Their Properties

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Abstract The interval-valued Pythagorean fuzzy set (IVPFS) presents a novel approach to tackling vagueness and uncertainty, while neutrosophic sets, a broader concept encompassing of fuzzy sets and intuitionistic fuzzy sets, are tailored to depict real-world data characterized by uncertainty, imprecision, inconsistency, and incompleteness. Additionally, the development of Interval Value Neutrosophic Sets (IVNS) enhances precision in handling problems involving a range of numbers within the real unit interval, rather than focusing solely on a single value. However, despite these advancements, there is a deficiency in research addressing practical implementation challenges, conducting comparative analyses with existing methods, and applying these concepts across various fields. This study aims to bridge this research gap by proposing a novel concept based on the Interval Valued Pythagorean Neutrosophic Set (IVPNS), which is a generalization of the IVPFS and INS. The development of IVPNS provides a more comprehensive framework for handling uncertainty, ambiguity, and incomplete information in various fields, leading to more robust decision-making processes, improved problem-solving capabilities, and better management of complex systems. Furthermore, this research introduces the algebraic operations for IVPNS, including addition, multiplication, scalar multiplication, and exponentiation and provides a comparative analysis with IVPFS and IVNS. The study incorporates illustrative numerical examples to demonstrate these operations in practice. Additionally, this study provides and rigorously proves the algebraic properties of IVPNS, specifically discussing their commutative and associative properties. This validation ensures compliance with established conditions for IVPNS, reinforcing their theoretical soundness and practical applicability.

Keywords: Interval valued pythagorean fuzzy set, interval valued neutrosophic set, interval valued pythagorean neutrosophic set, algebraic operations.

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Introduction

In many complicated problems, such as engineering, social and economic contexts, computer science, medical science, and more, the data involved may not always have crisp, precise, and deterministic attributes due to their inherent vagueness. A variety of theories have been employed to address most of these complex problems. To cater to the problems, Zadeh [40] introduced the concept of fuzzy set, where the idea of a membership function is denoted by μ, which gives the range [0,1] to indicate the degree of belongingness to the set under consideration. A membership function in a fuzzy set is defined to characterize the extent to which an element belongs to a class. The number of membership values is a number between 0 and 1, where 0 indicates that the element does not belong to a class, 1 indicates that it does, and other values indicate the degree of membership in a class.

Atanassov and Stoeva [3] proposed the Intuitionistic Fuzzy Set (IFS) as a generalization of the fuzzy set, where it caters to the concept of the degree of membership function $\mu_A(x) \in [0,1]$ of each element $x \in X$,and also considers the degree of non-membership function $\mu_A(x) \in [0,1]$, but such that $0 \le \mu_A(x) + v_A(x) \le 1$.

Atanasov and Gargov [2], and Atanassov [4] generalized the notion of intuitionistic fuzzy sets in terms of interval-valued fuzzy sets. The new models are called Interval-Valued Intuitionistic Fuzzy Sets (IVIFS). They also present some basic preliminaries of IVIFS theory. Using this concept, the truth membership and falsity – membership was extended to interval numbers in the IVIFS; only incomplete information, not ambiguous or inconsistent information, can be handled by IFSs and IVIFSs. In some other situations, $T_A(x) + F_A(x) \ge 1$, it cannot be captured by the IFS concept. To address this problem, researchers developed the Pythagorean fuzzy set (PFS) as a generalized version of IFS. The idea of PFS is more adequate and correct than IFS and can be utilized to characterize uncertain information.

In 2013, Yager [36] proposed PFSs to deal with vagueness. It considers the membership grade and non-membership. Yager, [37] constructs both memberships where π is the Pythagorean fuzzy index satisfy the conditions $T_A(x) + F_A(x) \le 1$ and $T_A(x) + F_A(x) \ge 1$, it follows that $[T_A(x)]^2 + [F_A(x)]^2 + \pi^2 = 1$. Yager and Abbasov [38] presented some concepts associated with PFSs to investigate the relationship between complex numbers and Pythagorean Fuzzy Numbers (PFNs). Interval-valued Pythagorean fuzzy sets (IVPFSs) presented by Garg [13] to improve score function in ranking order. Li and Zeng [21], proposed a variety of distance measures designed for Pythagorean fuzzy sets and Pythagorean fuzzy numbers, considering all four parameters of Pythagorean fuzzy sets and provided a numerical example to demonstrate the effectiveness and practicality of these distance measures. The cosine function is measured by considering the degree of membership, degree of non-membership, and degree of hesitation in Pythagorean fuzzy sets (PFSs) defined by wei and wei [35], and its applications are studied and applied in pattern recognition and medical diagnosis. The same year (2018) other applications based on similarity measures in concept PFSs were studied by Li [21] and Sarkar and Biswas.

In dealing with multicriteria problems, experts may find it challenging to assess their evaluations by a precise value accurately. To solve this issue, Peng, and Yang [24] generalized the PFSs model to interval-valued Pythagorean fuzzy sets (IVPFSs) and discussed the new operators and scores on its. The maximizing deviation approach for the multi-criteria Group decision-making (MCGDM) problem was developed by Liang *et al*. [22] and is based on an interval-valued Pythagorean fuzzy-weighted aggregating operator. Garg [14] introduced a new accuracy function based on interval-valued Pythagorean fuzzy sets for solving the Multi Criteria Decision Making (MCDM) problem and in 2017, he proposed a new improved score function for an interval-valued Pythagorean fuzzy set-based TOPSIS method. Rahman *et al*. [27] discussed interval-valued Pythagorean fuzzy geometric aggregation operators and their application to the MCGDM problem.

Some point operator-based similarity measures of PFSs are developed by Biswas and Sarkar [6], and a MADM problem is solved using the proposed similarity measures. Several distance and similarity measures of PFSs were developed by Zeng *et al*. [41] which were further utilized in the MADM problem. Mohd *et al*, [23], proposed the integration of Pythagorean fuzzy AHP to find the weight for each criterion and identify the most important criteria and the Pythagorean fuzzy VIKOR approach used to rank the alternatives of the green supplier development programs and suggest which program is the best. Pythagorean fuzzy sets have attracted great attention from many researchers, and subsequently, the concept has been applied to many application areas such as decision-making, aggregation operators, and information measures. Ho, Lin, and Chen [15] propose a comprehensive IVPF correlation-based closeness index to balance the consequences between ultra-approach orientation and ultra-avoidance orientation and acquire the ultimate compromise solution for decision support and aid. The feasibility and practicability of the developed methodology are illustrated by a practical MCDA problem of rehabilitation treatment for hospitalized patients with acute stroke. To control the spreading rate of the COVID-19 virus from place to place, Rahman *et al*. [28] provide novel mathematical techniques based on complex Pythagorean Fuzzy Sets (CPyFSs) and their operators, such as the Complex Pythagorean Fuzzy Einstein weighted geometric operator and the induced complex Pythagorean fuzzy Einstein hybrid geometric operator. Most of the established measures possess linearity in the Pythagorean fuzzy set, this concept cannot be incorporated to approximate the nonlinear nature of information as it might lead to counter-intuitive results. To cater to this limitation Dutta *et al.* [12] define two nonlinear distances, namely generalized chordal distance and non-Archimedean chordal distance, for PFSs and apply it in medical diagnosis including the COVID-19 problem.

However, the IFs, IVFs set and PFSs are only able to handle incomplete values that only consider both the truth membership (membership) and falsity membership (non-membership) values. It is unable to deal with the ambiguity and unpredictability of information found in belief systems. A mathematical approach that addresses issues of handling imprecise, indeterminate, and inconsistent data is the theory of the neutrosophic set. The concepts of the neutrosophic set were introduced by Smarandache [31] in 2006 as an innovative mathematical tool for dealing with issues involving inconsistent, imprecise, and indeterminate data specifically. Neutrosophic sets are generalizations of fuzzy sets and intuitionistic fuzzy sets. In 2010, Smarandache [32] further demonstrated how intuitionistic fuzzy sets are a continuation of the neutrosophic set. The neutrosophic set has three elements truth (T), indeterminacy (I) (whether true or false), and falsity (F). It is important to note that T, I, and F are not restricted to intervals, but they can be various sets, including discrete, continuous, open, closed, half-open – open and half–closed intervals, or combinations of these sets through intersections or unions. The triplet (T, I, F) that stands for the truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively, provides the neutrosophic set, where] 0, 1 + [represents the non-standard interval. The concept of a neutrosophic set effectively manages indeterminate data, whereas fuzzy sets and intuitionistic fuzzy sets fail when dealing with indeterminate relationships.

Broumi *et al*. [10] extended the concept of rough neutrosophic sets in 2024 and explained the fundamental idea of neutrosophic sets and how they work. In 2015 [7,8], they combined the concepts of the interval-valued neutrosophic set and the interval-valued rough set and proposed a hamming distance between the lower and upper approximations of the interval-valued neutrosophic set. Dung *et al*. [11] applied the interval neutrosphic set in the TOPSIS techniques to express the importance of weights and ratings of the personnel selection criterion. Iram and Cengiz [16] developed the Neutrosophic Fuzzy Analytic Network for multicriteria decision-making. Kahraman *et al*. [18] proposed the Neutrosphic Analytic Hierarchy Process (NAHP) integrated with neutrosophic Data Envelopment Analysis (DEA), where the weighted input and outputs of the DA method were calculated using Neutrosophic AHP. Smarandache *et al*. [33] studied the notion and some properties of interval neutrosophic sets and defined the concept of a rough neutrosophic set. Quynh *et al*. [26] employ the Technique for Order of Preference by Similarity to the Ideal Solution (TOPSIS) approach based on single-valued complex neutrosophic sets (SVCNSs), where the importance weights of criteria, the ratings of alternatives, and their aggregated values are assessed and evaluated using SVCNSs. Poonia and Bajaj [25] offer a complicated neutosophic matrix to address the uncertainty issue in 2021. Based on the recommended matrix, they provided several algebraic operations, such as addition, union, subtraction, and many others. Khatter [20], introduced the concept of an interval–valued trapezoidal neutrosophic set and presented the score and accuracy of the function. Khan *et al*. [19] defined different operations on the neutrosophic soft matrix and applied it to decision–making problems. A few ideas and characteristics of neutrosophic soft matrices are examined by ShebaMaybell and Shanmugapriya [29]. They provide a unique fuzzy neutrosophic soft complement matrix, the trace, and some complement functions of the fuzzy neutrosophic soft matrices. The idea of the Pythagorean neutrosophic set is a mixture of the concepts of the Pythagorean fuzzy set and the neutrosophic set, as proposed by Ajay and Chellamani [1]. In this idea, they extend the concept of soft sets and define a new concept of Pythagorean Neutrosophic soft sets. The definition and some properties have been discussed and given in this paper.

The triangular fuzzy neutrosophic number grey relational analysis method is proposed by Yao and Ran [39] where this method is a hybrid between the traditional grey relational analysis and triangular fuzzy neutrosophic sets. The neutrosophic environment with interval numbers can handle the situations efficiently. By considering a de-neutrosophication technique in crisp numbers, Sinika and Ramesh [30 utilize the interval-valued trapezoidal neutrosophic numbers for more flexibility. The multi-valued neutrosophic matrix was discussed by Jene Seles Martina and Deepa [17]. This discussion involved the operations and properties of the suggested matrix, and the linguistic variable was created and applied in the neutrosophic simplified TOPSIS method. The new operations denoted by Fermatean, neutrosophic matrices (FNMs), and some standard properties of addition, including scalar multiplication and exponentiation, were already discussed by Broumi and Prabha [9]. The Analytic Hierarchy Process (AHP) has been used in numerous MCDM scenarios. To solve the MCDM problem of factors affecting floods, Awang *et al.* [5] defined a new combination of Interval Interval Neutrosophic Weighted Averaging (INWA) aggregation operators into the AHP method.

Stephen and Helen [34] introduced research on interval-valued Pythagorean sets and neutrosophic sets. However, their paper has a drawback based on the comprehensive definition of IVPNS, as it fails to satisfy the Pythagorean condition. Our study, on the other hand, cater the lack of Stephen and Helen

[34] focuses on the practical application of interval-valued combinations of Pythagorean sets and neutrosophic sets, specifically examining interval-valued Pythagorean sets and interval-valued neutrosophic sets. The improvised IVPNS will contribute to the advancement of both theory and methodologies for addressing uncertainty and imprecision across diverse decision-making and modelling contexts. It, provide a means to represent uncertainty through intervals, thereby facilitating more informed and resilient analyses and solutions. Our paper aims to present the algebraic function and characteristics of the Interval Valued Pythagorean Neutrosophic Set (IVPNS). Certain properties of the operations defined on IVPNS have been established through proof.

Preliminaries

In this section, the basic concepts, general definitions, and operations related to Pythagorean Fuzzy Numbers (PFNs), Interval Valued Pythagorean Fuzzy Numbers (IVPFNs), Neutrosophic set, and Interval Neutrosophic Set (INS) related to the study are presented.

Pythagorean Fuzzy Number

Definition 2.1[42]: Let X be a fixed set, then a PFS in X is an object defined as

$$P = \{< x, (\mu_P(x), \nu_P(x)) > / x \in X\}$$

Where the function $\mu_p : X \to [0,1]$ represents the membership degree and $\nu_P : X \to [0,1]$ is the non-membership degree of the element $x \in X : [0,1]$ to P, and it holds that $0 \le (\mu_P(x))^2 + (\nu_P(x))^2 \le 1$, in addition, $\pi P(x) = \sqrt{1 - \mu_p^2(x) - \nu_p^2(x)}$ is named the degree of indeterminacy of x to the set P, and a PFS is defined by Zhang and Xu [74] as $P = (\mu_P(x), \nu_P(x))$ denoted by $\beta = (\mu\beta, \nu\beta)$ for convenience.

Interval – Valued Pythagorean Fuzzy Number

Definition 2.2 [42]: Let X be an ordinary finite nonempty set, then a IVPFS $\tilde{p}$ in X is a defined as:

$$\tilde{P} = \left\{\left\langle x, [\tilde{\mu}^L_{\tilde{P}}(x), \tilde{\mu}^U_{\tilde{P}}(x)], [\tilde{\nu}^L_{\tilde{P}}(x), \tilde{\nu}^U_{\tilde{P}}(x) \middle| x \in X \right\rangle\right\}$$

Where the function $\tilde{\mu}_{\tilde{P}}(x) \subseteq [0,1]$ and $\tilde{\nu}_{\tilde{P}}(x) \subseteq [0,1]$ are interval values, the lower and upper interval $\tilde{\mu}_{\tilde{P}}(x)$ are $\tilde{\mu}^L_{\tilde{P}}(x)$ and $\tilde{\mu}^U_{\tilde{P}}(x)$, while the lower and upper interval $\tilde{\nu}_{\tilde{P}}(x)$ are $\tilde{\nu}^L_{\tilde{p}}(x)$ and $\tilde{\nu}^U_{\tilde{p}}(x)$,respectively and satisfy $\tilde{\mu}^U_{\tilde{P}}(x) + \tilde{\nu}^U_{\tilde{P}}(x) \le 1$ and $(\tilde{\mu}^U_{\tilde{P}}(x))^2 + (\tilde{\nu}^U_{\tilde{P}}(x))^2 \le 1$. The interval valued membership values for every $x \in X$, $\tilde{\pi}_{\tilde{P}}(x) = [\tilde{\pi}^U_{\tilde{P}}(x), \tilde{\pi}^L_{\tilde{P}}(x)]$ is called Interval valued Pythagorean set of x to $\tilde{P}$,where

$$\tilde{\pi}_{\tilde{P}}(x) = [\tilde{\pi}^U_{\tilde{P}}(x), \tilde{\pi}^L_{\tilde{P}}(x)]$$
$$= \left[\sqrt{1 - (\tilde{\mu}^L_{\tilde{P}}(x))^2 - (\tilde{\nu}^L_{\tilde{P}}(x))^2}, \sqrt{1 - (\tilde{\mu}^U_{\tilde{P}}(x))^2 - (\tilde{\nu}^U_{\tilde{P}}(x))^2}\right]$$

Basic Operations of IVPFS

Definition 2.3 [42]: $\tilde{\eta} = \tilde{P}([\tilde{b}^L_{\tilde{\eta}}, \tilde{b}^U_{\tilde{\eta}}], [\tilde{s}^L_{\tilde{\eta}}, \tilde{s}^U_{\tilde{\eta}}])$, $\tilde{\eta}_1 = \tilde{P}([\tilde{b}^L_{\eta_1}, \tilde{b}^U_{\eta_1}], [\tilde{s}^L_{\tilde{\eta}_1}, \tilde{s}^U_{\tilde{\eta}_1}])$, and $\tilde{\eta}_2 = \tilde{P}([\tilde{s}^L_{\tilde{\eta}_1}, \tilde{s}^U_{\tilde{\eta}_1}], [\tilde{s}^L_{\tilde{\eta}_2}, \tilde{s}^U_{\tilde{\eta}_2}])$ be three IVPNS, the basic algebra operation is given as follows:

1) $\tilde{\eta}_1 \oplus \tilde{\eta}_2 = [\sqrt{(\tilde{b}^L_{\eta_1})^2 + (\tilde{b}^L_{\eta_2})^2 - (\tilde{b}^L_{\eta_1})^2 (\tilde{\mu}^L_{\eta_2})^2}, \sqrt{(\tilde{b}^U_{\eta_1})^2 + (\tilde{b}^U_{\eta_2})^2 - (\tilde{b}^U_{\eta_1})^2 (\tilde{b}^U_{\eta_2})^2}], [\tilde{s}^L_{\tilde{\eta}_1} \tilde{s}^L_{\tilde{\eta}_{21}}, \tilde{s}^U_{\tilde{\eta}_1} \tilde{s}^U_{\tilde{\eta}_{21}}]$

2) $\tilde{\eta}_1 \otimes \tilde{\eta}_2 = [\tilde{b}^L_{\eta_1} \tilde{b}^L_{\eta_2} \tilde{b}^U_{\eta_1} \tilde{\mu}^U_{\eta_2}], [\sqrt{(\tilde{s}^L_{\eta_1})^2 + (\tilde{s}^L_{\eta_2})^2 - (\tilde{s}^L_{\eta_1})^2 (\tilde{s}^L_{\eta_2})^2}, \sqrt{(\tilde{s}^U_{\eta_1})^2 + (\tilde{s}^U_{\eta_2})^2 - (\tilde{s}^U_{\eta_1})^2 (\tilde{s}^U_{\eta_2})^2}]$

3) $[\sqrt{1 - \left(1 - (\tilde{b}^L_{\tilde{\eta}})^2\right)^k}, \sqrt{1 - \left(1 - (\tilde{b}^U_{\tilde{\eta}})^2\right)^k}], [(\tilde{s}^L_{\tilde{\eta}})^k, (\tilde{s}^U_{\tilde{\eta}})^k]$, $(k > 0)$

4) $\tilde{\eta}^k = [(\tilde{b}^L_{\tilde{\eta}})^k, (\tilde{b}^U_{\tilde{\eta}})^k], [\sqrt{(1-(1-\tilde{s}^L_{\tilde{\eta}})^2)^k}, \sqrt{(1-(1-\tilde{s}^U_{\tilde{\eta}})^2)^k}], (k>0)$

Neutrosophic Set

Definition 2.4 [8]: Let U be a Universe discourse; then the neutrosophic sets A is an object having the form:

$$A = \{x, \langle b_A(x), I_A(x), s_A(x) \rangle : x \in U\}$$

Which is characterized by the degree of truth-membership function $b_A(x)$, indeterminacy membership function $I_A(x)$ and falsity-membership function $s_A(x)$ where $b_A(x), I_A(x), s_A(x) : U \to]^- 0, 1^+[$ of the elements $x \in U$ to the set A with the condition. $0^- \le b_A(x) + I_A(x) + s_A(x) \le 3^+$

From a philosophical point of view, the neutrosophic set takes the value from real standard or nonstandard, so instead of $]^- 0, 1^+[$ we need to take the interval $[0,1]$ for technical application, since $]^- 0, 1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems. The $b_A(x)$ and $I_A(x)$ are dependent on neutrosophic components and $s_A(x)$ is an independent component.

Interval Valued Neutrosophic Set (IVNS)

Definition 2.5 [43]: Let U be a universe of discourse (objects) with generic elements in U denote by x .Then the INS A in x is characteristics by truth-membership function $b_A(x)$, indeterminacy membership function $I_A(x)$ and falsity-membership function $s_A(x)$, where, $b_A(x)$ and $I_A(x)$ are dependent neutrosophic component and $s_A(x)$ is independent component. Every point x in X, we have that $b_A(x) = [\inf b_A(x), \sup b_A(x)]$, $I_A(x) = [\inf I_A(x), \sup I_A(x)]$, $s_A(x) = [\inf s_A(x), \sup s_A(x)] \subset [0,1]$ and $0 \le \sup b_A(x) + \sup I_A(x) + \sup s_A(x) \le 3, x \in U$ and every function lies between [0,1] in U .

For convenience, can use $x = [b^U_A, b^L_A], [I^U_A, I^L_A], [s^U_A, s^L_A]$.

Basic operations of Interval Valued Neutrosophic set (IVNS)

Definition 2.6 [43]: Let, $N_1 = ([b^L_{N_1}, b^U_{N_1}], [I^L_{N_1}, I^U_{N_1}[s^L_{N_1}, s^U_{N_1}])$ $N_2 = ([b^L_{N_2}, b^U_{N_2}], [I^L_{N_2}, I^U_{N_2}[s^L_{N_2}, s^U_{N_2}])$ be two IVNS, the basic algebra operation is given as follows:

1) $N_1 \oplus N_2 = [b^L_{N_1} + b^L_{N_2} - b^L_{N_1} b^L_{N_2}, b^U_{N_1} + b^U_{N_2} - b^U_{N_1} b^U_{N_2}], [I^L_{N_1} I^L_{N_2}, I^U_{N_1} I^U_{N_2}], [s^L_{N_1} s^L_{N_2}, s^U_{N_1} s^U_{N_2}]$
2) $N_1 \otimes N_2 = [b^L_{N_1} b^L_{N_2}, b^U_{N_1} b^U_{N_2}], [I^L_{N_1} + I^L_{N_2} - I^L_{N_1} I^L_{N_2}, I^U_{N_1} + I^U_{N_2} - I^U_{N_1} I^U_{N_2}],\ [s^L_{N_1} + s^L_{N_2} - s^L_{N_1} s^L_{N_2}, s^U_{N_1} + s^U_{N_2}$
3) $kN = [1-(1-b^L_N)^k, 1-(1-b^U_N)^k], [(I^L_N)^k, (I^U_N)^k][(s^L_N)^k, (s^U_N)^k], (k>0)$
4) $N^k = [s^L_N, s^U_N], [1-I^L_N, 1-I^U_N], [b^L_N, b^U_N](k>0)$

Interval Valued Pythagorean Neutrosophic Set ((Stephen and Helen [34])

Definition 3.1: An interval – valued neutrosophic Pythagorean set having truth T and Falsity F as a dependent interval – valued neutrosophic components and indeterminacy I as independent interval – valued neutrosophic components in a universe can be defined as $N = \{< s, [T^L_N, T^U_N], [I^L_N, I^U_N], [F^L_N, F^U_N] >; s \in S\}$ on S having the form $\left[\frac{T^L_N + T^U_N}{2}\right] + \left[\frac{I^L_N + I^U_N}{2}\right] + \left[\frac{F^L_N + F^U_N}{2}\right] \le 2.$ Here $[T^L_N, T^U_N], [I^L_N, I^U_N], [F^L_N, F^U_N]$ represents the lower and upper bound of the truth, indeterminacy, and falsity membership degrees.

According to **Definition 3.1** by Stephen and Helen [34], it does not consider the condition in Pythagorean for example we have an IVPNS number $[T_N^L, T_N^U],[I_N^L, I_N^U],[F_N^L, F_N^U] = [0.10,0.30], [0.50,0.60], [0.70,0.80]$, this interval must satisfy the condition as follows:

i. $0 \le T_N^U(x) + F_N^U(x) \le 1$

$T_N^U(x) + F_N^U(x) = 0.3 + 0.8 = 1.10$,

ii. $0 \le T_N^U(x) + I_N^U(x) + F_N^U(x) \le 2$

$T_N^U(x) + I_N^U(x) + F_N^U(x) = 0.3 + 0.6 + 0.8 = 1.70$

In this example, Stephen and Helen [34] only consider the condition of INS, leaving the condition of IVPS. Where the $T_N^U(x) + F_N^U(x) = 0.3 + 0.8 = 1.10 > 1$ is not satisfied with the condition of IVPFS $0 \le T_N^U(x) + F_N^U(x) \le 1$. To resolve this drawback, we introduce the development of the new definition of IVPNS which satisfies the condition of both IVPS and IVNS presented in Definition 3.2.

Improvised Interval Valued Pythagorean Neutrosophic Set

In this section, we present the new definition of the development of IVPNS, its algebraic operations and their properties.

Definition 3.2 Let U be a universe of discourse (objects) with generic elements in U denote by x. Then the Interval Valued Pythagorean Neutrosophic Sets (IVPNS) $\widetilde{A}$ is an object having the form

$$\widetilde{A} = \left\{ < x, [(\widetilde{b}_{\widetilde{A}}^L(x), \widetilde{b}_{\widetilde{A}}^U(x)], [I_{\widetilde{A}}^L(x), I_{\widetilde{A}}^U(x)], [\widetilde{s}_{\widetilde{A}}^L(x), \widetilde{s}_{\widetilde{A}}^U(x)] > \middle| x \in X \right\}$$

Then the IVPNS $\widetilde{A}$ in x is characterized by the truth-membership function $\widetilde{b}_{\widetilde{A}}(x)$, indeterminacy membership function $I_{\widetilde{A}}(x)$ and falsity-membership function $\widetilde{s}_{\widetilde{A}}(x)$. For each point x in X, the lower and upper intervals are $\widetilde{b}_{\widetilde{A}}(x) = [\widetilde{b}_{\widetilde{A}}^L((x), \widetilde{b}_{\widetilde{A}}^U(x)] \subseteq [0,1]$, $I_{\widetilde{A}}(x) = [I_{\widetilde{A}}^L(x), I_{\widetilde{A}}^U(x)] \subseteq [0,1]$, and $\widetilde{s}_{\widetilde{A}}(x) = [\widetilde{s}_{\widetilde{A}}^L(x), \widetilde{s}_{\widetilde{A}}^U(x)] \subseteq [0,1]$, respectively. The $b_A(x)$ and $I_A(x)$ are dependent on neutrosophic components and $s_A(x)$ is an independent component. This satisfies all the conditions:

i. $0 \le \widetilde{b}_{\widetilde{A}}^U(x) + \widetilde{s}_{\widetilde{P}}^U(x) \le 1$

ii. $0 \le \left(\widetilde{b}_{\widetilde{A}}^U(x)\right)^2 + \left(\widetilde{b}_{\widetilde{A}}^U(x)\right)^2 \le 1$

iii. $0 \le (\widetilde{b}_{\widetilde{A}}^U(x))^2 + (I_{\widetilde{A}}^U(x))^2 + (\widetilde{s}_{\widetilde{A}}^U(x))^2 \le 2$, and

iv. $0 \le \widetilde{b}_{\widetilde{A}}^U(x) + I_{\widetilde{A}}^U(x) + \widetilde{s}_{\widetilde{A}}^U(x) \le 2, x \in U$.

For every $x \in X$, $\widetilde{\pi}_{\widetilde{A}}(x) = \left[\widetilde{\pi}_{\widetilde{A}}^U(x), \widetilde{\pi}_{\widetilde{A}}^L(x)\right]$ is called IVPNS index of x to $\widetilde{P}$,where $\widetilde{\pi}_{\widetilde{A}}^L(x) = \sqrt{1 - \widetilde{b}_{\widetilde{A}}^U(x))^2 - (I_{\widetilde{A}}^U(x))^2 - (\widetilde{s}_{\widetilde{A}}^U(x))^2}$ $\widetilde{\pi}_{\widetilde{A}}^L(x) = \sqrt{1 - \widetilde{b}_{\widetilde{A}}^L(x))^2 - (I_{\widetilde{A}}^L(x))^2 - (\widetilde{s}_{\widetilde{A}}^L(x))^2}$.

Algebraic Operations of IVPNS

Definition 3.3 Let $C = ([\widetilde{b}_C^L, \widetilde{b}_C^U], [\widetilde{I}_C^L, \widetilde{I}_C^U], [\widetilde{s}_C^L, \widetilde{s}_C^U]$ and $D = ([\widetilde{b}_D^L \widetilde{b}_D^U], [\widetilde{I}_D^L, \widetilde{I}_D^U], [\widetilde{s}_D^L, \widetilde{s}_D^U]$ be two IVPNS, and k is any scalar multiplication, then algebraic operation is given as follows:

1. $C \oplus D = [\sqrt{(\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2}, \sqrt{(\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2}], [I_C^L I_D^L, I_C^U I_D^U]$ $, [\tilde{s}_C^L \tilde{s}_D^L, \tilde{s}_C^U \tilde{s}_D^U]$
2. $C \otimes D = [\tilde{b}_C^L \tilde{b}_D^L, \tilde{b}_C^U \tilde{b}_D^U], [I_C^L + I_D^L - I_C^L I_D^L, I_C^U + I_D^U - I_C^U I_D^U],$ $[\sqrt{(\tilde{s}_C^L)^2 + (\tilde{s}_D^L)^2 - (\tilde{s}_C^L)^2(\tilde{s}_D^L)^2}, \sqrt{(\tilde{s}_C^U)^2 + (\tilde{s}_D^U)^2 - (\tilde{s}_C^U)^2(\tilde{s}_D^U)^2}]$
3. $kC = [\sqrt{1-(1-(\tilde{b}_C^L)^2)^k}, \sqrt{1-(1-(\tilde{b}_C^U)^2)^k}], [(I_C^L)^k, (I_C^U)^k], [(\tilde{s}_C^L)^k, (\tilde{s}_C^U)^k], (k > 0)$
4. $C^k = [(\tilde{b}_C^L)^k, (\tilde{b}_C^U)^k], [1 - I_C^L, 1 - I_C^U], [\sqrt{1-(1-(\tilde{s}_C^L)^2)^k}, \sqrt{1-(1-(\tilde{s}_C^U)^2)^k}], (k > 0)$

Example 1: Let $C = ([0.45, 0.60], [0.40, 0.50], [0.30, 0.40]) \in [0,1]$ and $D = ([0.20, 0.40], [0.60, 0.70], [0.50, 0.60]) \in [0,1]$ as an IVPNS numbers and $k = 2$ as a scalar multiplication.

1. $A \oplus B = [\sqrt{(\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2}, \sqrt{(\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2}], [I_C^L I_D^L, I_C^U I_D^U]$ $, [\tilde{s}_C^L \tilde{s}_D^L, \tilde{s}_C^U \tilde{s}_D^U]$

$$= [\sqrt{(0.45)^2 + (0.20)^2 - (0.45)^2(0.20)^2}, \sqrt{(0.60)^2 + (0.40)^2 - (0.60)^2(0.40)^2}], [(0.40)(0.60), (0.50)(0.70)]$$
$$, [(0.30)(0.50), (0.40)(0.60)]$$
$$= ([0.48, 0.68], [0.24, 0.35], [0.15, 0.24]$$

2. $C \otimes D = [\tilde{b}_C^L \tilde{b}_D^L, \tilde{b}_C^U \tilde{b}_D^U], [I_C^L + I_D^L - I_C^L I_D^L, I_C^U + I_D^U - I_C^U I_D^U]$

$$[\sqrt{(\tilde{s}_C^L)^2 + (\tilde{s}_D^L)^2 - (\tilde{s}_C^L)^2(\tilde{s}_D^L)^2}, \sqrt{(\tilde{s}_C^U)^2 + (\tilde{s}_D^U)^2 - (\tilde{s}_C^U)^2(\tilde{s}_D^U)^2}]$$
$$= [(0.45)(0.20), (0.60)(0.40)], [(0.40 + 0.60 - (0.40)(0.60), (0.50) + (0.70) - (0.50)(0.70)]$$
$$\sqrt{(0.3)^2 + (0.5)^2 - (0.3)^2(0.5)^2}, \sqrt{(0.4)^2 + (0.6)^2 - (0.4)^2(0.6)^2}$$
$$= ([0.09, 0.24], [0.76, 0.85], [0.56, 0.68])$$

3. $kC = [\sqrt{1-(1-(\tilde{b}_C^L)^2)^k}, \sqrt{1-(1-(\tilde{b}_C^U)^2)^k}], [(I_C^L)^k, (I_C^U)^k], [(\tilde{s}_C^L)^k, (\tilde{s}_C^U)^k]$

$$= [\sqrt{1-(1-(0.45)^2)^2}, \sqrt{1-(1-(0.60)^2)^2}], [(0.40)^2, 0.50)^2], [(0.3)^2, (0.4)^2]$$
$$= ([0.60, 0.77], [0.16, 0.25], [0.09, 0.16])$$

4. $C^k = [(\tilde{b}_C^L)^k, (\tilde{b}_C^U)^k], [1 - I_C^L, 1 - I_C^U], [\sqrt{1-(1-(\tilde{s}_C^L)^2)^k}, \sqrt{1-(1-(\tilde{s}_C^U)^2)^k}], (k > 0)$

$$= [(0.45)^2, (0.60)^2], [(1 - 0.40), (1 - 0.50)], [\sqrt{(1-)(1-(0.3)^2)^2}, \sqrt{1-(1-(0.4)^2)^2}]$$
$$= ([0.20, 0.36], [0.60, 0.50], [0.41, 0.54])$$

Comparative Analysis of IVPNS

In this section, we discuss the comparative analysis of the algebraic operations of IVPNS according to the definition 3.3, which originate from the combination of the algebraic operations of IVPFS and IVNS.

The analysis of Algebraic properties of IVPNS with, Let $C = ([\tilde{b}_C^L, \tilde{b}_C^U], [\tilde{I}_C^L, \tilde{I}_C^U], [\tilde{s}_C^L, \tilde{s}_C^U]$ and $D = ([\tilde{b}_D^L \tilde{b}_D^U], [\tilde{I}_D^L, \tilde{I}_D^U], [\tilde{s}_D^L, \tilde{s}_D^U]$ be two IVPNS, and k is any scalar multiplication, then the algebraic operation of addition (1), Multiplication (2), Scalar Multiplication (3) and Power of k (4) is given in Table 1 as follows:

Table 1. The comparative analysis of algebraic operation IVPNS

1.	IVPFS	$C \oplus D = [\sqrt{(\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2}, \sqrt{(\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2}],$

		$[\tilde{s}_C^L\tilde{s}_D^L, \tilde{s}_C^U\tilde{s}_D^U]$
	IVNS	$C \oplus D = [(\tilde{b}_C^L) + (\tilde{b}_D^L) - (\tilde{b}_C^L)(\tilde{b}_D^L), (\tilde{b}_C^U) + (\tilde{b}_D^U) - (\tilde{b}_C^U)(\tilde{b}_D^U)], [I_C^L I_D^L, I_C^U I_D^U]$ $, [\tilde{s}_C^L\tilde{s}_D^L, \tilde{s}_C^U\tilde{s}_D^U]$
	IVPNS	$C \oplus D = [\sqrt{(\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2}, \sqrt{(\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2}],$ $[I_C^L I_D^L, I_C^U I_D^U], [\tilde{s}_C^L\tilde{s}_D^L, \tilde{s}_C^U\tilde{s}_D^U]$
2.	IVPFS	$C \otimes D = [\tilde{b}_C^L\tilde{b}_D^L, \tilde{b}_C^U\tilde{b}_D^U], [\sqrt{(\tilde{s}_C^L)^2 + (\tilde{s}_D^L)^2 - (\tilde{s}_C^L)^2(\tilde{s}_D^L)^2},$ $\sqrt{(\tilde{s}_C^U)^2 + (\tilde{s}_D^U)^2 - (\tilde{s}_C^U)^2(\tilde{s}_D^U)^2}]$
	IVNS	$C \otimes D = [\tilde{b}_C^L\tilde{b}_D^L, \tilde{b}_C^U\tilde{b}_D^U], [I_C^L + I_D^L - I_C^L I_D^L, I_C^U + I_D^U - I_C^U I_D^U]$ $[(\tilde{s}_C^L) + (\tilde{s}_D^L) - (\tilde{s}_C^L)(\tilde{s}_D^L), (\tilde{s}_C^U) + (\tilde{s}_D^U) - (\tilde{s}_C^U)(\tilde{s}_D^U)]$
	IVPNS	$C \otimes D = [\tilde{b}_C^L\tilde{b}_D^L, \tilde{b}_C^U\tilde{b}_D^U], [I_C^L + I_D^L - I_C^L I_D^L, I_C^U + I_D^U - I_C^U I_D^U]$ $[\sqrt{(\tilde{s}_C^L)^2 + (\tilde{s}_D^L)^2 - (\tilde{s}_C^L)^2(\tilde{s}_D^L)^2}, \sqrt{(\tilde{s}_C^U)^2 + (\tilde{s}_D^U)^2 - (\tilde{s}_C^U)^2(\tilde{s}_D^U)^2}]$
3.	IVPFS	$kC = [\sqrt{1-(1-(\tilde{b}_C^L)^2)^k}, \sqrt{1-(1-(\tilde{b}_C^U)^2)^k}], [(\tilde{s}_C^L)^k, (\tilde{s}_C^U)^k]$
	IVNS	$kC = [1-(1-(\tilde{b}_C^L)^2)^k, 1-(1-(\tilde{b}_C^U)^2)^k], [(I_C^L)^k, (I_C^U)^k], [(\tilde{s}_C^L)^k, (\tilde{s}_C^U)^k]$
	IVPNS	$kC = [\sqrt{1-(1-(\tilde{b}_C^L)^2)^k}, \sqrt{1-(1-(\tilde{b}_C^U)^2)^k}], [(I_C^L)^k, (I_C^U)^k], [(\tilde{s}_C^L)^k, (\tilde{s}_C^U)^k]$ $, (k > 0)$
4.	IVPFS	$C^k = [(\tilde{b}_C^L)^k, (\tilde{b}_C^U)^k], [\sqrt{1-(1-(\tilde{s}_C^L)^2)^k}, \sqrt{1-(1-(\tilde{s}_C^U)^2)^k}], (k > 0)$
	IVNS	$C^k = [\tilde{b}_C^L, \tilde{b}_C^U], [1-I_C^L, 1-I_C^U], [\tilde{s}_C^L, \tilde{s}_C^U, (k > 0)$
	IVPNS	$C^k = [(\tilde{b}_C^L)^k, (\tilde{b}_C^U)^k], [1-I_C^L, 1-I_C^U], [\sqrt{1-(1-(\tilde{s}_C^L)^2)^k}, \sqrt{1-(1-(\tilde{s}_C^U)^2)^k}], (k > 0)$

Properties of IVPNS

In this section, we present the algebraic properties of IVPNS, specifically discussing their commutative and associative properties.

Proposition 4.1: Let, $C = ([\tilde{b}_C^L, \tilde{b}_C^U], [\tilde{I}_C^L, \tilde{I}_C^U], [\tilde{s}_C^L, \tilde{s}_C^U]$ and $D = ([\tilde{b}_D^L \tilde{b}_D^U], [\tilde{I}_D^L, \tilde{I}_D^U], [\tilde{s}_D^L, \tilde{s}_D^U]$ and $E = ([\tilde{b}_E^L, \tilde{b}_{M_3}^U], [I_E^L, I_E^U][\tilde{s}_E^L, \tilde{s}_E^U])$ be three IVPNS, and Let $k > 0$ is any scalar multiplication, then we have:

1. $C \oplus D = D \oplus C$
2. $C \otimes D = D \otimes C$
3. $k(C \oplus D) = k(D) \oplus k(C)$
4. $(C \otimes D)^k = (C)^k \otimes (D)^k$
5. $(C \oplus D) \oplus E = C \oplus (D \oplus E)$
6. $(C \otimes D) \otimes E = C \otimes (D \otimes E)$

Proof

1. $C \oplus D = D \oplus C$

$$C \oplus D = [\sqrt{(\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2}, \sqrt{(\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2}], [I_C^L I_D^L, I_C^U I_D^U], [\tilde{s}_C^L \tilde{s}_D^L, \tilde{s}_C^U \tilde{s}_D^U]$$

$$D \oplus C = [\sqrt{(\tilde{b}_D^L)^2 + (\tilde{b}_C^L)^2 - (\tilde{b}_D^L)^2(\tilde{b}_C^L)^2}, \sqrt{(\tilde{b}_D^U)^2 + (\tilde{b}_C^U)^2 - (\tilde{b}_D^U)^2(\tilde{b}_C^U)^2}], [I_D^L I_C^L, I_D^U I_C^U], [\tilde{s}_D^L \tilde{s}_C^L, \tilde{s}_D^U \tilde{s}_C^U]$$

Therefore, $C \oplus D = D \oplus C$.

2. $C \otimes D = D \otimes C$

$$C \otimes D = [\tilde{b}_C^L \tilde{b}_D^L, \tilde{b}_C^U \tilde{b}_D^U], [I_C^L + I_D^L - I_C^L I_D^L, I_C^U + I_D^U - I_C^U I_D^U], [\sqrt{(\tilde{s}_C^L)^2 + (\tilde{s}_D^L)^2 - (\tilde{s}_C^L)^2(\tilde{s}_D^L)^2}, \sqrt{(\tilde{s}_C^U)^2 + (\tilde{s}_D^U)^2 - (\tilde{s}_C^U)^2(\tilde{s}_D^U)^2}]$$

$$D \otimes C = [\tilde{b}_D^L \tilde{b}_C^L, \tilde{b}_D^U \tilde{b}_C^U], [I_D^L + I_C^L - I_D^L I_C^L, I_D^U + I_C^U - I_D^U I_C^U], [\sqrt{(\tilde{s}_D^L)^2 + (\tilde{s}_C^L)^2 - (\tilde{s}_D^L)^2(\tilde{s}_C^L)^2}, \sqrt{(\tilde{s}_D^U)^2 + (\tilde{s}_C^U)^2 - (\tilde{s}_D^U)^2(\tilde{s}_C^U)^2}]$$

Therefore, $C \otimes D = D \otimes C$.

3. $k(C + D) = k(C) + k(D)$

$$k(C \oplus D) = k([\sqrt{(\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2}, \sqrt{(\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2}], [I_C^L I_D^L, I_C^U I_D^U], [\tilde{s}_C^L \tilde{s}_D^L, \tilde{s}_C^U \tilde{s}_D^U]$$

$$= [\sqrt{1 - (1 - ((\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L \tilde{b}_D^L)^2))^k}, \sqrt{1 - (1 - ((\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U \tilde{b}_D^U)^2))^k}], [(I_C^L I_D^L)^k, (I_C^U I_D^U)^k], [(\tilde{s}_C^L \tilde{s}_D^L)^k, (\tilde{s}_C^U \tilde{s}_D^U)^k]$$

$$k(C) \oplus k(D) = [\sqrt{1 - (1 - (\tilde{b}_C^L)^2)^k}, \sqrt{1 - (1 - (\tilde{b}_C^U)^2)^k}], [(I_C^L)^k, (I_C^U)^k], [(\tilde{s}_C^L)^k, (\tilde{s}_C^U)^k] \oplus [\sqrt{1 - (1 - (\tilde{b}_D^L)^2)^k}, \sqrt{1 - (1 - (\tilde{b}_D^U)^2)^k}], [(I_D^L)^k, (I_D^U)^k], [(\tilde{s}_D^L)^k, (\tilde{s}_D^U)^k]$$

$$= [\sqrt{1 - (1 - ((\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2))^k}, \sqrt{1 - (1 - ((\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2))^k}], [(I_C^L I_D^L)^k, (I_C^U I_D^U)^k], [(\tilde{s}_C^L \tilde{s}_D^L)^k, (\tilde{s}_C^U \tilde{s}_D^U)^k]$$

Therefore, $k(C + D) = k(C) + k(D)$.

4. $(C \otimes D)^k = C^k \otimes D^k$

$$(C \otimes D)^k = [(\tilde{b}_C^L \tilde{b}_D^L)^k, (\tilde{b}_C^U \tilde{b}_D^U)^k], [1 - (I_C^L + I_D^L - I_C^L I_D^L), 1 - (I_C^U + I_D^U - I_C^U I_D^U), [\sqrt{1 - (1 - ((\tilde{s}_C^L)^2 + (\tilde{s}_D^L)^2 - (\tilde{s}_C^L)^2(\tilde{s}_D^L)^2))^k}, \sqrt{1 - (1 - ((\tilde{s}_C^U)^2 + (\tilde{s}_D^U)^2 - (\tilde{s}_C^U)^2(\tilde{s}_D^U)^2))^k}]$$

$$(C^k) \otimes (D^k) = [(\tilde{b}_C^L)^k, (\tilde{b}_C^U)^k], [1 - I_C^L, 1 - I_C^U], [\sqrt{1 - (1 - (\tilde{s}_C^L)^2)^k}, \sqrt{1 - (1 - (\tilde{s}_C^U)^2)^k}] \otimes [(\tilde{b}_D^L)^k, (\tilde{b}_D^U)^k], [1 - I_D^L, 1 - I_D^U], [\sqrt{1 - (1 - (\tilde{s}_D^L)^2)^k}, \sqrt{1 - (1 - (\tilde{s}_D^U)^2)^k}]$$

$$= [(\tilde{b}_C^L \tilde{b}_D^L)^k, (\tilde{b}_C^U \tilde{b}_D^U)^k], [1 - (I_C^L + I_D^L - I_C^L I_D^L), 1 - (I_C^U + I_D^U - I_C^U I_D^U), [\sqrt{1 - (1 - ((\tilde{s}_C^L)^2 + (\tilde{s}_D^L)^2 - (\tilde{s}_C^L)^2(\tilde{s}_D^L)^2))^k}, \sqrt{1 - (1 - ((\tilde{s}_C^U)^2 + (\tilde{s}_D^U)^2 - (\tilde{s}_C^U)^2(\tilde{s}_D^U)^2))^k}]$$

Therefore, $(C \otimes D)^k = (C)^k \otimes (D)^k$.

5. $(C \oplus D) \oplus E = C \oplus (D \oplus E)$

$$(C \oplus D) \oplus E[\sqrt{(\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2}, \sqrt{(\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2}], [I_C^L I_D^L, I_C^U I_D^U], [\tilde{s}_C^L \tilde{s}_D^L, \tilde{s}_C^U \tilde{s}_D^U] \oplus ([\tilde{b}_E^L, \tilde{b}_E^U], [I_E^L, I_E^U], [\tilde{s}_E^L, \tilde{s}_E^U])$$

$$= [\sqrt{(\tilde{b}_C^L)^2 + (\tilde{b}_D^L)^2 + (\tilde{b}_E^L)^2 - (\tilde{b}_C^L)^2(\tilde{b}_D^L)^2(\tilde{b}_E^L)^2}, \sqrt{(\tilde{b}_C^U)^2 + (\tilde{b}_D^U)^2 + (\tilde{b}_E^L)^2 - (\tilde{b}_C^U)^2(\tilde{b}_D^U)^2(\tilde{b}_E^L)^2}], [I_C^L I_D^L I_E^L, I_C^U I_D^U I_E^U], [\tilde{s}_C^L \tilde{s}_D^L \tilde{s}_E^L, \tilde{s}_C^U \tilde{s}_D^U \tilde{s}_E^U]$$

$$C \oplus (D \oplus E) = [\tilde{b}_C^L, \tilde{b}_C^U], [I_C^L, I_C^U], [\tilde{s}_C^L, \tilde{s}_C^U] \oplus ([\sqrt{((\tilde{b}_D^L)^2 + (\tilde{b}_E^L)^2 - (\tilde{b}_D^L)^2(\tilde{b}_E^L)^2}, \sqrt{((\tilde{b}_D^U)^2 + (\tilde{b}_E^L)^2 - (\tilde{b}_D^U)^2(\tilde{b}_E^L)^2}], [I_C^L I_D^L, I_C^U I_D^U], [\tilde{s}_C^L \tilde{s}_D^L, \tilde{s}_C^U \tilde{s}_D^U])$$

$$=[\sqrt{(\tilde{b}_C^L)^2+(\tilde{b}_D^L)^2+(\tilde{b}_E^L)^2-(\tilde{b}_C^L)^2(\tilde{b}_D^L)^2(\tilde{b}_E^L)^2},$$
$$\sqrt{(\tilde{b}_C^U)^2+(\tilde{b}_D^U)^2+(\tilde{b}_E^L)^2-(\tilde{b}_C^U)^2(\tilde{b}_D^U)^2(\tilde{b}_E^L)^2}],$$
$$[I_C^L I_D^L I_E^L, I_C^U I_D^U I_E^U],[\tilde{s}_C^L\tilde{s}_D^L\tilde{s}_E^L,\tilde{s}_C^U\tilde{s}_D^U\tilde{s}_E^U]$$

Therefore, $(C\oplus D)\oplus E = C\oplus(D\oplus E)$.

6. $(C\otimes D)\otimes E = C\otimes(D\otimes E)$

$$(C\otimes D)\otimes E=[\tilde{b}_C^L\tilde{b}_D^L-\tilde{b}_C^U\tilde{b}_D^U],,[I_C^L+I_D^L-I_C^L I_D^L, I_C^U+I_D^U-I_C^U I_D^U],\ ,[\sqrt{(\tilde{s}_C^L)^2+(\tilde{s}_D^L)^2-(\tilde{s}_C^L)^2(\tilde{s}_D^L)^2}$$
$$,\sqrt{(\tilde{s}_C^U)^2+(\tilde{s}_D^U)^2-(\tilde{s}_C^U)^2(\tilde{s}_D^U)^2}])\otimes[\tilde{b}_E^L,\tilde{b}_E^U],[I_E^L,I_E^U],[\tilde{s}_E^L,\tilde{s}_E^U]$$
$$=[\tilde{b}_C^L\tilde{b}_D^L\tilde{b}_E^L-,\tilde{b}_C^U\tilde{b}_D^U\tilde{b}_E^U][I_C^L+I_D^L+I_E^L-I_C^L I_D^L I_E^L, I_C^U+I_D^U+I_E^U-I_C^U I_D^U I_E^U]$$
$$,[\sqrt{(\tilde{s}_C^L)^2+(\tilde{s}_D^L)^2+(\tilde{s}_E^L)^2-(\tilde{s}_C^L)^2(\tilde{s}_D^L)^2(\tilde{s}_E^L)^2}$$
$$,\sqrt{(\tilde{s}_C^U)^2+(\tilde{s}_D^U)^2+(\tilde{s}_E^U)^2-(\tilde{s}_C^U)^2(\tilde{s}_D^U)^2(\tilde{s}_E^U)^2}])$$
$$C\otimes(D\otimes E)=[\tilde{b}_C^L,\tilde{b}_C^U],[I_C^L I_C^U][\tilde{s}_C^L\tilde{s}_C^U]\otimes[\tilde{b}_D^L\tilde{b}_E^L,\tilde{b}_D^U\tilde{b}_E^U],[I_D^L+I_E^L-I_D^L I_E^L, I_D^U+I_E^U-I_D^U I_E^U]$$
$$,[\sqrt{(\tilde{s}_D^L)^2+(\tilde{s}_E^L)^2-(\tilde{s}_D^L)^2(\tilde{s}_E^L)^2},\sqrt{(\tilde{s}_D^U)^2+(\tilde{s}_E^U)^2-(\tilde{s}_D^U)^2(\tilde{s}_E^U)^2}])$$
$$=[\tilde{b}_C^L\tilde{b}_D^L\tilde{b}_E^L-,\tilde{b}_C^U\tilde{b}_D^U\tilde{b}_E^U][I_C^L+I_D^L+I_E^L-I_C^L I_D^L I_E^L, I_C^U+I_D^U+I_E^U-I_C^U I_D^U I_E^U]$$
$$,[\sqrt{(\tilde{s}_C^L)^2+(\tilde{s}_D^L)^2+(\tilde{s}_E^L)^2-(\tilde{s}_C^L)^2(\tilde{s}_D^L)^2(\tilde{s}_E^L)^2},$$
$$\sqrt{(\tilde{s}_C^U)^2+(\tilde{s}_D^U)^2+(\tilde{s}_E^U)^2-(\tilde{s}_C^U)^2(\tilde{s}_D^U)^2(\tilde{s}_E^U)^2}])$$

Therefore, $(C\otimes D)\otimes E = C\otimes(D\otimes E)$.

Conclusions

In this paper, we introduce the Interval Valued Pythagorean Neutrosophic Set (IVPNS) by combining Interval Valued Pythagorean Sets (IVPS) and Interval Valued Neutrosophic Sets (IVNS). The new algebraic operations for IVPNS, including addition, multiplication, scalar multiplication, and exponentiation were developed. We also explore and prove the commutative and associative properties of IVPNS, highlighting its unique characteristics. The development of IVPNS offers a comprehensive framework for managing complex systems, providing a detailed and precise method for handling uncertainty, ambiguity, and incomplete information across various fields. This framework enhances decision-making processes and problem-solving capabilities. Additionally, the flexibility of IVPNS allows for integration with other mathematical and computational models, broadening its applicability. This makes IVPNS a versatile tool for researchers and practitioners dealing with complex systems. Future research should focus on creating novel aggregation operators, developing efficient computational algorithms, integrating IVPNS with machine learning models, applying IVPNS to big data analytics, conducting comparative studies, exploring new application domains, and working towards the formalization and standardization of IVPNS.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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