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RESEARCH ARTICLE

Laplacian Spectrum and Energy of the Cyclic Order Product Prime Graph of Semi-dihedral Groups

Norlyda Mohamed[a,b], Nor Muhainiah Mohd Ali[a*], Muhammed Bello[c]

[a]Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia; [b]Mathematical Sciences Studies, College of Computing, Informatics and Mathematics, Universiti Teknologi MARA Cawangan Negeri Sembilan Kampus Seremban, 70300 Seremban, Negeri Sembilan, Malaysia; [c]Department of Mathematics and Statistics, Federal University of Kashere, P.M.B 0182 Gombe, Gombe State, Nigeria

Abstract This paper introduces cyclic order product prime graph to examine the spectral graph properties of semi-dihedral groups, specifically focusing on the Laplacian spectrum and energy. Combining principles from cyclic graphs and order product prime graphs enhances understanding of group algebraic structures. Let G be a finite group. In this context, the cyclic order product prime graph of G is defined as a simple undirected graph with vertex set G where two distinct vertices, x and y, are adjacent if and only if $\langle x, y\rangle$ is a proper cyclic subgroup of G and $|x||y| = p^{\alpha}, \alpha \in \mathbb{N}$ for some prime p. Our methodological approach begins with establishing a general presentation for these graphs within semi-dihedral groups. This foundational step is essential for deriving some properties, such as vertex degrees, the number of edges, and Laplacian characteristic polynomials. This information subsequently facilitates the determination of the Laplacian spectrum, characterized by seven eigenvalues of various multiplicities, and the computation of their Laplacian energy. The semi-dihedral group of order 16 serves as a particular example to illustrate the practicality and generality of our theorems.

Keywords: Graph of group, semi-dihedral group, cyclic order product prime graph, Laplacian spectrum, Laplacian energy.

Introduction

The study of graphs of groups establishes a close connection between algebraic graph theory and group theory. The graphs constructed from group elements and their respective connections help us to understand a group's algebraic structure visually and combinatorially. The intricate relationship between the properties of groups and the characteristics of their associated graphs provides a chance for theoretical exploration and practical application. By exploring these connections, researchers can uncover many aspects of the graphs of groups, including those related to spectral graph theory. The Laplacian spectrum and Laplacian energy, central to the spectral graph theory, have evolved significantly since their introduction in the mid-20th century. With their origins in electrical circuit theory [1], the principles of the Laplacian spectrum substantially impact group theory, particularly in graph analysis. The Laplacian spectrum refers to the collection of eigenvalues obtained from its Laplacian matrix, which provides important insights into the structure and properties of the graph. The Laplacian energy is an extension of the study of the Laplacian spectrum. The concept of graph energy was initially developed by Gutman in the 1970s, with a specific emphasis on its use in molecular chemistry. Subsequently, Gutman and Zhou [2] improved upon this concept in 2006, introducing a new concept of Laplacian energy. The energy of a graph is defined as the sum of the absolute value of the eigenvalues of the graph's adjacency matrix. In contrast, the concept of Laplacian energy in graph theory encompasses the Laplacian matrix of the graph by subtracting the adjacency matrix from the degree matrix.

***For correspondence:**
normuhainiah@utm.my

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Various graphs of groups, such as cyclic graphs and order product prime graphs, have been introduced and studied to explore their unique properties. Let G be a finite group with the identity element e. Cyclic graph, Γ^c, is a graph in which two vertices, x and y, are joined if and only if $\langle x, y\rangle$ is a cyclic subgroup of G [3]. In some literature, the cyclic graph is also known as the enhanced power graph, representing a graph between the power graph and the commuting graph [4]. In recent years, there has been increased research on the enhanced power graph of finite groups and their related properties. On the other hand, the order product prime graph, Γ^{op}, is a graph whose vertices are all elements of G and two vertices, x and y, are adjacent if and only $|x||y| = p^\alpha, \alpha \in \mathbb{N}$ for some prime p [5]. The order product prime graph expands upon the concept of prime graphs by integrating the multiplicative aspect of element orders. Bello *et al*. [5] also emphasized on the importance of establishing the structural general presentations of the graphs of groups before exploring the graph properties. Understanding the presentations is important as they provide insights into graph connectivity and facilitate the easier derivation of related properties. Hence, various graph properties can easily be obtained using the general presentation. Here, establishing relationships between the cyclic graph and the order product prime graph led us to introduce a graph consisting of these two types of restrictions simultaneously to provide a more comprehensive view of the group's structure. The practical applications of group theory can be expanded, and theoretical understanding of the subject matter can be improved in various fields, leading to new results of the Laplacian spectrum and energy.

In the past decade, significant research has been directed at the matrices associated with graphs of groups, including the Laplacian spectrum and energy. These spectra have been studied by some researchers for the commuting graph of certain finite groups [6, 7, 8, 9]. Comprehensive investigations of these spectra also covered for the power graph of some finite groups [10, 11, 12, 13]. Parveen *et al*. [14] continued this work for the enhanced power graph, focusing on the semi-dihedral, dihedral, and generalized quaternion groups. Researchers have extensively investigated the graph energy [15, 16, 17] and Laplacian energy [18, 19, 20] for various graphs of groups. Recently, Ugasini *et al*. [21] analysed the adjacency and Laplacian spectrum of certain non-simple standard graphs containing self-loops, calculated their energy and Laplacian energy, and derived lower bounds for these energies. Further exploration can be done on this topic by considering other types of graphs of groups.

This paper introduces a cyclic order product prime graph as the intersection graph of the cyclic graph and order product prime graph. By combining the restriction of cyclic graph and order product prime graph, this new graph has the potential to establish connections between the restrictions imposed by the cyclic graph and the order product prime graph and, consequently, uncover new properties and relationships. Then, we give general presentations of the graph, specifically on the semi-dihedral groups. The obtained general presentations will aid in finding the vertex degree, number of edges, their Laplacian characteristic polynomials, Laplacian spectrum, and Laplacian energy. The implications of our findings could extend beyond theoretical interest, offering valuable insights for applications in computational and physical sciences.

Preliminaries

This section presents the essential information, definitions, and theorems used throughout this research for the convenience of the readers.

Definition 1 [22] Semi-dihedral groups
The semi-dihedral group of order $2^n, SD_{2^n}$, is a group generated by two elements a and b such that

$$SD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = e, ba = a^{2^{n-2}-1}b\rangle \text{ for all } n \geq 4.$$

By letting $R = \{e, a, a^2, \cdots, a^{2^{n-1}-1}\} = \langle a\rangle$, $F_1 = \{b, a^2b, \cdots, a^{2^{n-1}-2}b\} = \sum_{j=0}^{2^{n-2}-1} a^{2j}b$ and $F_2 = \{ab, a^3b, \cdots, a^{2^{n-1}-1}b\} = \sum_{j=0}^{2^{n-2}-1} a^{2j+1}b$ where $j \in \mathbb{N}$, the semi-dihedral group of order $2^n, SD_{2^n}$, can also be presented as $SD_{2^n} = R \cup F_1 \cup F_2$. The centre of $SD_{2^n}, Z(SD_{2^n}) = \{e, a^{2^{n-2}}\}$ for all $n \geq 4$.

Theorem 1 [22] The orders of elements within the semi-dihedral group SD_{2^n}, are characterized as follows:

(1) The order of the identity element e is 1, denoted as $|e| = 1$,

(2) $|a^i| = \frac{2^{n-1}}{\gcd(i, 2^{n-1})}$ for any integer $0 \leq i < 2^{n-1}$,

(3) for any integer $0 \leq j < 2^{n-1}$, $|a^jb| = 2$ and $|a^jb| = 4$, when j is even and odd, respectively.

Definition 2 [23] Cyclic groups
The cyclic group of order n, $\mathbb{Z}_n$, is a group generated by a single element a such that $G = \langle a \rangle = \{a^n : n \in \mathbb{Z}\}$.

For the graph mentioned in this research, we consider finite simple undirected graphs. The vertex and edge sets of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$ respectively. The notation $x \sim y$ is used to indicate that vertices x and y are adjacent.

Definition 3 [24] Complete graph
A graph Γ is known as a complete graph, K_n, provided that every vertex $v \in V(\Gamma)$ is connected to other vertices in $V(\Gamma)$.

Definition 4 [25] Disjoint union and join of graphs
Let $\Gamma_1 = (\mathrm{V}_1, \mathrm{E}_1)$ and $\Gamma_2 = (\mathrm{V}_2, \mathrm{E}_2)$ be two subgraphs of Γ. The disjoint union of the graphs, $\Gamma_1 \cup \Gamma_2$ is a graph with the vertex set $\mathrm{V}_1 \cup \mathrm{V}_2$ and edge set $\mathrm{E}_1 \cup \mathrm{E}_2$, where V_1 and V_2 are disjoint sets of vertices. The join of the graphs, $\Gamma_1 + \Gamma_2$ is obtained from $\Gamma_1 \cup \Gamma_2$ by adding edges joining all vertices of Γ_1 to Γ_2.

Definition 5 [24] Degree of a vertex of a graph
For any vertex $v \in V(\Gamma)$, the degree of a vertex v in Γ, $deg(v)$, is the number of edges that are incident on v.

Lemma 1 [26] Handshaking Lemma
In any undirected graph Γ, the total number of edges is equal to half the sum of the degrees of all vertices in Γ that is $|E(\Gamma)| = \frac{1}{2}\sum_{v \in V(\Gamma)} deg(v)$.

Definition 6 [26] Average degree of a vertex of an undirected graph
The average degree of a vertex of a graph Γ, $\bar{d}(\Gamma) = \frac{2|E(\Gamma)|}{|V(\Gamma)|}$ where $|E(\Gamma)|$ is the number of edges and $|V(\Gamma)|$ is the number of vertices in the graph.

Definition 7 [27] Laplacian matrix of a graph
The Laplacian matrix of a graph, $L(\Gamma)$, is defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$ where $D(\Gamma)$ is a degree matrix and $A(\Gamma)$ is an adjacency matrix of the graph. The degree matrix is a diagonal matrix with the degree of each vertex on the diagonal. The adjacency matrix, $A(\Gamma) = \left(a_{ij}\right)_{i,j=1}^{n}$ in which $a_{ij} = 1$ if the vertex v_i is adjacent to v_j or $a_{ij} = 0$ if it is otherwise.

Definition 8 [10] Laplacian characteristic polynomial of a graph
The Laplacian characteristic polynomial of a graph Γ, $p_L(\lambda)$, is defined as $p_L(\lambda) = \det\left(\lambda I - L(\Gamma)\right)$ where λ is a real number, I is the identity matrix, and $L(\Gamma)$ is the Laplacian matrix associated to the graph Γ.

This polynomial plays an important role in graph theory, as its roots provide valuable insights into the graph's properties, including its eigenvalues and structural characteristics.

Definition 9 [14] Laplacian spectrum of a graph
The Laplacian spectrum of a graph Γ, $L_{spec}(\Gamma)$, refers to the Laplacian eigenvalues, $\lambda_1, \lambda_2, \cdots, \lambda_n$ with their multiplicities, $m_1, m_2, \cdots, m_n$, respectively and can be presented as $L_{spec}(\Gamma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}$. These eigenvalues and multiplicities are obtained from the roots of the characteristic polynomial, $p_L(\lambda)$ given by $\det\left(\lambda I - L(\Gamma)\right) = 0$.

Definition 10 [9] Laplacian energy of a graph
The Laplacian energy of a graph Γ, $L_E(\Gamma) = \sum_{k=1}^{n}\left|\lambda_k - \bar{d}(\Gamma)\right|$ is the sum of the absolute value of differences between each eigenvalue in the Laplacian spectrum, $L_{spec}(\Gamma)$ and the average degree of vertices in a graph, $\bar{d}(\Gamma)$.

Results and Discussion

In this section, we introduce the cyclic order product prime graph and establish a general representation for these graphs on the semi-dihedral group of order $2^n, SD_{2^n}$. This representation enables the calculation of vertex degrees, number of edges and Laplacian characteristic polynomials, leading to further insights into their Laplacian spectrum and Laplacian energy.

Definition 11 Cyclic order product prime graph
Let G be a finite group. The cyclic order prime graph, $\Gamma^{cop}(G)$, is the graph having vertex set G where two distinct vertices, x and y, are adjacent if and only if $\langle x, y\rangle$ is a proper cyclic subgroup of G and $|x||y| = p^\alpha, \alpha \in \mathbb{N}$ for some prime p.

Proposition 1 outlines the criteria for selecting two vertices, x and y such that $\langle x, y\rangle$ forms the proper cyclic subgroup of G. The general presentation of the cyclic order product prime graph for semi-dihedral groups of order 2^n is then established by Theorem 2. A detailed illustration of this concept, specifically for the case where $n = 4$, is provided in Example 1.

Proposition 1 Let G be the semi-dihedral group of order 2^n, SD_{2^n} for $n \geq 4$. For all $x, y \in G, \langle x, y\rangle$ is equal to the proper cyclic subgroup of G if and only if

1. $x, y \in \langle a\rangle$,
2. $x = e$ and $y \in F_1 \cup F_2$,
3. $x = a^{2^{n-2}}$ and $y \in F_2$,
4. $x, y \in F_2$ where $x = a^{2j+1}b$ and $y = a^{2^{n-2}+2j+1}b$ for $0 \leq j < 2^{n-3} - 1$,

where $F_1 = \sum_{j=0}^{2^{n-2}-1} a^{2j}b$ and $F_2 = \sum_{j=0}^{2^{n-2}-1} a^{2j+1}b$ for $j \in \mathbb{N}$.

Proof. From Definition 1, consider the group presentation of $SD_{2^n} = R \cup F_1 \cup F_2$ and the centre of $SD_{2^n}, Z(SD_{2^n}) = \{e, a^{2^{n-2}}\}$ for $n \geq 4$. Obviously, for all $x = e, y \in SD_{2^n}$, then $\langle x, y\rangle = \langle e, y\rangle = \langle y\rangle$. Note that $R \cong \mathbb{Z}_{2^{n-1}} = \langle a\rangle$ which is always cyclic. Also, for all $x = a^{2^{n-2}}$, $\langle x, y\rangle = \langle a^{2^{n-2}}, y\rangle$ is a proper cyclic subgroup of G if $y \in SD_{2^n}\backslash\{F_1\}$ since for $y \in F_1$, y has unique elements of order 2 which cannot form the proper cyclic subgroups with any other elements except e. If $\langle x\rangle = \langle y\rangle$ for all $x, y \in F_2$, that is, $x = a^{2j+1}b$ and $y = a^{2^{n-2}+2j+1}b$ for $0 \leq j < 2^{n-3} - 1$, then $\langle x, y\rangle$ is a proper cyclic subgroup. Otherwise, $\langle x, y\rangle$ is not a proper cyclic subgroup.

Theorem 2 Let G be the semi-dihedral group of order 2^n, SD_{2^n} where $n \geq 4$. Then,

$$\Gamma^{cop}(G) = K_1 + \left(\left(K_1 + (K_{2^{n-1}-2} \cup 2^{n-3}K_2)\right) \cup \overline{K_{2^{n-2}}}\right).$$

Proof. Let $x, y \in G$. From Proposition 1, select two vertices, x and y, that form proper cyclic subgroup of G satisfying the first condition of $\Gamma^{cop}(G)$. Also, by Theorem 1, we have $|x||y| = 2^\alpha, \alpha \in \mathbb{N}$, since all elements has order power of 2, satisfying the second condition of $\Gamma^{cop}(G)$. Note that the identity element, e is always adjacent to every other element in G, while the nontrivial central element, $a^{2^{n-2}}$ is adjacent to all other elements except those in F_1. Since R is always cyclic, all elements of R form single clique of in $\Gamma^{cop}(G)$. For all $x, y \in F_2$ and $\langle x\rangle = \langle y\rangle$, each pair forms a clique of size 2 and there are 2^{n-3} such cliques. Therefore,

$$\begin{aligned}\Gamma^{cop}(G) &= K_{|e|} + \left(\left(K_{\left|a^{2^{n-2}}\right|} + \left(K_{\left|R\backslash\{e, a^{2^{n-2}}\}\right|} \cup \frac{|F_2|}{2}K_2\right)\right) \cup \overline{K_{|F_1|}}\right) \\ &= K_1 + \left(\left(K_1 + (K_{2^{n-1}-2} \cup 2^{n-3}K_2)\right) \cup \overline{K_{2^{n-2}}}\right).\end{aligned}$$

■

Example 1 Let $SD_{16} = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\} = \langle a\rangle \cup \sum_{i=0}^{3} a^{2i}b \cup \sum_{j=0}^{3} a^{2j+1}b$. From Theorem 2, we have $\langle x, y\rangle$ is a proper cyclic subgroup of G if $x = e, y \in SD_{16}$; $x = a^4, y \in SD_{16}\backslash\{b, a^2b, a^4b, a^6b\}$; $x, y \in \langle a\rangle$; $x = ab, y = a^5b$ and $x = a^3b, y = a^7b$. For all $x, y \in SD_{16}$, the order of distinct element, x and y is either $1, 2, 4$ or 8 , hence, $|x||y| = 2^\alpha, \alpha \in \mathbb{N}$. Therefore, $\Gamma^{cop}(SD_{16}) = K_1 + \left(K_1 + (K_6 \cup 2K_2) \cup \overline{K_4}\right)$ and can be drawn as in Figure 1.

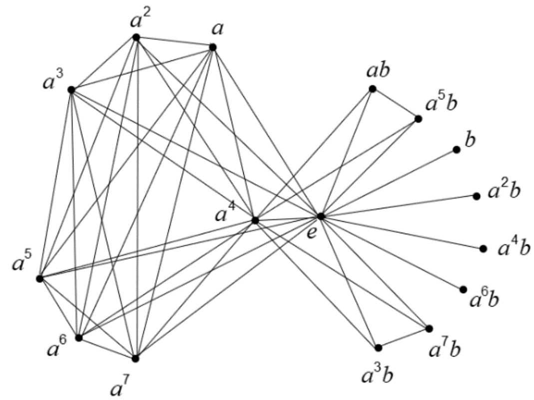

Figure 1. Cyclic order product prime graph of SD_{16}, $\Gamma^{cop}(SD_{16})$

Remarks The group SD_{16} will be consistently employed to illustrate subsequent theorems.

Proposition 2 describes the degrees of vertices corresponding to the elements of semi-dihedral groups which can be used in investigating other properties of the graph.

Proposition 2 Let G be the semi-dihedral group of order 2^n, SD_{2^n} for $n \geq 4$. In $\Gamma^{cop}(SD_{2^n})$,

1. $\deg(e) = |G| - 1$,
2. for all $1 \leq i < 2^{n-1}$, $\deg(a^i) = \begin{cases} \frac{3|G|}{4} - 1, & i = 2^{n-2}, \\ \frac{|G|}{2} - 1, & i \neq 2^{n-2}, \end{cases}$
3. for $0 \leq j < 2^{n-1}$, $\deg(a^j b) = \begin{cases} 1, & j \text{ is even}, \\ 3, & j \text{ is odd}. \end{cases}$

Proof. Let G be the semi-dihedral group of order 2^n, SD_{2^n} for $n \geq 4$ which can be presented as $SD_{2^n} = R \cup F_1 \cup F_2$ as in Definition 1. From Theorem 2 and Definition 5, it is clear that

1. the set R have $\frac{|G|}{2}$ elements and interconnected. The identity element, $e \in R$ is adjacent to every other element in G as they form a cyclic subgroup with e. Hence, $\deg(e) = |G| - 1$,
2. the nontrivial central element of G, $a^{2^{n-2}} \in R$ is also adjacent to all those elements in F_2 that have $\frac{|G|}{4}$ elements and thus, $\deg(a^{2^{n-2}}) = \frac{3|G|}{4} - 1$. For all $a^i \in R \backslash \{e, a^{2^{n-2}}\}$, $\deg(a^i) = \frac{|G|}{2} - 1$ as reflect the interconnected within the set R.
3. the reflection elements in F_1 are only connected to the identity element, e but the reflection elements in F_2 are connected to $e, a^{2^{n-2}}$ and its inverse elements. Therefore, $\deg(a^j b) = 1$ and $\deg(a^j b) = 3$ if j is even and j is odd, respectively.

■

Proposition 3 Let SD_{2^n} be the semi-dihedral group of order 2^n for $n \geq 4$. Then, the number of edges, $|E(\Gamma^{cop}(G))| = \frac{2^n(5+2^n)}{8}$ and the average degree of a vertex, $\bar{d}(\Gamma^{cop}(G)) = \frac{5+2^n}{4}$, accordingly.

Proof. For all $n \geq 4$, the number of vertices, $V(\Gamma^{cop}(SD_{2^n})) = 2^n$ and the sum of the degrees of all vertices in $\Gamma^{cop}(SD_{2^n})$,

$$\sum_{v \in V(\Gamma^{cop})} \deg(v) = \deg(e) + \sum_{i=1}^{2^{n-1}-1} \deg(a^i) + \sum_{j=0}^{2^{n-1}-1} \deg(a^j b)$$
$$= (2^n - 1) + (3(2^{n-2}) - 1) + (2^{n-1} - 2)(2^{n-1} - 1) + 2^{n-2} + 3(2^{n-2})$$

which arised from Proposition 2 and lead us to obtain

$$\sum_{v \in V(\Gamma^{cop})} \deg(v) = \frac{2^n(5 + 2^n)}{4}.$$

By applying Lemma 1, the number of edges in $\Gamma^{cop}(SD_{2^n})$ can be obtained with simple substitution,

$$\left|E\left(\Gamma^{cop}(G)\right)\right| = \frac{1}{2}\sum_{v\in V(\Gamma^{cop})} \deg(v) = \frac{2^n(5+2^n)}{8}.$$

Then, the desired result for the average degree of a vertex of the graph, $\bar{d}\left(\Gamma^{cop}(G)\right)$ can be obtained with formula from Definition 6 that is

$$\bar{d}\left(\Gamma^{cop}(G)\right) = \frac{2\left|E\left(\Gamma^{cop}(G)\right)\right|}{\left|V\left(\Gamma^{cop}(G)\right)\right|} = \frac{5+2^n}{4}.$$

■

Now, the Laplacian spectrum of the cyclic order product prime graph ,$L_{spec}\left(\Gamma^{cop}(SD_{2^n})\right)$ is provided together with their Laplacian characteristic polynomial, $p_{L\left(\Gamma^{cop}(SD_{2^n})\right)}$.

Theorem 3 For all $n \geq 4$, the Laplacian characteristic polynomial of $\Gamma^{cop}(SD_{2^n})$ is given by

$$p_{L\left(\Gamma^{cop}(SD_{2^n})\right)}(\lambda) = \lambda\left(\lambda - 3(2^{n-2})\right)(\lambda - 2^n)(\lambda-2)^{2^{n-3}}(\lambda-4)^{2^{n-3}}(\lambda-1)^{2^{n-2}}(\lambda-2^{n-1})^{2^{n-1}-3}$$

and the Laplacian spectrum of $\Gamma^{cop}(SD_{2^n})$ is

$$L_{spec}\left(\Gamma^{cop}(SD_{2^n})\right) = \begin{pmatrix} 0 & 3(2^{n-2}) & 2^n & 2 & 4 & 1 & 2^{n-1} \\ 1 & 1 & 1 & 2^{n-3} & 2^{n-3} & 2^{n-2} & 2^{n-1}-3 \end{pmatrix}.$$

Proof. Consider the cyclic order product prime graph of semi-dihedral group of order 2^n, $\Gamma^{cop}(SD_{2^n})$ for $n \geq 4$. Utilizing Theorem 2 and Proposition 3, the Laplacian matrix, $L\left(\Gamma^{cop}(SD_{2^n})\right)$ of size $2^n \times 2^n$ is constructed. The rows and columns of , $L\left(\Gamma^{cop}(SD_{2^n})\right)$ are indexed in order by the vertices $e, a^{2^{n-2}}, a, a^2, \dots, a^{2^{n-2}-1}, a^{2^{n-2}+1}, \dots, a^{2^{n-1}-1}$ and then all element in F_2 and F_1 as defined in Definition 1 that can be represented as follows

$$L\left(\Gamma^{cop}(SD_{2^n})\right) = \begin{pmatrix} A_{2\times 2} & B_{2\times 2^{n-1}-2} & C_{2\times 2^{n-2}} & D_{2\times 2^{n-2}} \\ P_{2^{n-1}-2\times 2} & Q_{2^{n-1}-2\times 2^{n-1}-2} & O_{k_1} & O_{k_1} \\ R_{2^{n-2}\times 2} & O_{k_2} & S_{2^{n-2}\times 2^{n-2}} & O_{k_3} \\ T_{2^{n-2}\times 2} & O_{k_2} & O_{k_3} & I_{2^{n-2}\times 2^{n-2}} \end{pmatrix}$$

where

$$A_{2\times 2} = \begin{pmatrix} 2^n-1 & -1 \\ -1 & 2^{n-1}-1+2^{n-2} \end{pmatrix}, \quad B_{2\times 2^{n-1}-2} = \begin{pmatrix} -1 & \cdots & -1 \\ -1 & \cdots & -1 \end{pmatrix}, C_{2\times 2^{n-2}} = \begin{pmatrix} -1 & \cdots & -1 \\ -1 & \cdots & -1 \end{pmatrix},$$

$$D_{2\times 2^{n-2}} = \begin{pmatrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \end{pmatrix}, \quad P_{2^{n-1}-2\times 2} = \begin{pmatrix} -1 & -1 \\ \vdots & \vdots \\ -1 & -1 \end{pmatrix},$$

$$Q_{2^{n-1}-2\times 2^{n-1}-2} = \begin{pmatrix} 2^{n-1}-1 & -1 & \cdots & -1 \\ -1 & 2^{n-1}-1 & -1 & \vdots \\ \vdots & \vdots & \ddots & -1 \\ -1 & -1 & \cdots & 2^{n-1}-1 \end{pmatrix}, R_{2^{n-2}\times 2} = \begin{pmatrix} -1 & -1 \\ \vdots & \vdots \\ -1 & -1 \end{pmatrix},$$

$$S_{2^{n-2}\times 2^{n-2}} = \begin{pmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}, T_{2^{n-2}\times 2} = \begin{pmatrix} -1 & 0 \\ \vdots & \vdots \\ -1 & 0 \end{pmatrix}.$$

Note that $I_{2^{n-2}\times 2^{n-2}}$ is the identity matrix of size $2^{n-2} \times 2^{n-2}$. Also, O_{k_1}, O_{k_2} and O_{k_3} are the zero matrices of size $2^{n-1}-2 \times 2^{n-2}, 2^{n-2} \times 2^{n-1}-2$ and $2^{n-2} \times 2^{n-2}$, respectively. Using Definition 8, the Laplacian characteristic polynomial, $p_{L\left(\Gamma^{cop}(SD_{2^n})\right)}(\lambda)$ can be written as

$$p_{L\left(\Gamma^{cop}(SD_{2^n})\right)}(\lambda) = |\lambda I_{2^n} - L(\Gamma^{cop}(SD_{2^n}))|$$

that is

$$p_{L\left(\Gamma^{cop}(SD_{2^n})\right)}(\lambda) = \begin{vmatrix} \lambda I_{2\times 2} - A_{2\times 2} & -B_{2\times 2^{n-1}-2} & -C_{2\times 2^{n-2}} & -D_{2\times 2^{n-2}} \\ -P_{2^{n-1}-2\times 2} & \lambda I_{2^{n-1}-2\times 2^{n-1}-2} - Q_{2^{n-1}-2\times 2^{n-1}-2} & O_{k_1} & O_{k_1} \\ -R_{2^{n-2}\times 2} & O_{k_2} & \lambda I_{2^{n-2}\times 2^{n-2}} - S_{2^{n-2}\times 2^{n-2}} & O_{k_3} \\ -T_{2^{n-2}\times 2} & O_{k_2} & O_{k_3} & I_{2^{n-2}\times 2^{n-2}}(\lambda - 1) \end{vmatrix}.$$

The desired Laplacian characteristic polynomial of $\Gamma^{cop}(SD_{2^n}), p_{L\left(\Gamma^{cop}(SD_{2^n})\right)}(\lambda)$ is obtained by applying some elementary row operations such as $R_1 \to (\lambda - 1)R_1 - R_2 - \cdots - R_{2^n}$ and same steps are repeated for each subsequent row until the matrix is completely transformed into upper triangular form. Then, using Definition 9, the result of the Laplacian spectrum of $\Gamma^{cop}(SD_{2^n})$, $L_{spec}\left(\Gamma^{cop}(SD_{2^n})\right)$ holds.

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Example 2 For the cyclic order product prime graph of the semi-dihedral group of order 16 $\Gamma^{cop}(SD_{16})$, the Laplacian matrix, $L\left(\Gamma^{cop}(SD_{16})\right)$ of size 16×16 can be represented as

$$L\left(\Gamma^{cop}(SD_{16})\right) = \begin{pmatrix} A_{2\times 2} & B_{2\times 6} & C_{2\times 4} & D_{2\times 4} \\ P_{6\times 2} & Q_{6\times 6} & O_{6\times 4} & O_{6\times 4} \\ S_{4\times 2} & O_{4\times 6} & T_{4\times 4} & O_{4\times 6} \\ W_{4\times 2} & O_{4\times 6} & O_{O_{4\times 4}} & I_{4\times 4} \end{pmatrix}$$

where $A_{2\times 2} = \begin{pmatrix} 15 & -1 \\ -1 & 11 \end{pmatrix}, B_{2\times 6} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}, C_{2\times 4} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}$,

$$D_{2\times 4} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -0 \end{pmatrix}, P_{6\times 2} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}, Q_{6\times 6} = \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 \\ -1 & 7 & -1 & -1 & -1 & -1 \\ -1 & -1 & 7 & -1 & -1 & -1 \\ -1 & -1 & -1 & 7 & -1 & -1 \\ -1 & -1 & -1 & -1 & 7 & -1 \\ -1 & -1 & -1 & -1 & -1 & 7 \end{pmatrix},$$

$$S_{4\times 2} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}, \; T_{4\times 4} = \begin{pmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}, W_{4\times 2} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ -1 & 0 \\ -1 & 0 \end{pmatrix}, I_{4\times 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Also, $O_{6\times 4}, O_{4\times 6}$ and $O_{4\times 4}$ are the zero matrices of size 6×4, 4×6 and 4×4, respectively. Now, the Laplacian characteristic polynomial of $\Gamma^{cop}, p_{L(\Gamma^{cop}(SD_{16}))}(\lambda)$ is given by

$$\begin{aligned} p_{L(\Gamma^{cop}(SD_{16}))}(\lambda) &= |\lambda I_{16} - \Gamma^{cop}(SD_{16})| \\ &= \begin{vmatrix} \lambda I_{2\times 2} - A_{2\times 2} & -B_{2\times 6} & -C_{2\times 4} & -D_{2\times 4} \\ -P_{6\times 2} & \lambda I_{6\times 6} - Q_{6\times 6} & O_1 & O_{6\times 4} \\ -S_{4\times 2} & O_{4\times 6} & \lambda I_{4\times 4} - T_{4\times 4} & O_{4\times 6} \\ -W_{4\times 2} & O_{4\times 6} & O_{4\times 6} & I_{4\times 4}(\lambda - 1) \end{vmatrix}. \end{aligned}$$

By applying some elementary row operations, we have

$$p_{L(\Gamma^{cop}(SD_{16}))}(\lambda) = \lambda(\lambda - 12)(\lambda - 16)(\lambda - 2)^2(\lambda - 4)^2(\lambda - 1)^4(\lambda - 8)^5.$$

Using Definition 9, the Laplacian spectrum of $\Gamma^{cop}(SD_{16})$,

$$L_{spec}\left(\Gamma^{cop}(SD_{16})\right) = \begin{pmatrix} 0 & 12 & 16 & 2 & 4 & 1 & 8 \\ 1 & 1 & 1 & 2 & 2 & 4 & 5 \end{pmatrix}.$$

Proposition 4 builds upon previous results on the Laplacian spectrum of Theorem 3 by determining the Laplacian energy of $\Gamma^{cop}(G), L_E\left(\Gamma^{cop}(G)\right)$. This significant finding provides the graph's energy, complementing our spectral analysis.

Proposition 4 Let SD_{2^n} be the semi-dihedral group of order 2^n for $n \geq 4$. Then, the Laplacian energy of $\Gamma^{cop}(G)$,

$$L_E\big(\Gamma^{cop}(G)\big) = 4^{n-1} + \tfrac{10-2^n}{4}.$$

Proof. The Laplacian energy of the graph $\Gamma^{cop}(G), L_E\big(\Gamma^{cop}(G)\big)$ is derived using the formula provided in Definition 10. By substituting the results from Proposition 3 and Theorem 3, we have

$$\begin{aligned} L_E\big(\Gamma^{cop}(G)\big) &= \sum_{k=1}^{n} \left|\lambda_k - \bar{d}\big(\Gamma^{cop}(G)\big)\right| \\ &= \left|0 - \frac{5+2^n}{4}\right| + \left|3(2^{n-2}) - \frac{5+2^n}{4}\right| + \left|2^n - \frac{5+2^n}{4}\right| + (2^{n-3})\left|2 - \frac{5+2^n}{4}\right| \\ &\quad + (2^{n-3})\left|4 - \frac{5+2^n}{4}\right| + (2^{n-2})\left|1 - \frac{5+2^n}{4}\right| + (2^{n-1}-3)\left|2^{n-1} - \frac{5+2^n}{4}\right|. \end{aligned}$$

Simplifying the absolute value expressions, we have

$$\begin{aligned} L_E\big(\Gamma^{cop}(G)\big) &= \frac{5+2^n}{4} + 3(2^{n-2}) - \frac{5+2^n}{4} + 2^n - \frac{5+2^n}{4} + (2^{n-3})\left(\frac{5+2^n}{4} - 2\right) + (2^{n-3})\left(\frac{5+2^n}{4} - 4\right) \\ &\quad + (2^{n-2})\left(\frac{5+2^n}{4} - 1\right) + (2^{n-1}-3)\left(2^{n-1} - \frac{5+2^n}{4}\right) \end{aligned}$$

The desired result holds by some simple simplification. ∎

Example 3 Since the Laplacian spectrum of $\Gamma^{cop}(SD_{16}), L_{spec}\big(\Gamma^{cop}(SD_{16})\big) = \begin{pmatrix} 0 & 12 & 16 & 2 & 4 & 1 & 8 \\ 1 & 1 & 1 & 2 & 2 & 4 & 5 \end{pmatrix}$ Then, the Laplacian energy of $\Gamma^{cop}(SD_{16})$,

$$\begin{aligned} L_E\big(\Gamma^{cop}(SD_{16})\big) &= \sum_{i=1}^{n} \left|\lambda_i - \bar{d}_{\Gamma^{cop}(SD_{16})}\right| \\ &= \left|0 - \frac{21}{4}\right| + \left|12 - \frac{21}{4}\right| + \left|16 - \frac{21}{4}\right| + 2\left|2 - \frac{21}{4}\right| + 2\left|4 - \frac{21}{4}\right| + 4\left|1 - \frac{21}{4}\right| + 5\left|8 - \frac{21}{4}\right| \\ &= 62.5 \end{aligned}$$

which is equivalent to the results when we substitute $n = 4$ in Proposition 4.

Conclusions

In this paper, the cyclic order product prime graph is introduced and used to explore the Laplacian spectrum and energy in semi-dihedral groups, an important development in spectral graph analysis. Establishing general presentations to obtain some graph properties, including vertex degrees, the number of edges, and Laplacian characteristic polynomials, was also significant. These efforts helped show the graph's adjacencies and the Laplacian spectrum's complexity, which has seven eigenvalues of different multiplicities. The presence of a single zero eigenvalue of $L_{spec}\big(\Gamma^{cop}(SD_{2^n})\big)$ shows that the graph is connected, while the presence of the second-smallest eigenvalue (1 in this case) shows that it is stable. The range of the spectrum, from 0 to 2^n and the multiplicities of the eigenvalues tell us about the degree distribution and structure complexity of the graph. This spectrum not only helps us understand how connected and resilient the graph is, but it also provides foundations for more in-depth studies of its characteristics and behavior in network dynamics. Moreover, the Laplacian energy was also formulated with the aid of those graph properties and the Laplacian spectrum. This research significantly contributes to algebraic graph and group theory and indicates potential applications in wider scientific areas.

Conflicts of Interest

This paper's authors claim there is no conflict of interest.

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References

[1] Yang, H., & Hu, G. (2003). Laplacian spectrum analysis and spanning tree algorithm for circuit partitioning problems. *Science in China Series F: Information Sciences*, *46*, 459-465.

[2] Gutman, I., & Zhou, B. (2006). Laplacian energy of a graph. *Linear Algebra and Its Applications, 414*(1), 29-37.

[3] Ma, X. L., Wei, H. Q., & Zhong, G. (2013). The cyclic graph of a finite group. *Algebra*, *2013*, 1-7.

[4] Aalipour, G., Akbari, S., Cameron, P. J., Nikandish, R., & Shaveisi, F. (2017). On the structure of the power graph and the enhanced power graph of a group. *The Electronic Journal of Combinatorics, 24*(3), 1-22.

[5] Bello, M., Ali, N. M. M., & Zulkifli, N. (2020). A systematic approach to group properties using its geometric structure. *European Journal of Pure and Applied Mathematics*, *13*(1), 84-95.

[6] Dutta J., and Nath, R. K. (2017). Spectrum of commuting graphs of some classes of finite groups. *Matematika*, *33*(1), 87-95.

[7] Torktaz, M. and Ashrafi, A. (2019). Spectral properties of the commuting graphs of certain groups. *AKCE International Journal of Graphs and Combinatorics*,*16*(3), 300-309.

[8] Kumar, J., Dalal, S. and Baghel, V. (2021). On the commuting graph of semidihedral group. *Bulletin of the Malaysian Mathematical Sciences Society*, *44*(5), 3319-3344.

[9] Cheng, T., Dehmer, M., Emmert-Streib, F., Li, Y., & Liu, W. (2021). Properties of commuting graphs over semidihedral groups. *Symmetry*, *13*(1), 103.

[10] Chattopadhyay, S., & Panigrahi, P. (2015). On Laplacian spectrum of power graphs of finite cyclic and dihedral groups. *Linear and Multilinear Algebra*, *63*(7), 1345-1355.

[11] Mehranian, Z., Gholami, A., & Ashrafi, A. R. (2017). The spectra of power graphs of certain finite groups. *Linear and Multilinear Algebra*, *65*(5), 1003-1010.

[12] Hamzeh, A., & Ashrafi, A. R. (2017). Spectrum and L-spectrum of the power graph and its main supergraph for certain finite groups. *Filomat*, *31*(16), 5323-5334.

[13] Panda, R. P. (2019). Laplacian spectra of power graphs of certain finite groups. *Graphs and Combinatorics*, *35*(5), 1209-1223.

[14] Parveen, Dalal, S., & Kumar, J. (2023). Enhanced power graphs of certain non-abelian groups. *Discrete Mathematics, Algorithms and Applications*, *2350063*, 1-19.

[15] Mahmoud, R., Sarmin, N. H., & Erfanian, A. (2017). The conjugacy class graph of some finite groups and its energy. *Malaysian Journal of Fundamental and Applied Sciences*, *13*(4), 659-665.

[16] Alimon, N. I., Sarmin, N. H., & Fadzil, A. F. A. (2018). The energy of four graphs of some metacyclic 2-groups. *Malaysian Journal of Fundamental and Applied Sciences*, *14*(1), 59-66.

[17] Sharafdini, R., Nath, R. K., & Darbandi, R. (2022). Energy of commuting graph of finite AC-groups. *Proyecciones (Antofagasta)*, *41*(1), 263-273.

[18] Mahmoud, R., Fadzil, A. F. A., Sarmin, N. H., & Erfanian, A. (2019). The Laplacian energy of conjugacy class graph of some finite groups. *Matematika*, *35*(1), 59-65.

[19] Das, K. C., Alazemi, A., & Anđelić, M. (2020). On energy and Laplacian energy of chain graphs. *Discrete Applied Mathematics*, *284*, 391-400.

[20] Sharma, M., & Nath, R. K. (2023). Signless Laplacian energies of non-commuting graphs of finite groups and related results. *arXiv preprint arXiv:2303.17795*.

[21] Ugasini P. P., Suresh, M., & Bonyah, E. (2023). On the spectrum, energy and Laplacian energy of graphs with self-loops. *Heliyon*, *9*(7), 1-18.

[22] Schaefer, J. and Schlechtweg, K. (2017). On the structure of symmetric spaces of semidihedral groups. Involve, *10*(4), 665-676.

[23] Fraleigh, J. B. (2003). *A first course in abstract algebra*. Pearson Education India.

[24] West, D. B. *et al.* (2001). *Introduction to graph theory*. Vol. 2. Prentice Hall Upper Saddle River.

[25] Bello, M. (2021). *Order product prime graph and its variations of some finite groups* (Doctoral dissertation, Ph. D. Report, Universiti Teknologi Malaysia).

[26] Delen, S., Demirci, M., Cevik, A. S., & Cangul, I. N. (2021). On omega index and average degree of graphs. *Journal of Mathematics*, *2021*, 1-5.

[27] Jahanbani, A., Sheikholeslami, S. M., & Khoeilar, R. (2021). On the spectrum of Laplacian matrix. *Mathematical Problems in Engineering*, *2021*, 1-4.