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RESEARCH ARTICLE

The Effects of Impurities on Discrete Nonlinear Schrödinger Equation

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Abstract Understanding the effect that impurities may have on the soliton propagation process, particularly during the interaction process involving the Nonlinear Schrödinger Equation (NLSE), has become a major research focus in recent years. This paper studied the phenomenon of soliton scattering when it interacts with a localized impurity of the Delta potential under the discrete case of NLSE. Using an analytical approach, i.e., the variational approximation (VA) method, the equations of soliton parameters for the width, center-of-mass position, and linear and quadratic phase-front corrections are derived in order to describe the soliton evolutions throughout the scattering process. The VA method results were validated by the direct numerical simulation of the discrete NLSE, provided that the soliton is initially set at a distance from the Delta potential. When the nonlinearity was taken to be of the cubic and quintic types, it was shown that the soliton of the Discrete Cubic-Quintic NLSE could be either reflected or transmitted by the Delta potential with different potential strengths and a constant soliton's initial velocity. The results suggested that the VA method is an effective and useful approach to investigate the scattering process of discrete NLSE in the presence of impurities.

Keywords: Soliton, nonlinear Schrödinger equation, nonlinear equation, discrete system, partial differential equation.

Introduction

The heightened attention towards researching nonlinear structures of waves in physical systems stems from the realization that the interplay between nonlinearity and dispersion effects leads to intriguing wave phenomena. Solitons, which are localized wave structures, arise as a result of the delicate balance between the steepening effect of nonlinearity and the spreading effect of dispersion during propagation in a nonlinear medium. These waves possess remarkable features, preserving their shape and velocity over long distances and exhibiting resilience even after colliding with other solitons. Understanding the behaviour of solitons when they encounter external potentials is of great importance in elucidating their propagation dynamics and potential applications.

In discrete systems, where solitons propagate through a series of discrete sites or points, the influence of external potentials on the soliton behaviour becomes particularly fascinating and opens up avenues for studying soliton interactions in diverse physical contexts. In particular, discrete systems encompass a wide range of applications such as in optical switching [1], Bose-Einstein condensates in a deep optical lattice [2, 3], DNA molecular chains [4, 5] and electrical transmission lines [6], where discrete solitons become apparent. Indeed, the earliest experimental discovery of discrete solitons in nonlinear waveguide arrays was recorded by Eisenberg *et al*. [7] who observed the mechanism by which discrete solitons are generated in the Aluminium Gallium Arsenide (AlGaAs) waveguide arrays below the half-band-gap.

The process of soliton interaction with external potential can be examined within the framework of the Discrete Nonlinear Schrödinger Equation (NLSE), which serves as a mathematical model for describing soliton's behaviour in the presence of external potential in the context of discrete systems. In fact, the discreteness of the lattice itself leads to the existence of the commonly referred Peierls-Nabarro potential

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(PNP) which influences the dynamic of discrete solitons throughout the lattices [8]. PNP can be conceptualized as the difference between the energy of those on-site, i.e., stable soliton, centered at the waveguide, and inter-site modes, i.e., unstable soliton, centered between two waveguides [9]. In other words, it is like a barrier that a discrete soliton needs to overcome in order to move across lattices. The greater the differences, the more the mobility of the solitons will be restricted. The PNP becomes significant enough to stop and lock the soliton to the lattice at high power. Brazhnyi *et al*. [10] suggested that considering the frequency of $-0.4 \le \omega < 0$ for soliton solutions, it is feasible to analyze the scattering features of both small-amplitude and high-amplitude solitons while ensuring the soliton remains intact and free from dissipating or getting trapped in the lattice.

To examine the impact on the moving solitons, it is possible to generate external potential in the lattices using several different methods. Based on Morales-Molina and Vicencio [11], modifying the refractive index of a specific waveguide locally by doping it with different material properties or by adjusting its geometry is a straightforward technique to produce defects in waveguide arrays. Another form of defect in waveguide arrays can be generated by locally changing the coupling coefficient of two neighbouring waveguides as applied by Morandotti *et al*. [9]. In addition, Sakaguchi and Tamura [12] stated that external periodic potentials are created by the interference patterns of the laser beam field.

The analysis of the interaction between discrete breathers and impurities has demonstrated that even a slight modification in the mass defects, ranging from 5% to 10%, can effectively impact the behaviour of breathers within a lattice, causing them to be trapped [13]. This occurs due to the similarities between the impurity mode and the breather, particularly their overlapping frequency domains. Meanwhile, the trapping process of discrete solitons as they traverse through linear and nonlinear impurities has been discovered to be the consequence of the resonant energy transfer occurring between the soliton and the impurity mode [11, 14]. In addition to that, the investigation of discrete solitons interacting with a linear impurity in the structure of a Gaussian defect has found that increasing the initial amplitude of the solitons causes the regions of the transmission to disappear [10].

In the experiment setting, the scattering of discrete solitons with structural defects in waveguide arrays has revealed notable distinctions in their motion compared to linear propagation through similar structures. Even slight alterations in input conditions result in significant changes in output positions, as shown by the sudden switch of the soliton across a narrow repulsive defect [9]. In the interim, four essential regimes of the scattering process of discrete solitons, i.e., trapping, trapping and reflection, reflection with no trapping and transmission with no trapping, have been reported by Palmero *et al*. [15], considering attractive and repulsive cases. To the extent of current knowledge, the interaction between discrete solitons and external potentials within the context of the generalized discrete NLSE has yet to be studied. While Aklan *et al*. [16 – 18] have focused on the dynamic of the soliton with external potentials in the generalized NLSE within the continuous framework, this study is extended to the discrete case of such systems.

For this purpose, this paper examines the behaviour of the solitons in the generalized discrete NLSE where the nonlinearity is considered to be of cubic and quintic types and explores the scattering process when they interact with a localized impurity in the form of a Delta function. First, the variational approximation method is formulated to generate the variational equations for soliton parameters. Then, the direct numerical simulation of the governing equation is performed to validate the approximation results.

Materials and Methods

The Model of The Main Equation

The model of the main equation used is rooted in the discrete version of the generalized NLSE, which includes the generalized form of higher-order nonlinearity coefficients, specifically known as the Discrete Cubic-Quintic NLSE. This equation is called perturbed NLSE when there is an additional term of external potential which is considered as perturbation in this case. The corresponding main equation applied to a one-dimensional discrete system can be modelled by the following equation,

$$i\frac{\partial \psi_n}{\partial t} + c\left(\psi_{n+1} - 2\psi_n + \psi_{n-1}\right) + k\left|\psi_n\right|^2 \psi_n + g\left|\psi_n\right|^4 \psi_n + V_n\psi_n = 0, \tag{1}$$

where $\psi_n(t)$ is the complex-valued wave function of time *t* at the site *n*, where *n* takes the integer values, *c* is the coupling coefficient characterizes the strength between two adjacent sites of lattice, *k* and *g* correspond to the nonlinearity coefficients of the cubic and quintic terms, respectively, while V_n in the last term represents a localized impurity in the specific lattice site n. Since the lattice sites are identical and spaced equally apart from one another, the values of parameters c, k and g remain unchanged. In this setting, the quantity of the wave-field power is conserved where,

$$P = \sum_{n=-\infty}^{+\infty} |\psi_n|^2. \tag{2}$$

In the absence of external potential, $V_n = 0$, Equation (1) can be modified to obtain the stationary soliton solution by substituting the steady-state ansatz $\psi_n(t) = e^{i\omega t} u_n$ with ω as the frequency, and the stationary amplitude $u_n(t)$ satisfies the following equation:

$$-\omega u_n + c\left(u_{n+1} + u_{n-1} - 2u_n\right) + k|u_n|^2 u_n + g|u_n|^4 u_n = 0. \tag{3}$$

Without loss of generality, Equation (3) can be simplified by only taking into consideration the real-value-solutions since it must satisfy the condition $u_n \to 0$ when $n \to \pm\infty$ [19] such that,

$$-\omega u_n + c\left(u_{n+1} + u_{n-1} - 2u_n\right) + k u_n^{\ 3} + g u_n^{\ 5} = 0. \tag{4}$$

The study investigates the interaction process when a discrete soliton propagates towards an external potential and interacts with it. The demonstration of the scenario is depicted by the following figure:

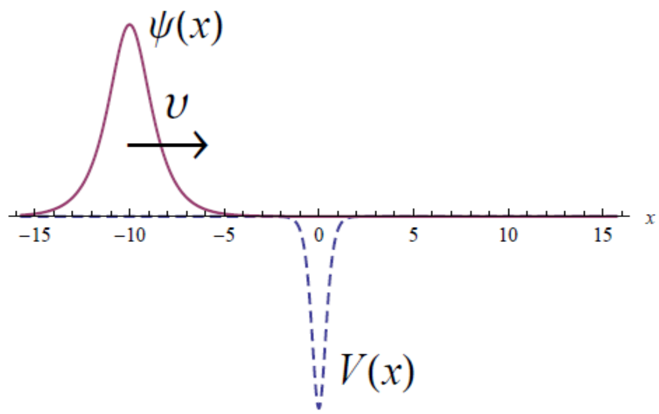

Figure 1. Schematic view of a moving soliton $\psi(x)$ towards the external potential $V(x)$ [20]

Figure 1 illustrates the wave function of a soliton $\psi(x)$ as it approaches the external potential $V(x)$, which is centred at $x = 0$. The soliton was initially set at a considerable distance to the left of the potential and is moving towards it with velocity v. As the soliton propagates, it begins to interact with the potential, influencing its dynamics and resulting in changes to the wave function as it enters the interaction region.

In the continuous systems, the Dirac Delta function, denoted by $\delta(x)$ [21] is frequently applied as one of the examples of external potential in studying the effect on soliton's dynamics. Nevertheless, the discrete analogue of the Dirac Delta function, known as the Kronecker Delta function, denoted by $\delta_{i,j}$ must be considered since the discrete case is the primary subject of this research. This function is typically defined within a discrete domain and takes the value of $\delta_{i,j} = 1$ if $i = j$ whereas $\delta_{i,j} = 0$ if $i \neq j$ [22]. Therefore, this study considers the interaction of soliton in discrete NLSE with a localized impurity which takes the form of a Kronecker Delta function as reported by Al-Marzoug [23],

$$V_n = V_0 \delta_{n,n_0}, \tag{5}$$

where V_0 characterizes the strength of the potential and n_0 indicates the location of the impurity.

The impurity is located at a fixed position where $n_0 = \frac{N}{2}$ in an array of $N = 200$ waveguides throughout the computation. Generally, the shape of the potential is determined by the sign of V_0, whereby the positive value of V_0 corresponds to the shape of a potential well and the negative value of V_0 corresponds to the shape of a potential wall. In the context of discrete waveguide arrays, an attractive defect or impurity characterized by a local increase in the coupling coefficient, is comparable to a potential wall. In contrast, a repulsive defect or impurity characterized by a local decrease in the coupling coefficient, is comparable to a potential well [9].

Methodology

Throughout this study, two types of methods are used which are the variational approximation (VA) method and the direct numerical method to analyze the behaviour of the solitons in the discrete cubic-quintic NLSE (1) when external potential exists. For the first step, the main equation is solved analytically using the VA method to obtain the equations of soliton parameters evolution that describe the soliton scattering process through graphical simulations. Next, the analytical results obtained from the VA method are compared with the simulation results from the direct numerical method of the main equation.

The VA method is one of the most significant theoretical tools for studying the dynamics of solitons governed by the NLSE. The NLSE is non-integrable in nature and does not have analytical solutions. However, by means of a variational approach, the objective of studying the soliton's behaviour and its scattering process can be achieved. The ability of the VA method to simplify the complex partial differential equation into a system of ordinary differential equations has led to its widespread use among researchers in the research of solitons.

Anderson [24] was the first to apply this method to analyze the evolution of soliton in a strongly perturbed NLSE within the context of nonlinear optics. Later, Sakaguchi and Tamura [12], Al-Marzoug *et al*. [25], Umarov *et al*. [20], Din *et al*. [26] and Aklan *et al*. [21], among others have used the same approach to study the interaction of the NLSE soliton in the presence of external potentials. The VA method allows for the derivation of approximate solutions for the soliton parameters of width, amplitude, center-of-mass position, nonlinear frequency chirp and some other parameters. These parameters are essential enough to provide insights into wave propagation, allowing the possibility to thoroughly analyze the scattering of the solitons.

The direct numerical method of the main equation is then performed to check the accuracy of the VA method. This method validates the approximation results and ensures the consistency of the solitons' physical behaviour, considering that certain assumptions were made during the analysis of the VA method. In particular, the partial differential equation of the soliton governing equation, the discrete cubic-quintic NLSE (1) is solved directly to find the exact solution of the soliton. First, Newton's method is applied to find the stationary amplitude of the soliton solution, $u_n(t)$. Then, the corresponding stationary solution is used to solve for the numerical soliton solution from the governing equation (1) using the Mathematica software package of NDSolve function with the given initial condition.

Results and Discussion

Variational Analysis of the Soliton Scattering

To formulate the VA method, the Lagrangian is derived first from the system (1) which can be represented as below,

$$L = \sum_{n=-\infty}^{\infty} \mathrm{L}_n = \sum_{n=-\infty}^{\infty} \left[\frac{i}{2}\left(\psi_n \frac{\partial \psi_n^*}{\partial t} - \psi_n^* \frac{\partial \psi_n}{\partial t} \right) - c\left(\psi_{n+1}\psi_n^* + \psi_{n+1}^*\psi_n - 2|\psi_n|^2 \right) - \frac{k}{2}|\psi_n|^4 - \frac{g}{3}|\psi_n|^6 - V_n|\psi_n|^2 \right], \tag{6}$$

where L_n is the Lagrangian density of the system. The discrete cubic-quintic NLSE is a variational problem that complies with the Lagrangian equation, L. Hence, it can be simply verified that the governing equation (1) can be obtained from the above Lagrangian (6) using the Euler-Lagrange equation expressed by

$$\frac{\delta \mathrm{L}_n}{\delta \psi_n^*} = \frac{d}{dt}\frac{\partial \mathrm{L}_n}{\partial\left(\frac{\partial \psi_n^*}{\partial t}\right)} - \frac{\partial \mathrm{L}_n}{\partial \psi_n^*} = 0, \tag{7}$$

which is in accordance with the variational principle for the discrete case,

$$\delta \sum_{n=-\infty}^{\infty} \mathrm{L}_n\left(\psi_n, \psi_n^*, \frac{\partial \psi_n}{\partial t}, \frac{\partial \psi_n^*}{\partial t}\right) = 0. \tag{8}$$

To simplify the Lagrangian (6), the following Poisson summation formula is applied [27],

$$\sum_{n=-\infty}^{\infty} F(n) = \int_{-\infty}^{\infty} F(x) \sum_{n=-\infty}^{\infty} e^{2\pi i n x} dx, \tag{9}$$

so that a more convenient form of Lagrangian is produced,

$$\begin{aligned} L = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Bigg[& \frac{i}{2}\left(\psi(x,t)\frac{\partial \psi^*(x,t)}{\partial t} - \psi^*(x,t)\frac{\partial \psi(x,t)}{\partial t}\right) - \frac{k}{2}|\psi(x,t)|^4 - \frac{g}{3}|\psi(x,t)|^6 \\ & - c\left(\psi(x+1,t)\psi^*(x,t) + \psi^*(x+1,t)\psi(x,t) - 2|\psi(x,t)|^2\right) \\ & - V(x)|\psi(x,t)|^2 \Bigg] e^{2\pi i n x} dx. \end{aligned} \tag{10}$$

The wave function here is now depending on the continuous variable *x* with time *t*. When applying the VA method, a crucial aspect is the appropriate selection of the trial function that will be used as the initial guess for the wave function since it will determine the success of this method. As in this case, the initial pulse is assumed to have the form of a Gaussian function, referred to in Livak *et al*. [28],

$$\psi(x,t) = \sqrt{\frac{P}{a\sqrt{\pi}}} \exp\left(-\frac{(x-x_0)^2}{2a^2} + i\gamma(x-x_0) + i\beta(x-x_0)^2\right), \tag{11}$$

where the parameters $a(t)$, $x_0(t)$, $\gamma(t)$ and $\beta(t)$ represent the soliton's effective width, center-of-mass position, linear and quadratic phase-front corrections, respectively.

It is notable that the form of the Kronecker Delta potential in Equation (5) should be modified to be compatible with the VA method since this method is based on the continuous approximation of the discrete NLSE. Based on Balakrishnan [22], it is known that for the Kronecker Delta function, the following holds,

$$\sum_{j=-\infty}^{\infty} \delta_{i,j} a_j = a_i, \tag{12}$$

where the normalization is $\sum_{j=-\infty}^{\infty} \delta_{i,j} = 1$ and $\delta_{i,j} = \delta_{j,i}$. Shifting the analysis to the continuous setting necessitates the replacement of the summation over *j* by an integration over *x*. Also, a specific point x_0 is assigned to the role of specific index *i*. As such, there exists a function $\delta(x-x_0)$ that satisfies

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0), \tag{13}$$

where the normalization is $\int_{-\infty}^{\infty} \delta(x-x_0)\, dx = 1$ and $\delta(x-x_0) = \delta(x_0-x)$. In light of this, it can be inferred that the equivalent form of Equation (5) is the Dirac Delta potential given by

$$V(x) = V_0 \delta(x-n_0). \tag{14}$$

Inserting the ansatz (11) and the external Delta potential (14) into the simplified Lagrangian (10), the total averaged Lagrangian is obtained,

$$
\begin{aligned}
L =& \frac{1}{2}P\sum_{n=-\infty}^{\infty}\left(-2n^2\pi^2a^4\frac{d\beta}{dt}-2\gamma\frac{dx_0}{dt}+a^2\left(\frac{d\beta}{dt}-2in\pi\left(2\beta\frac{dx_0}{dt}-\frac{d\gamma}{dt}\right)\right)\right)e^{n\pi(-n\pi a^2+2ix_0)} \\
&-\frac{kP^2}{2\sqrt{2\pi}a}\sum_{n=-\infty}^{\infty}e^{-\frac{1}{2}n\pi(n\pi a^2-4ix_0)}-\frac{gP^3}{3\sqrt{3}\pi a^2}\sum_{n=-\infty}^{\infty}e^{-\frac{1}{3}n\pi(n\pi a^2-6ix_0)} \\
&-\frac{V_0P}{\sqrt{\pi}a}e^{-\frac{(n_0-x_0)^2}{a^2}}\sum_{n=-\infty}^{\infty}e^{2in\pi n_0}-cPe^{-\frac{1}{4a^2}-\beta^2a^2+i\gamma}\sum_{n=-\infty}^{\infty}e^{-a^2(n^2\pi^2+2\beta n\pi)+i(2n\pi x_0-n\pi)} \\
&-cPe^{-\frac{1}{4a^2}-\beta^2a^2-i\gamma}\sum_{n=-\infty}^{\infty}e^{-a^2(n^2\pi^2-2\beta n\pi)+i(2n\pi x_0-n\pi)}+2cP\sum_{n=-\infty}^{\infty}e^{n\pi(-n\pi a^2+2ix_0)}.
\end{aligned}
\tag{15}
$$

Due to the exponential decrease of this series as *n* increases, the focus can be placed solely on the case where only the term corresponding to *n=0* is retained when the soliton has a width of $a(t) >> \sqrt{2}/\pi$ in order to adequately describe the process. Consequently, the reduced Lagrangian of the system is formulated,

$$L_c=\frac{Pa^2}{2}\frac{d\beta}{dt}-P\gamma\frac{dx_0}{dt}-\frac{kP^2}{2\sqrt{2\pi}a}-\frac{gP^3}{3\sqrt{3}\pi a^2}-\frac{V_0P}{\sqrt{\pi}a}e^{-\frac{(n_0-x_0)^2}{a^2}}-2cP\cos\gamma e^{-\frac{1}{4a^2}-\beta^2a^2}+2cP. \tag{16}$$

From the reduced Lagrangian above, a system of equations of motion can be obtained for the Gaussian parameters. Particularly, these equations are derived by means of the Euler-Lagrange equation given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_t}\right)-\frac{\partial L}{\partial q}=0, \tag{17}$$

where *q* are the ansatz parameters $a,\ x_0,\ \gamma,\ \beta$. Upon formulating this equation for each parameter, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial a_t}\right)-\frac{\partial L_c}{\partial a}=0,\ \frac{d}{dt}\left(\frac{\partial L_c}{\partial x_{0t}}\right)-\frac{\partial L_c}{\partial x_0}=0,\ \frac{d}{dt}\left(\frac{\partial L_c}{\partial \gamma_t}\right)-\frac{\partial L_c}{\partial \gamma}=0,\ \frac{d}{dt}\left(\frac{\partial L_c}{\partial \beta_t}\right)-\frac{\partial L_c}{\partial \beta}=0, \tag{18}$$

the following system of four ordinary differential equations for soliton parameters is deduced,

$$\frac{da}{dt}=4\beta ac\cos\gamma e^{-\frac{1}{4a^2}-\beta^2a^2}, \tag{19}$$

$$\frac{dx_0}{dt}=2c\sin\gamma e^{-\frac{1}{4a^2}-\beta^2a^2}, \tag{20}$$

$$\frac{d\gamma}{dt}=\frac{2V_0(n_0-x_0)}{\sqrt{\pi}a^3}e^{-\frac{(n_0-x_0)^2}{a^2}}, \tag{21}$$

$$\frac{d\beta}{dt}=-\frac{kP}{2\sqrt{2\pi}a^3}-\frac{2gP^2}{3\sqrt{3}\pi a^4}+\frac{V_0\left(2(n_0-x_0)^2-a^2\right)}{\sqrt{\pi}a^5}e^{-\frac{(n_0-x_0)^2}{a^2}}+\frac{1-4\beta^2a^4}{a^4}c\cos\gamma e^{-\frac{1}{4a^2}-\beta^2a^2}. \tag{22}$$

Hence, the evolution of a discrete soliton under the influence of Delta potential can be described by Equations (19) – (22), which represent the main results of this VA method. Now consider the stationary soliton solution in the case of a collimated wave beam [27] when it is far away from the external potential such that the external potential is absent, i.e., $V_0 = 0$. From the system of Equations (19) – (22), setting up $\beta = 0$ and $\frac{d\beta}{dt} = 0$ give the relationship of power P and velocity v with the width a and the linear phase-front correction γ as in the expressions below, respectively,

$$P = \frac{-3ka\sqrt{\frac{3\pi}{2}} + \sqrt{\frac{3\pi}{2}}\sqrt{9k^2a^2 + 64\sqrt{3}ce^{-\frac{1}{4a^2}}g\cos\gamma}}{8g}, \tag{23}$$

$$v = 2c\sin\gamma e^{-\frac{1}{4a^2}}. \tag{24}$$

Equations (19) – (22) are interpreted into numerical simulation which are then visualized through the graphical forms in order to observe the behaviour of the discrete soliton when it interacts with Delta potential, taking into account different values of potential strengths. The number of free parameters for the numerical interpretation is reduced by setting the initial values of the soliton parameters to be $a(0) = 3.5,\ x_0(0) = 50,\ \gamma(0) = 0.1$ and $\beta(0) = 0$. The wave-field power is $P = 1.16019$ with a constant initial velocity at $v = 0.1956$, based on the correlation in Equation (23) and Equation (24), respectively. Additionally, the coupling strength between neighbouring sites of the lattice is set to be $c = 1$ and similar value is considered for the coefficient of the cubic and quintic nonlinearity, $k = g = 1$. The scattering process that takes place between a discrete soliton and Delta potential with different potential strengths are shown in Figure 2.

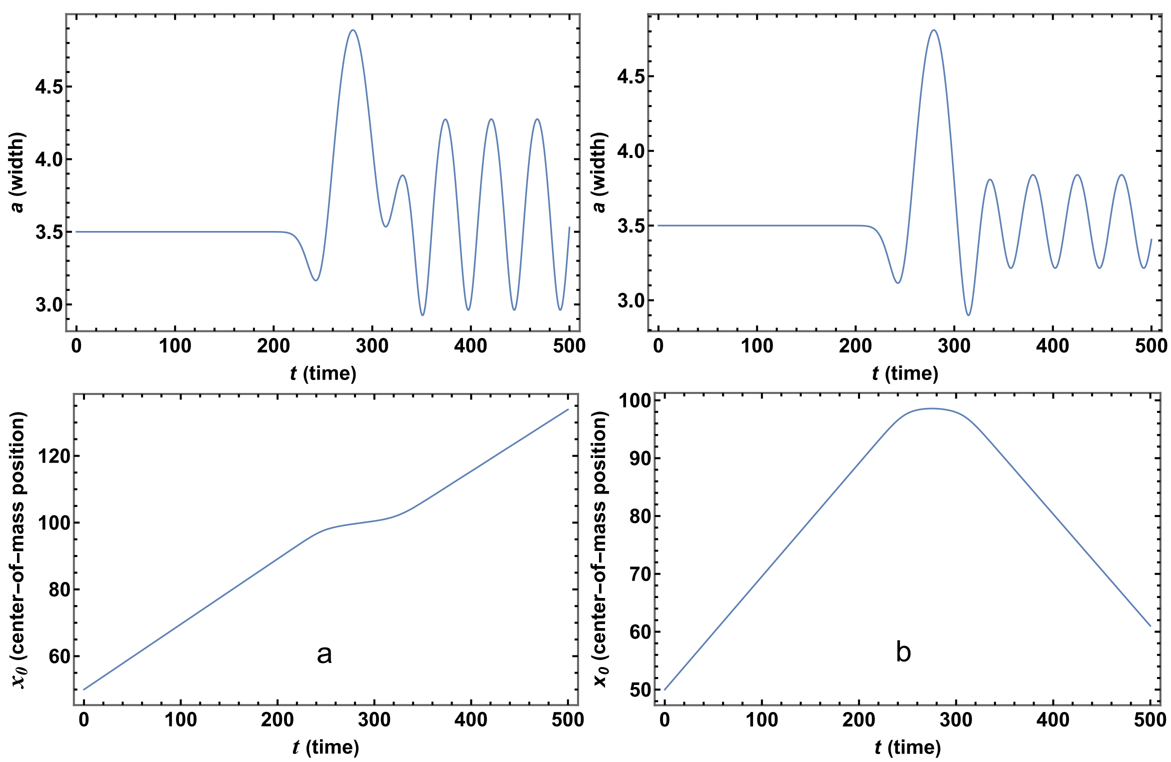

Figure 2. The evolution of the width (the above panel) and center-of-mass position (the below panel) of a discrete soliton over time *t*, as interacting with a Delta potential (14) when $V_0 = -0.06$ (a) and $V_0 = -0.07$ (b), described by the ODE system of Equations (19) and (20)

Figure 2(a) reveals that the soliton is able to pass through the Delta potential when the potential strength $V_0 = -0.06$ is applied, considering the potential barrier is not strong enough to prevent the soliton from being transmitted over it. The soliton then continues to propagate coherently after leaving the potential behind. In addition, the width of the soliton starts to exhibit oscillation right after the collision with the

Delta potential at approximately $t = 240$ as illustrated in the above panel of Figure 2(a). On the other hand, the soliton is reflected by the Delta potential when the potential strength is $V_0 = -0.07$ as manifested in Figure 2(b). At this moment, the Delta potential has enough energy to restrict the propagation of the soliton and reflect it to its initial position. The soliton then manages to regain its initial velocity and eventually preserve its coherence after the reflection from the potential barrier. The width of the soliton also starts to be perturbed immediately after it collides with the Delta potential at around $t = 240$ and remains to oscillate constantly afterward as demonstrated in the above panel of Figure 2(b).

Direct Numerical Simulation

The first step in this method accounts for the stationary soliton solution, $u_n(t)$ corresponds to the lattice equation (4), which is calculated using the Newton's method. The profile of the stationary soliton solution is demonstrated in Figure 3 below.

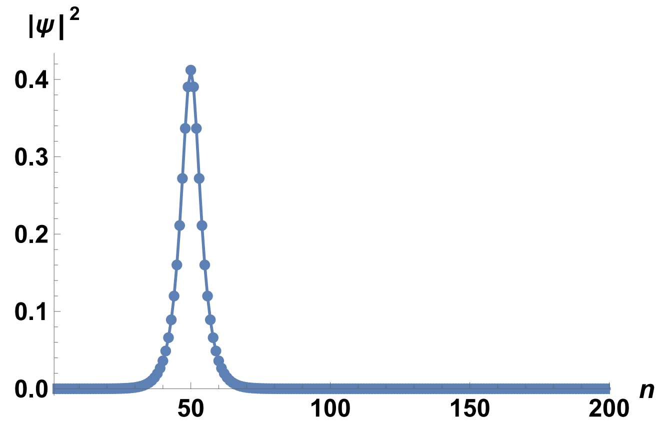

Figure 3. The stationary soliton solution $u_n(t)$ according to the lattice equation (4)

From the figure, it is noticeable that the soliton solution takes the form of a single elevation, moving as a distinct entity with a given velocity. The soliton $\psi_n(t)$ is then considered to be placed at a considerable distance from the external potential V_0 located at n_0, and subsequently set into motion towards that direction with some velocity *v*. The initial condition of the soliton at the input system is designated as follows,

$$\psi_n(t=0) = u_n \exp\left(\frac{1}{2} ivn\right), \tag{25}$$

where u_n denotes the corresponding stationary soliton solution while *v* denotes the initial velocity of the soliton.

For the sake of comparison, a similar range of parameters as in the variational analysis is used to run the direct numerical simulation of the main equation (1). The initial velocity remains constant at $v = 0.1956$ and the initial condition takes the form of Equation (25). Figure 4 depicts the density plots of the soliton's behaviour interacted with Delta potential for two different potential strengths. At $V_0 = -0.06$ (Figure 4(a)), the soliton exhibits transmission through the Delta potential which is accompanied by radiation loss that appears as a result of the collision that happened between them at approximately $t = 240$, while the soliton seems to be reflected by the Delta potential at $V_0 = -0.07$ (Figure 4(b)). Once more, it can be observed that radiative effects are apparent following the scattering process of the discrete soliton by the Delta potential. The widespread orange colour displayed in the density plots of both Figure 4(a) and Figure 4(b) serves as clear indications of these effects.

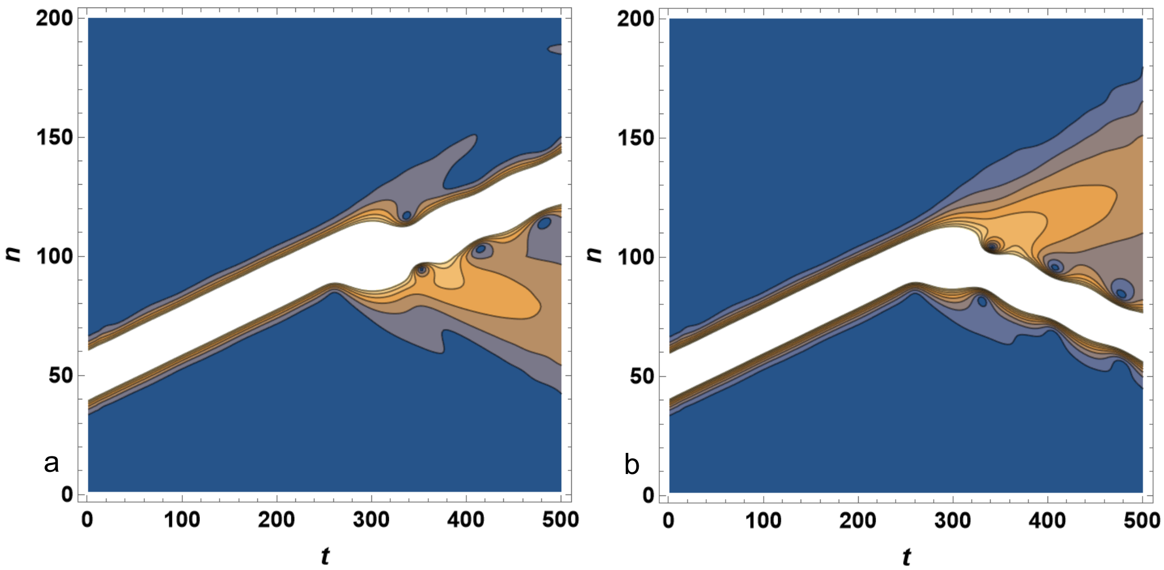

Figure 4. The evolution of a discrete soliton described by the PDE of the main equation (1), in the presence of Delta potential (5) with soliton initial condition as in Equation (25) with two different potential strength, $V_0 = -0.06$ (a) and $V_0 = -0.07$ (b)

Comparing all the results obtained from both methods, it is evident that the VA method produces quite similar results as the direct numerical method. The behaviour of discrete soliton resembling that of a classical particle is verified. In details, when the discrete soliton crosses paths with an external potential, it will undergo either transmission or reflection like classical particles do. This implies that the VA method appears to be a promising complementary tool to carry out the investigation on soliton scattering when subjected to the external potential.

Conclusions

The scattering process of a discrete soliton governed by the discrete cubic-quintic NLSE in the presence of a Delta potential function is addressed. This study examined the scattering process by using different values of the potential strengths while keeping the remaining parameters fixed. Two different methods are applied in this study where the analytical and numerical studies took placed and both results are compared. The systems of four ordinary differential equations for the soliton parameters were formulated and interpreted graphically in the VA method. On the other hand, the direct numerical method provided an exact soliton solution based on the soliton governing equation. It is found that both results revealed similar outcomes for the soliton's behaviours whereby the soliton is transmitted by the Delta potential when $V_0 = -0.06$ and it is reflected when $V_0 = -0.07$. Despite the fact that the VA method is based on approximation and some assumptions, it is worth noting that there is a clear correlation between the results obtained from the VA method and those obtained from the direct numerical method. These findings are useful in the application of soliton-based systems in various fields, including communication and transportation systems.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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