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Transformation of Matrix Presentation for Bieberbach Groups into Polycyclic Presentations

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Abstract A Bieberbach group is a torsion free crystallographic group that represents an extension of a free abelian lattice group by a finite point group. This research began by taking the group offered in the Crystallographic Algorithms and Tables (CARAT) package, which is in the matrix form. There are only four Bieberbach groups of dimension six to be isomorphic to the quaternion point group of order eight. In this study, three Bieberbach groups of dimension six with the quaternion point group of order eight that are considered as only the first group has been found its well-defined polycyclic presentation. Every group has eight generators that describe the group. However, the algorithm used in constructing the polycyclic presentation requires a new arbitrary generator to be added into the group. Then the consistency relations need to be checked and the polycyclic presentation is said to be a well-defined construction if it is consistent. Therefore, this study shows the construction of polycyclic presentation with the new arbitrary generator for all three groups. Furthermore, the polycyclic presentation for the second group has been proven to be consistent, which implies that the construction is well-defined.

Keywords: Crystallographic group, polycyclic presentations, quaternion point group, consistency relations.

Introduction

Crystallography deals with the principles that control the crystalline state of solid materials, the arrangement of atoms in crystals, and their physical and chemical characteristics, as well as their synthesis and growth. Bieberbach group is one of the crystallographic groups which is torsion free. A group G is said to be torsion free if for every $g \in G$, where g does not have finite order, i.e. $g^n \neq 1$, for some $n \in$ ¥ [14] and this group is an extension of a free abelian lattice group of finite rank by a finite point group. This crystallographic group can be transformed into polycyclic presentation and need to be checked for its consistency relations.

This research has received considerable attention over the years. Masri [5] has started the study on computing the polycyclic presentation for Bieberbach groups with cyclic point group of order two, C_2 and Bieberbach groups with the elementary abelian 2-group, $C_2 \times C_2$ as the extensions of polycyclic group. These groups are polycyclic since they are extensions of polycyclic groups. Later, Mat Hassim *et al.* [6] continued the research to find the homological functor of a group which is the exterior square. Abdul Ladi *et al.* [1] extended to compute the other homological invariants of some Bieberbach groups with elementary abelian 2-point group.

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Moreover, the nonabelian extension was taken into consideration in 2011. Mohd Idrus [11] started the research on transforming the Bieberbach group of dimension four with dihedral point group of order eight, D_4 and followed by Wan Mohd Fauzi *et al.* [15] who conducted the research on the same extension but with a different dimension of the group. Other than that, Tan *et al.* [16] continued the research but with a different extension namely the symmetric point group of order six, S_3.

Other than that, Mohammad *et al.* [9] conducted a study on computing the polycyclic presentations of the Bieberbach groups with quaternion point group of order eight, which is a torsion free crystallographic group. Crystallographic, Algorithms and Table (CARAT) package were used to find the list of isomorphism types of crystallographic groups. According to Mohammad [2], there are four groups of Bieberbach of dimension six with quaternion point group of order eight are found within the package which the groups are given in matrix form. The matrix representations are then being used to find the polycyclic presentation of the groups for all n^{th} Bieberbach groups where n = 1, 2, 3 and 4 with the help of Groups, Algorithms, and Programming (GAP) software. However, the matrix representation provided in the CARAT package by Opgenorth *et al.* [12] is no longer available online. Then, Mohammad *et al.* [10] shows the computation on achieving consistency for the first Bieberbach group, namely Q_1, for n = 1. The results for the Q_1 is found to be satisfying its consistency relations and the study is continued by claiming that the other three groups are consistent.

Mohammad [7] had extended the study to explicate the homological invariants for the Bieberbach groups of dimension six with the quaternion point group of order eight based on the polycyclic presentation from the previous literature. The researcher manages to explicate the homological invariant for the first Bieberbach group, where n = 1. However, Mohammad *et al.* [8] states several problems occurred in the computation of the homological invariants for the other three Bieberbach groups Q_2, Q_3 and Q_4 which concludes that the polycyclic presentations for the groups are not consistent. This happens because the new generator constructed in the presentation is not well-defined throughout the computation. The researchers recompute the polycyclic presentations for the first group with a new well-defined generator.

Hence, the main motivation of this research is to transform the other three matrix representations into polycyclic presentations as well as to satisfy the consistency relations for the second Bieberbach group namely as Q_2.

Methodology

According to Blyth & Morse [2], Ellis & Leonard [4], and Rocco [13] the homological invariants of a group G can be computed by an approach that involves $\nu(G)$.

Definition 1 Rocco [13]
Let G be a group with presentation $\langle G \mid R \rangle$ and let G^{φ} be an isomorphic copy of G through the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. The group is $\nu(G)$ defined as follows:

$$\nu(G) = \left\langle G, G^{\varphi} \mid R, R^{\varphi}, \left[g, h^{\varphi}\right]^{x} = \left[g^{x}, \left(h^{x}\right)^{\varphi}\right] = \left[g, h^{\varphi}\right]^{x^{\varphi}}, \forall x, g, h \in G \right\rangle$$

where $g^{h} = h^{-1}gh$ and $[g,h] = g^{-1}g^{h}$.

By Definition 1, it is shown that the computation of the group $\nu(G)$ required the group in the form of a group presentation. Thus, in this research, the Bieberbach groups of dimension six with the quaternion point group of order eight that are in the matrix representations are transformed into polycyclic presentations. As the Bieberbach groups with finite point group provided in the CARAT package are in the form of matrix representations that do not contain relations between each generator, Definition 2 and Definition 3 needs to be used in constructing the polycyclic presentation.

Definition 2: Polycyclic Presentation Eick & Nickel [3]
Let F_n be a free group on generators $g_i,\ldots,g_n$ and R be a set of relations of group F_n. The relations of a polycyclic presentation F_n/R have the following form:

$$g_i^{e_i} = g_{i+1}^{x_{i,i+1}} \ldots g_n^{x_{i,n}} \qquad \text{for } i \in I,$$
$$g_j^{-1} g_i g_j = g_{j+1}^{y_{i,j,j+1}} \ldots g_n^{y_{i,j,n}} \qquad \text{for } j < i,$$
$$g_j g_i g_j^{-1} = g_{j+1}^{z_{i,j,j+1}} \ldots g_n^{z_{i,j,n}} \qquad \text{for } j < i \text{ and } j \notin I$$

for some $I \subseteq \{1,\ldots n\}$, certain exponents $e_i \in N$ for $i \in I$ and $x_{i,j}, y_{i,j,k}, z_{i,j,k} \in Z$ for all i, j and k.

From the definition, it shows that the relation in the polycyclic presentation is made up from the product of generators and their inverses while some of the generators are raised to a certain power. After constructing the relations, it is essential that a new abstract generator to be constructed in order to check its consistency on the later part. Eick & Nickel [3] has introduced the construction of new abstract generators by simplifying the notation for the relations of G. It is written as relators and denoting it as $r_1, \ldots, r_l$, so that every relator r_j can be written in terms of generators $g_1,\ldots,g_n$, given by $r_j = r_j(g_1, \ldots, g_n)$.

Definition 3 Eick & Nickel [3]
Let l be the new abstract generators $t_1, \ldots, t_l$ and group $G^* \cong F_n / [R, F_n]$ is defined as the group generated by $g_1,\ldots,g_n$. Thus, $t_1, \ldots, t_l$ is subject to the following relators:

$$r_i(g_1, \ldots, g_n) t_i \qquad \text{for } 1 \le i \le l,$$
$$[t_i, g_j] \qquad \text{for } 1 \le j \le n, 1 \le i \le l,$$
$$[t_i, t_j] \qquad \text{for } 1 \le j < i \le l.$$

However, the construction of the polycyclic presentation based on the quotient F_n/R and adding some new arbitrary generators is possibly inconsistent. Hence, Definition 4 is used to show that the polycyclic presentation is consistent.

Definition 4: Consistent Polycyclic Presentation Eick & Nickel [3]
Let G be a group formed by $g_1,\ldots,g_n$ and the consistency relations in G may be assessed in the polycyclic presentation of G using the collection from the left, as shown below:

$$g_k(g_j g_i) = (g_k g_j) g_i \qquad \text{for } k > j > i,$$
$$(g_j^{e_j}) g_i = g_j^{e_j - 1}(g_j g_i) \qquad \text{for } j > i, j \in I,$$
$$g_j (g_i^{e_i}) = (g_j g_i) g_i^{e_i - 1} \qquad \text{for } j > i, i \in I,$$
$$(g_i^{e_i}) g_i = g_i (g_i^{e_i}) \qquad \text{for } i \in I,$$
$$g_j = (g_j g_i^{-1}) g_i \qquad \text{for } j > i, j \notin I$$

For some $I \subseteq \{1,\ldots n\}$, $e^i \in N$. Then, G is said to be provided by a consistent polycyclic presentation.

After the consistency has been checked, and if it is found that the polycyclic presentation of the group is consistent, it implies that the new abstract generator constructed is a well-defined generator and can be further used in explicating other homological invariants of the group. Figure 1 below shows the procedures in constructing the polycyclic presentation for Q_2, Q_3 and Q_4.

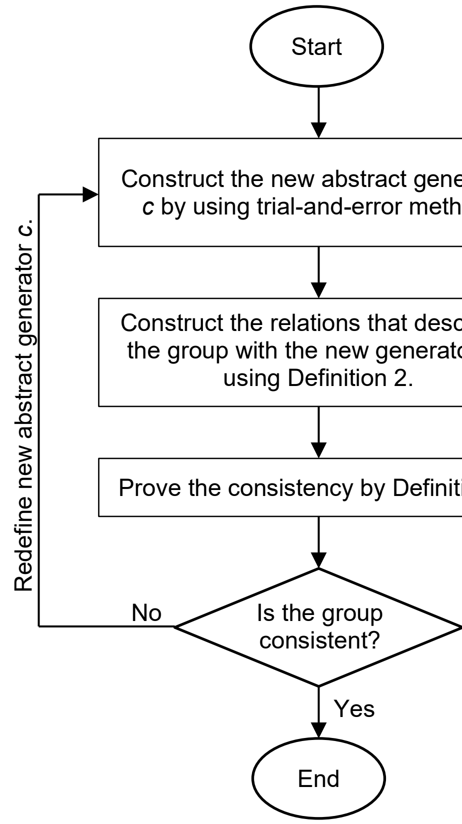

Figure 1. The flowchart on the construction of a consistent polycyclic presentation

Initially, there are 8 matrix representations that describes each group as their independent generators. Before constructing the polycyclic presentation for Q_2, Q_3 and Q_4, a new generator is added by combining the generators through product, inverse or power and it will be the relation that describes the new generator. The construction is done by using trial-and-error method. In this study, GAP software is used to verify the construction of the relation that describe the new generator. Next, with the construction of the new generator, the polycyclic presentation for the groups can be done by considering the new generator in other relation as in Definition 2. However, the new generator constructed may only satisfy several matrix operations within the relations in the polycyclic presentation, which implies that the presentation may be inconsistent. Thus, it is essential to check their consistency by Definition 4 to ensure that the new generator is well-defined. If it is inconsistent, the new generator needs to be reconstructed arbitrarily until the consistency is achieved.

Results and Discussion

The computation of homological invariants of a group requires a correct polycyclic presentation. A correct presentation is a presentation that consists of well-defined generators that satisfies several consistency relations. Once the polycyclic presentation has been constructed, it then can be used with important definitions, theorems, and algorithms in explicating the homological invariants of a particular group, accordingly.

The Construction of the Polycyclic Presentations

In this section, the computation of polycyclic presentations of three Bieberbach groups with the quaternion point group of order eight namely Q_2, Q_3 and Q_4 was shown. The construction of the polycyclic presentation starts with analysis of the matrix representation for each group that are obtained from Mohammad [7]. For each group, it consists of two generators a_0 and a_1, as well as six lattice generators, l_1, l_2, l_3, l_4, l_5 and l_6, where its basis matrix is an identity matrix.

Theorem 1:
Let $G_2 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$, be the second Bieberbach group of dimension six with quaternion point group of order eight, where;

$$a_0 = \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$l_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$l_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$l_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } l_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the polycyclic presentation of Q_2 is established as

$$\begin{aligned} Q_2 = \langle\, & a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \mid a^2 = cl_6, b^2 = c, c^2 = l_5^{-1} l_6^{-1}, b^a = bc^{-1} l_5^{-1}, c^a = c, c^b = c, l_1^a = l_3, \\ & l_1^b = l_1 l_2^{-1} l_4, l_1^c = l_1^{-1}, l_2^a = l_1 l_3 l_4, l_2^b = l_1 l_2^{-1} l_3^{-1}, l_2^c = l_2^{-1}, l_3^a = l_1^{-1}, l_3^b = l_2 l_3 l_4, l_3^c = l_3^{-1}, \\ & l_4^a = l_1 l_2^{-1} l_3^{-1}, l_4^b = l_1^{-1} l_3^{-1} l_4^{-1}, l_4^c = l_4^{-1}, l_5^a = l_5, l_5^b = l_6, l_5^c = l_5, l_6^a = l_6, l_6^b = l_5, l_6^c = l_6, \\ & l_2^{l_1} = l_2, l_3^{l_1} = l_3, l_4^{l_1} = l_4, l_5^{l_1} = l_5, l_6^{l_1} = l_6, l_3^{l_2} = l_3, l_4^{l_2} = l_4, l_5^{l_2} = l_5, l_6^{l_2} = l_6, l_4^{l_3} = l_4, l_5^{l_3} = l_5, \\ & l_6^{l_3} = l_6, l_5^{l_4} = l_5, l_6^{l_4} = l_6, l_6^{l_5} = l_6, l_2^{l_1^{-1}} = l_2, l_3^{l_1^{-1}} = l_3, l_4^{l_1^{-1}} = l_4, l_5^{l_1^{-1}} = l_5, l_6^{l_1^{-1}} = l_6, l_3^{l_2^{-1}} = l_3, \\ & l_4^{l_2^{-1}} = l_4, l_5^{l_2^{-1}} = l_5, l_6^{l_2^{-1}} = l_6, l_4^{l_3^{-1}} = l_4, l_5^{l_3^{-1}} = l_5, l_6^{l_3^{-1}} = l_6, l_5^{l_4^{-1}} = l_5, l_6^{l_4^{-1}} = l_6, l_6^{l_5^{-1}} = l_6 \,\rangle. \end{aligned}$$

Proof:
Take $\gamma : G_2 \to Q_2$ such that $\gamma(a_0) = a$, $\gamma(a_1) = b$ and the same mapping for the other generators. The mapping γ, is well-defined since γ maps the generators of G_2 to generators of Q_2. Now, under this mapping γ, all relations hold in Q_2 are constructed by using GAP software. The matrix representation of the group is first declared, and the command of the software is shown as follows:

```
gap> l1 := [
> [1,0,0,0,0,0,1],
> [0,1,0,0,0,0,0],
> [0,0,1,0,0,0,0],
> [0,0,0,1,0,0,0],
> [0,0,0,0,1,0,0],
> [0,0,0,0,0,1,0],
> [0,0,0,0,0,0,1]];
gap> l2 := [
> [1,0,0,0,0,0,0],
> [0,1,0,0,0,0,1],
> [0,0,1,0,0,0,0],
> [0,0,0,1,0,0,0],
> [0,0,0,0,1,0,0],
> [0,0,0,0,0,1,0],
> [0,0,0,0,0,0,1]];
gap> l3 := [
> [1,0,0,0,0,0,0],
> [0,1,0,0,0,0,0],
> [0,0,1,0,0,0,1],
> [0,0,0,1,0,0,0],
> [0,0,0,0,1,0,0],
> [0,0,0,0,0,1,0],
> [0,0,0,0,0,0,1]];
gap> l4 := [
> [1,0,0,0,0,0,0],
> [0,1,0,0,0,0,0],
> [0,0,1,0,0,0,0],
> [0,0,0,1,0,0,1],
> [0,0,0,0,1,0,0],
> [0,0,0,0,0,1,0],
> [0,0,0,0,0,0,1]];
gap> l5 := [
> [1,0,0,0,0,0,0],
> [0,1,0,0,0,0,0],
> [0,0,1,0,0,0,0],
> [0,0,0,1,0,0,0],
> [0,0,0,0,1,0,1],
> [0,0,0,0,0,1,0],
> [0,0,0,0,0,0,1]];
gap> l6 := [
> [1,0,0,0,0,0,0],
> [0,1,0,0,0,0,0],
> [0,0,1,0,0,0,0],
> [0,0,0,1,0,0,0],
> [0,0,0,0,1,0,0],
> [0,0,0,0,0,1,1],
> [0,0,0,0,0,0,1]];
gap> a0 := [
> [0 / 4 ,-4 / 4 ,4 / 4 ,-4 / 4 ,0 / 4 ,0 / 4 ,0/4],
> [0 / 4 ,0 / 4 ,0 / 4 ,4 / 4 ,0 / 4 ,0 / 4 ,0/4],
> [-4 / 4 ,-4 / 4 ,0 / 4 ,4 / 4 ,0 / 4 ,0 / 4 ,0/4],
> [0 / 4 ,-4 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0/4],
> [0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,4 / 4 ,0 / 4 ,-1/4],
> [0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,4 / 4 ,1/4],
> [0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,4/4]];
gap> a1 := [
> [-4 / 4 ,-4 / 4 ,0 / 4 ,4 / 4 ,0 / 4 ,0 / 4 ,0/4],
> [4 / 4 ,4 / 4 ,-4 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0/4],
> [0 / 4 ,4 / 4 ,-4 / 4 ,4 / 4 ,0 / 4 ,0 / 4 ,0/4],
> [-4 / 4 ,0 / 4 ,-4 / 4 ,4 / 4 ,0 / 4 ,0 / 4 ,0/4],
> [0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,4 / 4 ,-2/4],
> [0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,4 / 4 ,0 / 4 ,0/4],
> [0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,0 / 4 ,4/4]];
```

The new generator *c* is presumably constructed by using trial-and-error method. The generator is made up from possible combination between the generators of the group. For Q_2, let $c = a_0^2 l_6^{-1}$. In the GAP software, the generator *c* is declared as follows:

```
gap> c:=a0^2*l6^-1;
[ [ -1, 0, 0, 0, 0, 0, 0 ],
  [ 0, -1, 0, 0, 0, 0, 0 ],
  [ 0, 0, -1, 0, 0, 0, 0 ],
  [ 0, 0, 0, -1, 0, 0, 0 ],
  [ 0, 0, 0, 0, 1, 0, -1/2 ],
  [ 0, 0, 0, 0, 0, 1, -1/2 ],
  [ 0, 0, 0, 0, 0, 0, 1 ] ]
```

The matrix computation of $c = a_0{}^2 l_6{}^{-1}$ shows that

$$c = \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, by mapping $\gamma(a_0) = a$, $c = a_0{}^2 l_6{}^{-1} = a^2 l_6{}^{-1}$.

After the new generator c is constructed, the constructions of other relations that consists of raising to the power of certain exponents and its conjugation within the generators in the group is done.

Next, a_0^2 can be shown to be equal to cl_6, which the commands are as follows:

```
gap> a0^2;
[ [ -1, 0, 0, 0, 0, 0, 0 ],
[ 0, -1, 0, 0, 0, 0, 0 ],
[ 0, 0, -1, 0, 0, 0, 0 ],
[ 0, 0, 0, -1, 0, 0, 0 ],
[ 0, 0, 0, 0, 1, 0, -1/2 ],
[ 0, 0, 0, 0, 0, 1, 1/2 ],
[ 0, 0, 0, 0, 0, 0, 1 ] ]
```

```
gap> c*l6;
[ [ -1, 0, 0, 0, 0, 0, 0 ],
[ 0, -1, 0, 0, 0, 0, 0 ],
[ 0, 0, -1, 0, 0, 0, 0 ],
[ 0, 0, 0, -1, 0, 0, 0 ],
[ 0, 0, 0, 0, 1, 0, -1/2 ],
[ 0, 0, 0, 0, 0, 1, 1/2 ],
[ 0, 0, 0, 0, 0, 0, 1 ] ]
gap> a0^2=c*l6;
true
```

The matrix computation to show that a_0^2 is equal to cl_6 is shown as follows:

$$a_0^2 = \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$cl_6 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, by mapping $\gamma(a_0) = a$, $a_0^2 = a^2 = cl_6$. The next calculation shows that $c^{a_1} = c$.

$$c^{a_1} = a_1^{-1} c a_1$$

$$= \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= c.$$

Thus, $c^{a_1} = a_1^{-1} c a_1 = c$. By mapping $\gamma(a_1) = b$, $c^{a_1} = c^b = c$.

Hence, by following the same method for both matrix computation and GAP software, all possible relations formed by conjugation between each generator and the power of certain exponents have been computed. Therefore, $G_2 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ was shown to be isomorphic to $Q_2 = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ with $c = a_0{}^2 l_6{}^{-1}$. As a result of collecting all possible relations, the polycyclic presentation of   is established as

$$\begin{aligned} Q_2 = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \mid\ & a^2 = c l_6, b^2 = c, c^2 = l_5{}^{-1} l_6{}^{-1}, b^a = b c^{-1} l_5{}^{-1}, c^a = c, c^b = c, l_1{}^a = l_3, \\ & l_1{}^b = l_1 l_2{}^{-1} l_4, l_1{}^c = l_1{}^{-1}, l_2{}^a = l_1 l_3 l_4, l_2{}^b = l_1 l_2{}^{-1} l_3{}^{-1}, l_2{}^c = l_2{}^{-1}, l_3{}^a = l_1{}^{-1}, l_3{}^b = l_2 l_3 l_4, l_3{}^c = l_3{}^{-1}, \\ & l_4{}^a = l_1 l_2{}^{-1} l_3{}^{-1}, l_4{}^b = l_1{}^{-1} l_3{}^{-1} l_4{}^{-1}, l_4{}^c = l_4{}^{-1}, l_5{}^a = l_5, l_5{}^b = l_6, l_5{}^c = l_5, l_6{}^a = l_6, l_6{}^b = l_5, l_6{}^c = l_6, \\ & l_2{}^{l_1} = l_2, l_3{}^{l_1} = l_3, l_4{}^{l_1} = l_4, l_5{}^{l_1} = l_5, l_6{}^{l_1} = l_6, l_3{}^{l_2} = l_3, l_4{}^{l_2} = l_4, l_5{}^{l_2} = l_5, l_6{}^{l_2} = l_6, l_4{}^{l_3} = l_4, l_5{}^{l_3} = l_5, \\ & l_6{}^{l_3} = l_6, l_5{}^{l_4} = l_5, l_6{}^{l_4} = l_6, l_6{}^{l_5} = l_6, l_2{}^{l_1{}^{-1}} = l_2, l_3{}^{l_1{}^{-1}} = l_3, l_4{}^{l_1{}^{-1}} = l_4, l_5{}^{l_1{}^{-1}} = l_5, l_6{}^{l_1{}^{-1}} = l_6, l_3{}^{l_2{}^{-1}} = l_3, \\ & l_4{}^{l_2{}^{-1}} = l_4, l_5{}^{l_2{}^{-1}} = l_5, l_6{}^{l_2{}^{-1}} = l_6, l_4{}^{l_3{}^{-1}} = l_4, l_5{}^{l_3{}^{-1}} = l_5, l_6{}^{l_3{}^{-1}} = l_6, l_5{}^{l_4{}^{-1}} = l_5, l_6{}^{l_4{}^{-1}} = l_6, l_6{}^{l_5{}^{-1}} = l_6 \rangle. \end{aligned}$$

Theorem 2:
Let $G_3 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$, be the third Bieberbach group of dimension six with quaternion point group of order eight, where;

$$a_0=\begin{bmatrix}0&-1&1&-1&0&0&-\frac{1}{2}\\0&0&0&1&0&0&0\\-1&-1&0&1&0&0&\frac{1}{2}\\0&-1&0&0&0&0&0\\0&0&0&0&1&0&-\frac{1}{4}\\0&0&0&0&0&1&\frac{1}{4}\\0&0&0&0&0&0&1\end{bmatrix},\quad a_1=\begin{bmatrix}-1&-1&0&1&0&0&0\\1&1&-1&0&0&0&0\\0&1&-1&1&0&0&0\\-1&0&-1&1&0&0&0\\0&0&0&0&1&-\frac{1}{2}\\0&0&0&0&1&0&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$l_1=\begin{bmatrix}1&0&0&0&0&0&1\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix},\quad l_2=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&1\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$l_3=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&1\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix},\quad l_4=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&1\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$l_5=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&1\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix},\text{ and } l_6=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&1\\0&0&0&0&0&0&1\end{bmatrix}.$$

Proof:

By using the same method, let $G_3=\langle a_0,a_1,l_1,l_2,l_3,l_4,l_5,l_6\rangle$ was shown to be isomorphic to $Q_3=\langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6\rangle$ with $c=a_0{}^2l_3l_6{}^{-1}$. As a result of collecting all possible relations, the polycyclic presentation of Q_3 is established as:

$$\begin{aligned}Q_3=\langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6\mid\ & a^2=cl_3{}^{-1}l_6,b^2=c,c^2=l_5{}^{-1}l_6{}^{-1},b^a=l_4{}^{-1}l_5b^{-1}c^2,c^a=a^2l_6{}^{-1}l_1{}^{-1},\\ & c^b=c,l_1{}^a=l_3,l_1{}^b=l_1l_2{}^{-1}l_4,l_1{}^c=l_1{}^{-1},l_2{}^a=l_1l_3l_4,l_2{}^b=l_1l_2{}^{-1}l_3{}^{-1},l_2{}^c=l_2{}^{-1},l_3{}^a=l_1{}^{-1},\\ & l_3{}^b=l_2l_3l_4,l_3{}^c=l_3{}^{-1},l_4{}^a=l_1l_2{}^{-1}l_3{}^{-1},l_4{}^b=l_1{}^{-1}l_3{}^{-1}l_4{}^{-1},l_4{}^c=l_4{}^{-1},l_5{}^a=l_5,l_5{}^b=l_6,l_5{}^c=l_5,\\ & l_6{}^a=l_6,l_6{}^b=l_5,l_6{}^c=l_6,l_2{}^{l_1}=l_2,l_3{}^{l_1}=l_3,l_4{}^{l_1}=l_4,l_5{}^{l_1}=l_5,l_6{}^{l_1}=l_6,l_3{}^{l_2}=l_3,l_4{}^{l_2}=l_4,l_5{}^{l_2}=l_5,\\ & l_6{}^{l_2}=l_6,l_4{}^{l_3}=l_4,l_5{}^{l_3}=l_5,l_6{}^{l_3}=l_6,l_5{}^{l_4}=l_5,l_6{}^{l_4}=l_6,l_6{}^{l_5}=l_6,l_2{}^{l_1{}^{-1}}=l_2,l_3{}^{l_1{}^{-1}}=l_3,l_4{}^{l_1{}^{-1}}=l_4,\\ & l_5{}^{l_1{}^{-1}}=l_5,l_6{}^{l_1{}^{-1}}=l_6,l_3{}^{l_2{}^{-1}}=l_3,l_4{}^{l_2{}^{-1}}=l_4,l_5{}^{l_2{}^{-1}}=l_5,l_6{}^{l_2{}^{-1}}=l_6,l_4{}^{l_3{}^{-1}}=l_4,l_5{}^{l_3{}^{-1}}=l_5,l_6{}^{l_3{}^{-1}}=l_6,\\ & l_5{}^{l_4{}^{-1}}=l_5,l_6{}^{l_4{}^{-1}}=l_6,l_6{}^{l_5{}^{-1}}=l_6\rangle.\end{aligned}$$

Theorem 3:
Let $G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$, be the fourth Bieberbach group of dimension six with quaternion point group of order eight, where;

$$a_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$l_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$l_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$l_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } l_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof:
Therefore, by using the same method, let $G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ was shown to be isomorphic to $Q_4 = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ with $c = l_2^{-1} l_5^{-1} a_0^2$. In addition, G_4 was also shown to be isomorphic to Q_4 with another c where $c = a_0^2 l_5^{-1} l_3 l_1^{-1} l_4^{-1}$. As a result of collecting all possible relations, the polycyclic presentation of Q_4 is established as:

$$Q_4 = \langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \mid a^2 = l_2 l_5 c, b^2 = c, c^2 = l_1^{-1} l_3 l_5^{-1} l_6, b^a = l_6^{-1} b^{-1} c^2, c^a = l_1^{-1} l_2 b^2,$$
$$c^b = c, l_1^a = l_4, l_1^b = l_3, l_1^c = l_1^{-1}, l_2^a = l_3, l_2^b = l_4^{-1}, l_2^c = l_2^{-1}, l_3^a = l_2^{-1}, l_3^b = l_1^{-1}, l_3^c = l_3^{-1},$$
$$l_4^a = l_1^{-1}, l_4^b = l_2, l_4^c = l_4^{-1}, l_5^a = l_1 l_3^{-1} l_5, l_5^b = l_2^{-1} l_3^{-1} l_6^{-1}, l_5^c = l_1 l_2 l_3^{-1} l_4 l_5, l_6^a = l_2 l_4 l_6,$$
$$l_6^b = l_2^{-1} l_3 l_5^{-1}, l_6^c = l_1^{-1} l_2 l_3 l_4 l_6, l_2^{l_1} = l_2, l_3^{l_1} = l_3, l_4^{l_1} = l_4, l_5^{l_1} = l_5, l_6^{l_1} = l_6, l_3^{l_2} = l_3, l_4^{l_2} = l_4,$$
$$l_5^{l_2} = l_5, l_6^{l_2} = l_6, l_4^{l_3} = l_4, l_5^{l_3} = l_5, l_6^{l_3} = l_6, l_5^{l_4} = l_5, l_6^{l_4} = l_6, l_6^{l_5} = l_6, l_2^{l_1^{-1}} = l_2, l_3^{l_1^{-1}} = l_3,$$
$$l_4^{l_1^{-1}} = l_4, l_5^{l_1^{-1}} = l_5, l_6^{l_1^{-1}} = l_6, l_3^{l_2^{-1}} = l_3, l_4^{l_2^{-1}} = l_4, l_5^{l_2^{-1}} = l_5, l_6^{l_2^{-1}} = l_6, l_4^{l_3^{-1}} = l_4, l_5^{l_3^{-1}} = l_5,$$
$$l_6^{l_3^{-1}} = l_6, l_5^{l_4^{-1}} = l_5, l_6^{l_4^{-1}} = l_6, l_6^{l_5^{-1}} = l_6 \rangle.$$

Here, it shows that the new generator *c* can be varied. The key here is to make sure that $c = b^2$, so that the generator *c* and *b* is commute to each other and the rest of the calculation will follow.

Consistency Check for Q_2

It is vital to check whether the polycyclic presentation constructed is the correct presentation. Incorrect presentation will lead to failure in computing the algebraic properties of a group. The validity of the presentation will be checked using Definition 4. The aim is to prove that the polycyclic presentation constructed is consistent. From the previous result on Theorem 1, the theorem on its consistency relations is presented as follows:

Theorem 4:
Let $Q_2(6)$ be a Bieberbach group of dimension six with quaternion point group of order eight, and its polycyclic presentation is found to be;

$$Q_2 = \langle a,b,c,l_1,l_2,l_3,l_4,l_5,l_6 \mid a^2 = c l_6, b^2 = c, c^2 = l_5^{-1} l_6^{-1}, b^a = b c^{-1} l_5^{-1}, c^a = c, c^b = c, l_1^a = l_3,$$
$$l_1^b = l_1 l_2^{-1} l_4, l_1^c = l_1^{-1}, l_2^a = l_1 l_3 l_4, l_2^b = l_1 l_2^{-1} l_3^{-1}, l_2^c = l_2^{-1}, l_3^a = l_1^{-1}, l_3^b = l_2 l_3 l_4, l_3^c = l_3^{-1},$$
$$l_4^a = l_1 l_2^{-1} l_3^{-1}, l_4^b = l_1^{-1} l_3^{-1} l_4^{-1}, l_4^c = l_4^{-1}, l_5^a = l_5, l_5^b = l_6, l_5^c = l_5, l_6^a = l_6, l_6^b = l_5, l_6^c = l_6,$$
$$l_2^{l_1} = l_2, l_3^{l_1} = l_3, l_4^{l_1} = l_4, l_5^{l_1} = l_5, l_6^{l_1} = l_6, l_3^{l_2} = l_3, l_4^{l_2} = l_4, l_5^{l_2} = l_5, l_6^{l_2} = l_6, l_4^{l_3} = l_4, l_5^{l_3} = l_5,$$
$$l_6^{l_3} = l_6, l_5^{l_4} = l_5, l_6^{l_4} = l_6, l_6^{l_5} = l_6, l_2^{l_1^{-1}} = l_2, l_3^{l_1^{-1}} = l_3, l_4^{l_1^{-1}} = l_4, l_5^{l_1^{-1}} = l_5, l_6^{l_1^{-1}} = l_6, l_3^{l_2^{-1}} = l_3,$$
$$l_4^{l_2^{-1}} = l_4, l_5^{l_2^{-1}} = l_5, l_6^{l_2^{-1}} = l_6, l_4^{l_3^{-1}} = l_4, l_5^{l_3^{-1}} = l_5, l_6^{l_3^{-1}} = l_6, l_5^{l_4^{-1}} = l_5, l_6^{l_4^{-1}} = l_6, l_6^{l_5^{-1}} = l_6 \rangle.$$

Then, $Q_2(6)$ is consistent.

Proof:
$Q_2(6)$ is generated by *a*, *b*, *c*, l_1, l_2, l_3, l_4, l_5, l_6. By referring to Definition 4, let $g_1 = a$, $g_2 = b$, $g_3 = c$, $g_4 = l_1$, $g_5 = l_2$, $g_6 = l_3$, $g_7 = l_4$, $g_8 = l_5$, $g_9 = l_6$.

For the first consistency relation, i.e. $g_k(g_j g_i) = (g_k g_j) g_i$ for $i < j < k$, the following 84 relations hold:

(i) $c(ba) = (cb)a$
(ii) $l_1(cb) = (l_1 c)b$
(iii) $l_1(ca) = (l_1 c)a$
(iv) $l_1(ba) = (l_1 b)a$
(v) $l_2(l_1 c) = (l_2 l_1)c$
(vi) $l_2(l_1 b) = (l_2 l_1)b$
(vii) $l_2(l_1 a) = (l_2 l_1)a$
(viii) $l_2(cb) = (l_2 c)b$
(ix) $l_2(ca) = (l_2 c)a$
(x) $l_2(ba) = (l_2 b)a$
(xi) $l_3(l_2 l_1) = (l_3 l_2)l_1$
(xii) $l_3(l_2 c) = (l_3 l_2)c$
(xiii) $l_3(l_2 b) = (l_3 l_2)b$
(xiv) $l_3(l_2 a) = (l_3 l_2)a$
(xv) $l_3(l_1 c) = (l_3 l_1)c$
(xvi) $l_3(l_1 b) = (l_3 l_1)b$
(xvii) $l_3(l_1 a) = (l_3 l_1)a$
(xviii) $l_3(cb) = (l_3 c)b$
(xix) $l_3(ca) = (l_3 c)a$
(xx) $l_3(ba) = (l_3 b)a$
(xxi) $l_4(l_3 l_2) = (l_4 l_3)l_2$
(xxii) $l_4(l_3 l_1) = (l_4 l_3)l_1$

(xxiii) $l_4(l_3c) = (l_4l_3)c$

(xxiv) $l_4(l_3b) = (l_4l_3)b$

(xxv) $l_4(l_3a) = (l_4l_3)a$

(xxvi) $l_4(l_2l_1) = (l_4l_2)l_1$

(xxvii) $l_4(l_2c) = (l_4l_2)c$

(xxviii) $l_4(l_2b) = (l_4l_2)b$

(xxix) $l_4(l_2a) = (l_4l_2)a$

(xxx) $l_4(l_1c) = (l_4l_1)c$

(xxxi) $l_4(l_1b) = (l_4l_1)b$

(xxxii) $l_4(l_1a) = (l_4l_1)a$

(xxxiii) $l_4(cb) = (l_4c)b$

(xxxiv) $l_4(ca) = (l_4c)a$

(xxxv) $l_4(ba) = (l_4b)a$

(xxxvi) $l_5(l_4l_3) = (l_5l_4)l_3$

(xxxvii) $l_5(l_4l_2) = (l_5l_4)l_2$

(xxxviii) $l_5(l_4l_1) = (l_5l_4)l_1$

(xxxix) $l_5(l_4c) = (l_5l_4)c$

(xl) $l_5(l_4b) = (l_5l_4)b$

(xli) $l_5(l_4a) = (l_5l_4)a$

(xlii) $l_5(l_3l_2) = (l_5l_3)l_2$

(xliii) $l_5(l_3l_1) = (l_5l_3)l_1$

(xliv) $l_5(l_3c) = (l_5l_3)c$

(xlv) $l_5(l_3b) = (l_5l_3)b$

(xlvi) $l_5(l_3a) = (l_5l_3)a$

(xlvii) $l_5(l_2l_1) = (l_5l_2)l_1$

(xlviii) $l_5(l_2c) = (l_5l_2)c$

(xlix) $l_5(l_2b) = (l_5l_2)b$

(l) $l_5(l_2a) = (l_5l_2)a$

(li) $l_5(l_1c) = (l_5l_1)c$

(lii) $l_5(l_1b) = (l_5l_1)b$

(liii) $l_5(l_1a) = (l_5l_1)a$

(liv) $l_5(cb) = (l_5c)b$

(lv) $l_5(ca) = (l_5c)a$

(lvi) $l_5(ba) = (l_5b)a$

(lvii) $l_6(l_5l_4) = (l_6l_5)l_4$

(lviii) $l_6(l_5l_3) = (l_6l_5)l_3$

(lix) $l_6(l_5l_2) = (l_6l_5)l_2$

(lx) $l_6(l_5l_1) = (l_6l_5)l_1$

(lxi) $l_6(l_5c) = (l_6l_5)c$

(lxii) $l_6(l_5b) = (l_6l_5)b$

(lxiii) $l_6(l_5a) = (l_6l_5)a$

(lxiv) $l_6(l_4l_3) = (l_6l_4)l_3$

(lxv) $l_6(l_4l_2) = (l_6l_4)l_2$

(lxvi) $l_6(l_4l_1) = (l_6l_4)l_1$

(lxvii) $l_6(l_4c) = (l_6l_4)c$

(lxviii) $l_6(l_4b) = (l_6l_4)b$

(lxix) $l_6(l_4a) = (l_6l_4)a$

(lxx) $l_6(l_3l_2) = (l_6l_3)l_2$

(lxxi) $l_6(l_3l_1) = (l_6l_3)l_1$

(lxxii) $l_6(l_3c) = (l_6l_3)c$

(lxxiii) $l_6(l_3b) = (l_6l_3)b$

(lxxiv) $l_6(l_3a) = (l_6l_3)a$

(lxxv) $l_6(l_2l_1) = (l_6l_2)l_1$

(lxxvi) $l_6(l_2c) = (l_6l_2)c$

(lxxvii) $l_6(l_2b) = (l_6l_2)b$

(lxxviii) $l_6(l_2a) = (l_6l_2)a$

(lxxix) $l_6(l_1c) = (l_6l_1)c$

(lxxx) $l_6(l_1b) = (l_6l_1)b$

(lxxxi) $l_6(l_1a) = (l_6l_1)a$

(lxxxii) $l_6(cb) = (l_6c)b$

(lxxxiii) $l_6(ca) = (l_6c)a$

(lxxxiv) $l_6(ba) = (l_6b)a$

Hence, by the polycyclic presentation of $Q_2(6)$:

For (i),

$$c(ba) = c\left(ab^a\right) = cabc^{-1}l_5^{-1} = abcc^{-1}l_5^{-1} = abl_5^{-1}$$

$$(cb)a = \left(bc^b\right)a = bac = abc^{-1}l_5^{-1}c = abl_5^{-1}c^{-1}c = abl_5^{-1}$$

For (ii),

$$l_1(cb) = l_1\left(bc^b\right) = l_1bc = bl_1l_2^{-1}l_4c = bl_1l_2^{-1}cl_4^{-1} = bl_1cl_2l_4^{-1} = bcl_1^{-1}l_2l_4^{-1}$$

$$(l_1c)b = \left(cl_1^c\right)b = cl_1^{-1}b = cbl_4^{-1}l_2l_1^{-1} = bcl_4^{-1}l_2l_1^{-1} = bcl_4^{-1}l_1^{-1}l_2 = bcl_1^{-1}l_4^{-1}l_2 = bcl_1^{-1}l_2l_4^{-1}$$

By using the same method, the other 82 possible relations for the first consistency relations are proved to satisfy both sides of the equation.

For the second consistency relation, i.e. $\left(g_j^{e_i}\right)g_i = g_j^{e_i-1}\left(g_j g_i\right)$ for $j > i$, $j \in I$, the following 3 relations are satisfied:

(i) $b^2a = b(ba)$

(ii) $c^2b = c(cb)$

(iii) $c^2a = c(ca)$

Therefore, by the polycyclic presentation of $Q_2(6)$:

For (i),

$b^2a = ca = ac$

$b(ba) = b\left(ab^a\right) = babc^{-1}l_5^{-1} = abc^{-1}l_5^{-1}bc^{-1}l_5^{-1} = abc^{-1}bl_6^{-1}c^{-1}l_5^{-1} = abc^{-1}bc^{-1}l_6^{-1}l_5^{-1}$

$= abc^{-1}bc^{-1}l_5^{-1}l_6^{-1} = abbc^{-1}c^{-1}l_5^{-1}l_6^{-1} = ab^2\left(c^2\right)^{-1}l_5^{-1}l_6^{-1} = ac\left(l_5^{-1}l_6^{-1}\right)^{-1}l_5^{-1}l_6^{-1}$

$= acl_6l_5l_5^{-1}l_6^{-1} = acl_6l_6^{-1} = ac$

For (ii),

$c^2b = l_5^{-1}l_6^{-1}b = l_5^{-1}bl_5^{-1} = bl_6^{-1}l_5^{-1} = bl_5^{-1}l_6^{-1}$

$c(cb) = cbc = bcc = bc^2 = bl_5^{-1}l_6^{-1}$

For (iii),

$c^2a = l_5^{-1}l_6^{-1}a = l_5^{-1}al_6^{-1} = al_5^{-1}l_6^{-1}$

$c(ca) = cac = acc = ac^2 = al_5^{-1}l_6^{-1}$

For the third consistency relation, i.e. $g_j\left(g_i^{e_i}\right) = \left(g_j g_i\right)g_i^{e_i-1}$ for $j > i$, $i \in I$, the following 21 relations hold:

(i) $b\left(a^2\right) = (ba)a$

(ii) $c\left(b^2\right) = (cb)b$

(iii) $c\left(a^2\right) = (ca)a$

(iv) $l_1\left(c^2\right) = (l_1c)c$

(v) $l_1\left(b^2\right) = (l_1b)b$

(vi) $l_1\left(a^2\right) = (l_1a)a$

(vii) $l_2\left(c^2\right) = (l_2c)c$

(viii) $l_2\left(b^2\right) = (l_2b)b$

(ix) $l_2\left(a^2\right) = (l_2a)a$

(x) $l_3\left(c^2\right) = (l_3c)c$

(xi) $l_3\left(b^2\right) = (l_3b)b$

(xii) $l_3\left(a^2\right) = (l_3a)a$

(xiii) $l_4\left(c^2\right) = (l_4c)c$

(xiv) $l_4\left(b^2\right) = (l_4b)b$

(xv) $l_4\left(a^2\right) = (l_4a)a$

(xvi) $l_5\left(c^2\right) = (l_5c)c$

(xvii) $l_5\left(b^2\right) = (l_5b)b$

(xviii) $l_5\left(a^2\right) = (l_5a)a$

(xix) $l_6\left(c^2\right) = (l_6c)c$

(xx) $l_6\left(b^2\right) = (l_6b)b$

(xxi) $l_6\left(a^2\right) = (l_6a)a$

Thus, by the polycyclic presentation of $Q_2(6)$:

For (i),

$b\left(a^2\right) = bcl_6$

$(ba)a = \left(ab^a\right)a = abc^{-1}l_5^{-1}a = abc^{-1}al_5^{-1} = abac^{-1}l_5^{-1} = aabc^{-1}l_5^{-1}c^{-1}l_5^{-1}$

$= a^2bc^{-1}l_5^{-1}c^{-1}l_5^{-1} = a^2bc^{-1}c^{-1}l_5^{-1}l_5^{-1} = a^2b\left(c^2\right)^{-1}l_5^{-1}l_5^{-1} = a^2bl_6l_5l_5^{-1}l_5^{-1}$

$= a^2bl_6l_5^{-1} = a^2bl_5^{-1}l_6 = cl_6bl_5^{-1}l_6 = cbl_5l_5^{-1}l_6 = cbl_6 = bcl_6$

For (ii),

$c\left(b^2\right) = cc = c^2 = l_5^{-1}l_6^{-1}$

$(cb)b = \left(bc^b\right)b = bcb = bbc = b^2c = cc = c^2 = l_5^{-1}l_6^{-1}$

By using the same method, the other 19 possible relations for the third consistency relations are proved to satisfy both sides of the equation.

For the fourth consistency relation, i.e. $\left(g_i^{e_i}\right)g_i = g_i\left(g_i^{e_i}\right)$ for $i \in I$, the following 3 relations hold:

(i) $\left(a^2\right)a = a\left(a^2\right)$

(ii) $\left(b^2\right)b = b\left(b^2\right)$

(iii) $\left(c^2\right)c = c\left(c^2\right)$

Therefore, by the polycyclic presentation of $Q_2(6)$:

For (i),

$\left(a^2\right)a = cl_6a = cal_6 = acl_6$

$a\left(a^2\right) = acl_6$

For (ii),

$\left(b^2\right)b = cb = bc$

$b\left(b^2\right) = bc$

For (iii),

$\left(a^2\right)a = cl_6a = cal_6 = acl_6$

$a\left(a^2\right) = acl_6$

For the fifth consistency relation, i.e. $g_j = \left(g_jg_i^{-1}\right)g_i$ for $j > i$, $i \notin I$, the following 15 relations hold:

(i) $l_2 = \left(l_2l_1^{-1}\right)l_1$

(ii) $l_3 = \left(l_3l_1^{-1}\right)l_1$

(iii) $l_4 = \left(l_4l_1^{-1}\right)l_1$

(iv) $l_5 = \left(l_5l_1^{-1}\right)l_1$

(v) $l_6 = \left(l_6l_1^{-1}\right)l_1$

(vi) $l_3 = \left(l_3l_2^{-1}\right)l_2$

(vii) $l_4 = \left(l_4l_2^{-1}\right)l_2$

(viii) $l_5 = \left(l_5l_2^{-1}\right)l_2$

(ix) $l_6 = \left(l_6 l_2^{-1}\right) l_2$

(x) $l_4 = \left(l_4 l_3^{-1}\right) l_3$

(xi) $l_5 = \left(l_5 l_3^{-1}\right) l_3$

(xii) $l_6 = \left(l_6 l_3^{-1}\right) l_3$

(xiii) $l_5 = \left(l_5 l_4^{-1}\right) l_4$

(xiv) $l_6 = \left(l_6 l_4^{-1}\right) l_4$

(xv) $l_6 = \left(l_6 l_5^{-1}\right) l_5$

Thus, by the polycyclic presentation of $Q_2(6)$:
For (i),

$$l_2 = \left(l_2 l_1^{-1}\right) l_1 = \left[\left(l_1^{-1}\right)^{l_2}\right] l_1 = l_1^{-1} l_2 l_1 = l_1^{-1} l_1 l_2 = l_2$$

For (ii),

$$l_3 = \left(l_3 l_1^{-1}\right) l_1 = \left[\left(l_1^{-1}\right)^{l_3}\right] l_1 = l_1^{-1} l_3 l_1 = l_1^{-1} l_1 l_3 = l_3$$

By using the same method, the other 13 possible relations for the fifth consistency relations are proved to satisfy both sides of the equation.

Since the polycyclic presentation of $Q_2(6)$ satisfies the consistency relations as shown in Definition 4, hence, it is concluded that $Q_2(6)$ is a correct polycyclic presentation with its new well-defined generator $c = a_0{}^2 l_6{}^{-1}$.

Conclusions

In this research, the polycyclic presentations of the second, third and fourth Bieberbach group of dimension six with quaternion point group of order eight, namely Q_2, Q_3 and Q_4 have been constructed with the help of Groups, Algorithms, and Programming (GAP) software. The second polycyclic presentation, Q_2 have been proved to satisfy its consistency relation, and the other two groups namely, Q_3 and Q_4 might be tested to meet its consistency relations later. The findings of this study could be applied for further research to find the homological invariants such as the nonabelian tensor square, the kernel of homomorphism of the nonabelian tensor square, or the Schur multiplier by using these group presentations.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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