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Absolutely Fuzzy Lipschitz p-summing Maps between Fuzzy Pointed Metric Spaces

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Abstract Motivated by Lipschitz ideals in the conventional (crisp) theory, we are constructing a new Lipschitz ideal in fuzzy theory. We introduce the notion of fuzzy Lipschitz ideals and give some elementary illustrations. The class of absolutely fuzzy Lipschitz p-summing maps $(1 < p < \infty)$ between arbitrary fuzzy pointed metric spaces is a significant category of fuzzy Lipschitz ideals. It is a logical extension of the concept of absolutely (crisp) Lipschitz p-summing maps between arbitrary pointed metric spaces, as established by Farmer Jeffrey and William Johnson. We establish that the fuzzy Lipschitz norm of the previously specified concept is a fuzzy real number. We demonstrate that a complete fuzzy normed fuzzy operator ideal is the resulting class of fuzzy Lipschitz operators between arbitrary fuzzy pointed metric spaces and complete fuzzy normed spaces. Next, we define a basic characterisation of a Lipschitz p-summing map that is completely fuzzy. By demonstrating a fuzzy variant of the nonlinear Pietsch Domination Theorem, this is accomplished. Lastly, we bring forth a few unsolved problems that we find intriguing.

Keywords: Lipschitz ideals, Fuzzy functional analysis, Fuzzy real analysis.

Notations and Preliminaries

It is well-known that the set of all positive real numbers is represented by the symbols $\mathbb{R}^+$, while the set of all real numbers is $\mathbb{R}$, and the set of all positive integers is $\mathbb{N}$. The Banach space is represented by the order pair $(F, \|\cdot\|)$. The ordered pairs denoting pointed metric spaces are (X, d_X, x_0) and (Y, d_Y, y_0). If a non-negative constant C satisfies $d_Y(Sx, Sy) \leq C\ d_X(x, y)$ for all x and y in X, then a map S from (X, d_X, x_0) into (Y, d_Y, y_0) is said to be Lipschitz. The Lipschitz constant of S, represented by $\mathrm{Lip}(S)$, is the smallest possible value of C. $\mathcal{L}(X, Y)$ represents the class of all Lipschitz maps from (X, d_X, x_0) into (Y, d_Y, y_0). $X^{\#}$ is the Banach space of real-valued Lipschitz functions defined on (X, d_X, x_0) that, with the Lipschitz norm $\mathrm{Lip}(\cdot)$, send the special point x_0 into 0. To distinguish between the fuzzy norm of elements in fuzzy normed spaces and fuzzy Lipschitz maps, we shall use the symbols $\|\cdot\|_{\sim}$ and $\|\cdot\|^{\sim}$, respectively.

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$\alpha \in (0,1]$.

Definition 2 [1] Let $\eta \in \mathcal{F}$. If $t < 0$, $\eta(t) = 0$, then η is called a positive fuzzy real number. The set of all positive fuzzy real numbers is denoted by $\mathcal{F}^+$.

Lemma 3 [2] Let η and δ belong to $\mathcal{F}$. and let $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. Then

$$[\eta \oplus \delta]_\alpha = [\eta_\alpha^- + \delta_\alpha^-, \eta_\alpha^+ + \delta_\alpha^+],$$
$$[\eta \ominus \delta]_\alpha = [\eta_\alpha^- - \delta_\alpha^+, \eta_\alpha^+ - \delta_\alpha^-],$$
$$[\eta \odot \delta]_\alpha = [\eta_\alpha^- \cdot \delta_\alpha^-, \eta_\alpha^+ \cdot \delta_\alpha^+], \text{for } \eta, \delta \in \mathcal{F}^+.$$

Definition 4 [2] Let X be a non-void set and $\tilde{d}_X$ is a map from $X \times X$ to $\mathcal{F}^+$. The pair $(X, \tilde{d}_X)$ is said to be a fuzzy metric space with a fuzzy norm $\tilde{d}_X$ on $X \times X$ if the following conditions are satisfied:

1. $\tilde{d}_X(x, y) = \tilde{0}$ if and only if $x = y$.
2. $\tilde{d}_X(x, y) = \tilde{d}_X(y, x), \forall\ x, y \in X$.
3. $\tilde{d}_X(x, y) \leq \tilde{d}_X(x, w) \oplus \tilde{d}_X(w, y), \forall\ x, y, \text{and}\ w \in X$.

Introduction

Recall the definition of Lipschitz ideal concepts as follows [4]. Consider, for all pair of metric spaces X and Y, there is a subset $\mathcal{W}^L(X,Y)$ of $\mathcal{L}(X,Y)$. The class

$$\mathcal{W}^L := \cup_{X,Y} \mathcal{W}^L(X,Y)$$

is called a Lipschitz ideal, if the following requirements hold:

1. If $Y = F$, then $f \boxdot e \in \mathcal{W}^L(X,F)$ for $f \in X^{\#}$ and $e \in F$.
2. $BTA \in \mathcal{W}^L(X_0, Y_0)$ for $A \in \mathcal{L}(X_0, X)$, $T \in \mathcal{W}^L(X,Y)$, and $B \in \mathcal{L}(Y, Y_0)$, where X_0 and Y_0 be metric spaces.

One crucial example of Lipschitz ideal is the class of Lipschitz p-summing maps introduced by Farmer Jeffrey and William Johnson [5] as follows. A Lipschitz function S from (X, d_X, x_0) into (Y, d_Y, y_0) is said to be Lipschitz p-summing $(1 \le p < \infty)$ if and only if there is a constant $C \ge 0$ such that

$$\left[\sum_{j=1}^{m} d_Y(Sx_j, Sy_j)^p\right]^{\frac{1}{p}} \le C \cdot \sup_{f \in B_{X^{\#}}} \left[\sum_{j=1}^{m} \left|fx_j - fy_j\right|^p\right]^{\frac{1}{p}}$$

for arbitrary sequences $(x_j)_{j=1}^m$, $(y_j)_{j=1}^m$ in X, and $m \in \mathbb{N}$. The Lipschitz p-summing norm $P_p^L(T)$ is then the smallest possible constant C.

Theorem 5 [5] Let $1 \le p < \infty$. For every Lipschitz function S from (X, d_X, x_0) into (Y, d_Y, y_0) and $\tau \ge 0$, the following are equivalent:

1. $P_p^L(S) \le \tau$.
2. There is a probability measure ν on $B_{X^{\#}}$ such that

$$d_Y(Sx, Sy)^p \le \tau^p \cdot \int_{B_{X^{\#}}} |f(x) - f(y)|^p d\nu(f).$$

More detailed information about Lipschitz ideals can be pointed out in the manuscripts [6, 7, 4, 8].
To describe the content of the manuscript. In the next Section 3, we present a fuzzy Lipschitz map concept between arbitrary fuzzy pointed metric spaces. We define a fuzzy Lipschitz norm of its and prove that it is a fuzzy real number. We show that the class of all fuzzy Lipschitz operators between arbitrary fuzzy pointed metric spaces and complete fuzzy normed spaces is a complete fuzzy normed space. In Section 4, we construct a fuzzy Lipschitz ideals terminology between arbitrary fuzzy pointed metric spaces and present some basic examples. The class of absolutely fuzzy Lipschitz p-summing maps between arbitrary fuzzy pointed metric spaces $(1 < p < \infty)$ is a significant category of fuzzy Lipschitz ideals. It naturally expands the absolutely (crisp) Lipschitz p-summing maps between arbitrary metric spaces that were identified by Farmer Jeffrey and William Johnson. Consequently, the fuzzy Lipschitz version of Pietsch Domination Theorem is proven, which is the fundamental characterization of absolutely fuzzy Lipschitz p-summing maps. Finally, we raise some interesting open problems in Section 5.

3 Fuzzy Lipschitz norm of fuzzy Lipschitz maps between fuzzy pointed metric spaces

We slightly modify Definition 4 to introduce the following terminology.

Definition 6 Let X be a non-empty set and x_0 be a special point in X and let $\tilde{d}_X$ be a function from $X \times X$ into $\mathcal{F}^+$. An ordered triple $(X, \tilde{d}_X, x_0)$ is said to be a fuzzy pointed metric space if the requirements mentioned below are satisfied:

1. $\inf_{0<\delta\le 1} d_X(a,b)_\delta^- > 0$ whenever $a \ne b$.
2. $\tilde{d}_X(x,y) = \tilde{0}$ if and only if $x = y$.
3. $\tilde{d}_X(x,y) = \tilde{d}_X(y,x), \forall\ x, y \in X$.
4. $\tilde{d}_X(x,y) \le \tilde{d}_X(x,w) \oplus \tilde{d}_X(w,y), \forall\ x, y, \text{and}\ w \in X$.

Definition 7 A map S from $(X, \tilde{d}_X, x_0)$ into $(Y, \tilde{d}_Y, y_0)$ is called fuzzy Lipschitz, if there exists a fuzzy number $\eta \in \mathcal{F}^+$ such that

$$\tilde{d}_Y(Sx, Sy) \leq \eta \odot \tilde{d}_X(x, y), \forall\ x, y \in X. \quad (1)$$

The class of all fuzzy Lipschitz maps from $(X, \tilde{d}_X, x_0)$ into $(Y, \tilde{d}_Y, y_0)$ that send the special point x_0 in X into the special point y_0 in Y is denoted by $\mathrm{FLip}(X, Y)$.

Definition 8 A fuzzy real number-valued function $\mathrm{FLip}(\cdot)^\sim$ on $\mathrm{FLip}(X, Y)$ defined by,

$$[\mathrm{FLip}(S)^\sim]_\alpha = [\mathrm{FLip}(S)_\alpha^-, \mathrm{FLip}(S)_\alpha^+], \forall \alpha \in (0,1],$$

where $\mathrm{FLip}(S)_\alpha^- := \sup_{\beta<\alpha} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx,Sy)_\beta^-}{d_X(x,y)_\beta^-}$ and $\mathrm{FLip}(S)_\alpha^+ := \inf\{\eta_\alpha^+ : (1) \text{ holds}\}$.

The special case $Y = \mathbb{R}$ of Definition 7 gives the following terminology.

Definition 9 A map f from $(X, \tilde{d}_X, x_0)$ into $(\mathbb{R}, |\cdot|_\sim)$ is said to be a fuzzy Lipschitz map on X, if there is a fuzzy number $\eta \in \mathcal{F}^+$ such that

$$|f(x) - f(y)|_\sim \leq \eta \odot \tilde{d}_X(x, y), \forall\ x, y \in X. \quad (2)$$

The class of all fuzzy Lipschitz functionals on X that send the special point x_0 in X into 0 in $\mathbb{R}$ is denoted by $FX^\#$. We define the fuzzy real number-valued function $\mathrm{FLip}(\cdot)^\sim$ on $FX^\#$ by

$$[\mathrm{FLip}(f)^\sim]_\alpha = [\mathrm{FLip}(f)_\alpha^-, \mathrm{FLip}(f)_\alpha^+], \forall \alpha \in (0,1],$$

where $\mathrm{FLip}(f)_\alpha^- := \sup_{\beta<\alpha} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{|f(x)-f(y)|_\beta^-}{d_X(x,y)_\beta^-}$ and $\mathrm{FLip}(f)_\alpha^+ := \inf\{\eta_\alpha^+ : (2) \text{ holds}\}$.

Theorem 10 If $S \in \mathrm{FLip}(X, Y)$, then $\mathrm{FLip}(S)^\sim \in \mathcal{F}^+$.

Proof. First we show that $[\mathrm{FLip}(S)^\sim]_\alpha$ is a nonempty interval for all $\alpha \in (0,1]$. Let $\alpha \in (0,1]$ and let $\beta < \alpha$ and η satisfy $\tilde{d}_Y(Sx, Sy) \leq \eta \odot \tilde{d}_X(x, y), \forall\ x, y \in X$. Thus $\frac{d_Y(Sx,Sy)_\beta^-}{d_X(x,y)_\beta^-} \leq \eta_\beta^-\ \ \forall x \neq y$. Since $\eta_\beta^- \leq \eta_\alpha^-$ and $\eta_\alpha^- \leq \eta_\alpha^+$ we obtain

$$\sup_{\beta<\alpha} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx, Sy)_\beta^-}{d_X(x, y)_\beta^-} \leq \eta_\beta^- \leq \eta_\alpha^- \leq \eta_\alpha^+.$$

Therefore

$$\mathrm{FLip}(S)_\alpha^- \leq \inf\{\eta_\alpha^+ : (1) \text{ holds}\} =: \mathrm{FLip}(S)_\alpha^+.$$

Now we prove that $[\mathrm{FLip}(S)^\sim]_\alpha$ satisfies the conditions of [9, Lemma 2.9]:

1. Let $0 < \alpha_1 \leq \alpha_2 \leq 1$. To show that $[\mathrm{FLip}(S)^\sim]_{\alpha_2} \subset [\mathrm{FLip}(S)^\sim]_{\alpha_1}$. We have

$$\mathrm{FLip}(S)_{\alpha_1}^- := \sup_{\beta<\alpha_1} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{\tilde{d}_Y(Sx, Sy)_\beta^-}{\tilde{d}_X(x, y)_\beta^-} \leq \sup_{\beta<\alpha_2} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{\tilde{d}_Y(Sx, Sy)_\beta^-}{\tilde{d}_X(x, y)_\beta^-} =: \mathrm{FLip}(S)_{\alpha_2}^-.$$

Since $0 < \alpha_1 \leq \alpha_2 \leq 1$ we obtain $\eta_{\alpha_2}^+ \leq \eta_{\alpha_1}^+$ and then

$$\mathrm{FLip}(S)_{\alpha_2}^+ := \inf\{\eta_{\alpha_2}^+ : (1) \text{ holds}\} \leq \inf\{\eta_{\alpha_1}^+ : (1) \text{ holds}\} =: \mathrm{FLip}(S)_{\alpha_1}^+.$$

2. Let $(\alpha_k)_{k\in\mathbb{N}}$ be an increasing sequence in $(0,1]$ converging to α. To show that $[\lim_{k\to\infty} \mathrm{FLip}(S)_{\alpha_k}^-, \lim_{k\to\infty} \mathrm{FLip}(S)_{\alpha_k}^+] = [\mathrm{FLip}(S)_\alpha^-, \mathrm{FLip}(S)_\alpha^+]$. We have $\alpha_k \leq \alpha_{k+1} \leq \alpha$ and thus

$$\sup_k \sup_{\beta<\alpha_k} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx,Sy)_\beta^-}{d_X(x,y)_\beta^-} \leq \sup_{\beta<\alpha} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx,Sy)_\beta^-}{d_X(x,y)_\beta^-}. \quad (3)$$

Suppose that $\epsilon > 0$ and $\beta_0 < \alpha$ then

$$\sup_{\beta<\alpha} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx, Sy)_\beta^-}{d_X(x, y)_\beta^-} - \epsilon < \sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx, Sy)_{\beta_0}^-}{d_X(x, y)_{\beta_0}^-}.$$

$$\sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx, Sy)_{\beta_0}^-}{d_X(x, y)_{\beta_0}^-} \leq \sup_{\beta<\alpha_{k_0}} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx, Sy)_\beta^-}{d_X(x, y)_\beta^-} \leq \sup_k \sup_{\beta<\alpha_k} \sup_{\substack{x \neq y \\ x,y \in X}} \frac{d_Y(Sx, Sy)_\beta^-}{d_X(x, y)_\beta^-}.$$

Therefore

$$\sup_{\beta<\alpha}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}-\epsilon<\sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}.$$

As $\epsilon\longrightarrow 0$, we have

$$\sup_{\beta<\alpha}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}\leq\sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}. \quad (4)$$

We obtain

$$\lim_{k\to\infty}\sup_{\beta<\alpha_k}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}=\sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}=\sup_{\beta<\alpha}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}=\mathrm{FLip}(S)^-_\alpha.$$

Therefore

$$\lim_{k\to\infty}\inf\{\eta^+_{\alpha_k}:(1)\text{ holds}\}=\inf_k\inf\{\eta^+_{\alpha_k}:(1)\text{ holds}\}=\inf\{\eta^+_\alpha:(1)\text{ holds}\}.$$

3. To show that $-\infty<\mathrm{FLip}(S)^-_\alpha\leq\mathrm{FLip}(S)^+_\alpha<\infty$, for all $\alpha\in(0,1]$. Since $0\leq\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}$ for all $x\neq y\in X$ and all $\beta\in(0,1]$. Then $0\leq\mathrm{FLip}(S)^-_\alpha$. Let $\eta\in\mathcal{F}$ such that (1) holds for all x and y in X . It follows that $\eta^+_\alpha<\infty$ for all $\alpha\in(0,1]$. Hence $\mathrm{FLip}(S)^+_\alpha<\infty$. Thus we obtain $\mathrm{FLip}(S)^\sim$ is a fuzzy real number.

Proposition 11 If $S\in\mathrm{FLip}(X,Y)$, then $\tilde{d}_Y(Sx,Sy)\preceq\mathrm{FLip}(S)^\sim\odot\tilde{d}_X(x,y),\forall x,y\in X$.

Proof. If $x=y$ the statement is obvious. When $x\neq y$. Suppose that $(\beta_k)_{k\in\mathbb{N}}$ be an increasing sequence in $(0,1]$ converging to $\alpha\in(0,1]$. Since

$$\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}\leq\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}\leq\mathrm{FLip}(S)^-_\alpha.$$

Then $d_Y(Sx,Sy)^-_\beta\leq\mathrm{FLip}(S)^-_\alpha\cdot d_X(x,y)^-_\beta$. Since $\beta_k\nearrow\alpha$, it follows from [9, Lemma 2.9 (b)] that

$$d_Y(Sx,Sy)^-_\alpha=\lim_{k\to\infty}d_Y(Sx,Sy)^-_{\beta_k}\leq\mathrm{FLip}(S)^-_\alpha\cdot\lim_{k\to\infty}d_X(x,y)^-_{\beta_k}\leq\mathrm{FLip}(S)^-_\alpha\cdot d_X(x,y)^-_\alpha.$$

Hence

$$d_Y(Sx,Sy)^-_\alpha\leq\mathrm{FLip}(S)^-_\alpha\cdot d_X(x,y)^-_\alpha. \quad (5)$$

From our hypothesis we have $d_Y(Sx,Sy)^+_\alpha\leq\eta^+_\alpha\cdot d_X(x,y)^+_\alpha$. Then

$$d_Y(Sx,Sy)^+_\alpha\leq\inf\{\eta^+_\alpha:(1)\text{ holds}\}\cdot d_X(x,y)^+_\alpha. \quad (6)$$

From Inequalities (5) and (6), we fulfill the requirement.

Proposition 12 If $S:(X,\tilde{d}_X,x_0)\longrightarrow(Y,\tilde{d}_Y,y_0)$ be a fuzzy Lipschitz map satisfies (1), then $\mathrm{FLip}(S)^\sim\preceq\eta$.

Proof. Suppose that η satisfy $\tilde{d}_Y(Sx,Sy)\preceq\eta\odot\tilde{d}_X(x,y),\forall\ x,y\in X$. Let $\alpha\in(0,1]$ and $\beta<\alpha$ we have $\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}\leq\eta^-_\beta,\ \forall x,y\in X$ with $x\neq y$. Then $\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}\leq\eta^-_\beta\leq\eta^-_\alpha,\ \forall\beta<\alpha$. Therefore

$$\mathrm{FLip}(S)^-_\alpha:=\sup_{\beta<\alpha}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{d_Y(Sx,Sy)^-_\beta}{d_X(x,y)^-_\beta}\leq\eta^-_\alpha.$$

Since $\mathrm{FLip}(S)^+_\alpha:=\inf\{\eta^+_\alpha:(1)\text{ holds}\}\leq\eta^+_\alpha$. Thus, we conclude that $\mathrm{FLip}(S)^\sim\preceq\eta$.

Theorem 13 The ordered pair $(\mathrm{FLip}(X,F),\mathrm{FLip}(\cdot)^\sim)$ is a fuzzy normed space.

Proof. It is obvious that $\mathrm{FLip}(X,F)$ be a linear space. To prove Condition (FN_0). Let $0\neq T\in\mathrm{FLip}(X,F)$. Then there is a point $0\neq\zeta\in X$ such that $T(\zeta)\neq 0$. Suppose that $\sup_{0<\alpha\leq 1}d_X(\zeta,y)^-_\alpha<\infty,\forall\ y\in Y$. We have

$$\inf_{0<\beta\leq 1}\|T(\zeta)-T(y)\|^-_\beta\leq\|T(\zeta)-T(y)\|^-_\alpha\leq\mathrm{FLip}(T)^-_\alpha\cdot d_X(\zeta,y)^-_\alpha\leq\mathrm{FLip}(T)^-_\alpha\cdot\sup_{0<\alpha\leq 1}d_X(\zeta,y)^-_\alpha,\forall\alpha\in(0,1].$$

Hence $\inf_{0<\beta\leq 1}\|T(\zeta)-T(y)\|^-_\beta\leq\sup_{0<\alpha\leq 1}d_X(\zeta,y)^-_\alpha\cdot\inf_{0<\alpha\leq 1}\mathrm{FLip}(T)^-_\alpha$. Since $T(\zeta)\neq 0$, it follows that $0<\inf_{0<\alpha\leq 1}\|T(\zeta)-T(y)\|^-_\alpha$. Thus $0<\inf_{0<\alpha\leq 1}\mathrm{FLip}(T)^-_\alpha$. To show Condition (FN_1). When $T=0$ the statement is true. Conversely, let $\mathrm{FLip}(T)^\sim=\tilde{0}$. Since $\|T(x)-T(y)\|_\sim\preceq\tilde{0}\odot\tilde{d}_X(x,y),\forall\ x,y\in X$, it follows that $\|T(x)-T(y)\|_\sim=\tilde{0},\forall\ x,y\in X$. Hence $T(x)=T(y),\forall\ x,y\in X$ we conclude that $T(x)=0,\forall\ x\in X$, therefore $T=0$. To show Condition (FN_2). Let $r\in\mathbb{R}$ and $T\in\mathrm{FLip}(X,F)$ and let $\alpha\in(0,1]$. When $r=0$ the statement is obvious. If $r\neq 0$ we have

$$[\mathrm{FLip}(rT)^\sim]_\alpha=\left[\sup_{\beta<\alpha}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{\|rT(x)-rT(y)\|^-_\beta}{d_X(x,y)^-_\beta},\inf\{\eta^+_\alpha:\|rT(x)-rT(y)\|_\sim\preceq\eta\odot\tilde{d}_X(x,y)\}\right]$$

$$=\left[|r|\sup_{\beta<\alpha}\sup_{\substack{x\neq y\\ x,y\in X}}\frac{\|T(x)-T(y)\|^-_\beta}{d_X(x,y)^-_\beta},\inf\left\{\eta^+_\alpha:\|T(x)-T(y)\|_\sim\preceq\frac{\tilde{1}}{|r|_\sim}\odot\eta\odot\tilde{d}_X(x,y)\right\}\right]$$

$$= \left[|r| \sup_{\beta<\alpha} \sup_{\substack{x\neq y \\ x,y\in X}} \frac{\|T(x)-T(y)\|_{\beta}^{-}}{d_X(x,y)_{\beta}^{-}}, \inf\{|r|\cdot\gamma_{\alpha}^{+}: \|T(x)-T(y)\|_{\sim} \preceq \gamma \odot \tilde{d}_X(x,y)\} \right]$$

$$= \left[|r| \sup_{\beta<\alpha} \sup_{\substack{x\neq y \\ x,y\in X}} \frac{\|T(x)-T(y)\|_{\beta}^{-}}{d_X(x,y)_{\beta}^{-}}, |r| \inf\{\gamma_{\alpha}^{+}: \|T(x)-T(y)\|_{\sim} \preceq \gamma \odot \tilde{d}_X(x,y)\} \right] = [|r|_{\sim} \odot FLip(T)^{\sim}]_{\alpha}.$$

Thus $FLip(rT)^{\sim} = |r|_{\sim} \odot FLip(T)^{\sim}$. To show Condition (FN_3). Let T and S in $FLip(X,F)$ we have

$$\begin{aligned} FLip(T+S)_{\alpha}^{-} &:= \sup_{\beta<\alpha} \sup_{\substack{x\neq y \\ x,y\in X}} \frac{\|(T+S)(x)-(T+S)(y)\|_{\beta}^{-}}{d_X(x,y)_{\beta}^{-}} \\ &\le \sup_{\beta<\alpha} \sup_{\substack{x\neq y \\ x,y\in X}} \left(\frac{\|T(x)-T(y)\|_{\beta}^{-}}{d_X(x,y)_{\beta}^{-}} + \frac{\|S(x)-S(y)\|_{\beta}^{-}}{d_X(x,y)_{\beta}^{-}} \right) \\ &\le \sup_{\beta<\alpha} \sup_{\substack{x\neq y \\ x,y\in X}} \frac{\|T(x)-T(y)\|_{\beta}^{-}}{d_X(x,y)_{\beta}^{-}} + \sup_{\beta<\alpha} \sup_{\substack{x\neq y \\ x,y\in X}} \frac{\|S(x)-S(y)\|_{\beta}^{-}}{d_X(x,y)_{\beta}^{-}} \\ &=: FLip(T)_{\alpha}^{-} + FLip(S)_{\alpha}^{-}. \end{aligned} \tag{7}$$

Also, we have

$$\begin{aligned} \|(T+S)(x)-(T+S)(y)\|_{\sim} &\preceq \|T(x)-T(y)\|_{\sim} \oplus \|S(x)-S(y)\|_{\sim} \\ &\preceq FLip(T)^{\sim} \odot \tilde{d}_X(x,y) \oplus FLip(S)^{\sim} \odot \tilde{d}_X(x,y) \\ &= (FLip(T)^{\sim} \oplus FLip(S)^{\sim}) \odot \tilde{d}_X(x,y). \end{aligned}$$

Therefore

$$\inf\{\eta_{\alpha}^{+}: \|(T+S)(x)-(T+S)(y)\|_{\sim} \preceq \eta \odot \tilde{d}_X(x,y)\} \le FLip(T)_{\alpha}^{+} + FLip(S)_{\alpha}^{+}.$$

Hence

$$FLip(T+S)_{\alpha}^{+} \le FLip(T)_{\alpha}^{+} + FLip(S)_{\alpha}^{+}. \tag{8}$$

From Inequalities (7) and (8), we fulfill the requirement.

4 Fuzzy Lipschitz ideals between fuzzy pointed metric spaces

Before introducing the nonlinear theory of fuzzy Lipschitz ideals between arbitrary fuzzy pointed metric spaces the reader can be aware of the nonlinear theory of fuzzy Lipschitz ideals between arbitrary fuzzy pointed metric spaces and fuzzy normed spaces in [3] and the linear theory of fuzzy operator ideals between arbitrary fuzzy normed spaces in [10]. Now we construct the terminology of fuzzy Lipschitz ideals between fuzzy pointed metric spaces as follows.

Definition 14 Assume that, for all fuzzy pointed metric spaces X and Y, a subset $F\mathfrak{W}^L(X,Y)$ of $FLip(X,Y)$. The class

$$F\mathfrak{W}^L := \bigcup_{X,Y} F\mathfrak{W}^L(X,Y)$$

is called a fuzzy Lipschitz ideal if the following requirements hold:

1. If $Y = F$, then $f \boxdot e \in F\mathfrak{W}^L(X,F)$ with $\mathcal{W}^L(f \boxdot e)^{\sim} \preceq \|e\|_{\sim}$ for $f \in \mathfrak{B}_{FX^{\#}}$ and $e \in F$.
2. $AT \in F\mathfrak{W}^L(X,Z)$ with $\mathcal{W}^L(AT)^{\sim} \preceq FLip(A)^{\sim} \odot \mathcal{W}^L(T)^{\sim}$ for $T \in F\mathfrak{W}^L(X,Y)$ and $A \in FLip(Y,Z)$, where $\mathcal{W}^L$ is a function from $F\mathfrak{W}^L$ into $\mathcal{F}^+$and $\mathfrak{B}_{FX^{\#}}$ stands for the unit ball of all fuzzy-real valued fuzzy Lipschitz maps defined on fuzzy pointed metric space X.

Essential Example of Fuzzy Lipschitz Ideals

Absolutely fuzzy Lipschitz p-summing maps

Definition 15 Let $1 < p < \infty$. A fuzzy Lipschitz map $S: (X, \tilde{d}_X, x_0) \longrightarrow (Y, \tilde{d}_Y, y_0)$ is said to be absolutely fuzzy Lipschitz p-summing if there is a fuzzy real number $\zeta \in \mathcal{F}^+$ such that for all $(x_j)_{j=1}^m$, $(y_j)_{j=1}^m$ in X and $m \in \mathbb{N}$,

$$\left[\sum_{j=1}^{m} \tilde{d}_Y(Sx_j, Sy_j)^p\right]^{\frac{1}{p}} \preceq \zeta \odot \sup_{FLip(f) \preceq \tilde{1}} \left[\sum_{j=1}^{m} \left|fx_j - fy_j\right|_{\sim}^{p}\right]^{\frac{1}{p}} \tag{9}$$

holds. The symbol $F\mathfrak{P}_p^L(X,Y)$ is designated by the class of all absolutely fuzzy Lipschitz p-summing maps from $(X, \tilde{d}_X, x_0)$ to $(Y, \tilde{d}_Y, y_0)$. The absolutely fuzzy Lipschitz p-summing norm $FP_p^L(S)^{\sim}$ of S is defined by, $[FP_p^L(S)^{\sim}]_{\alpha} = [FP_p^L(S)_{\alpha}^{-}, FP_p^L(S)_{\alpha}^{+}]$ for all $\alpha \in (0,1]$, where

$$FP_p^L(S)_\alpha^- := \sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}},$$

and

$$FP_p^L(S)_\alpha^+ := \inf\{\eta_\alpha^+ : (9)\text{ holds}\}.$$

Proposition 16 Let $1<p<\infty$. If $S\in F\mathfrak{P}_p^L(X,Y)$, then $FP_p^L(S)^\sim \in \mathcal{F}$.

Proof. To show that $[FP_p^L(S)^\sim]_\alpha$ be non-void interval for all $\alpha\in(0,1]$. Let $\alpha\in(0,1]$ and $\beta<\alpha$ and let $(x_j)_{j=1}^m$, $(y_j)_{j=1}^m$ in X and $m\in\mathbb{N}$. From our assumptions

$$\frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}} \leq \eta_\beta^-. \tag{10}$$

and $\eta_\beta^-\leq\eta_\alpha^-$. Since $\eta_\alpha^-\leq\eta_\alpha^+$ we get $\frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m |fx_j-fy_j|_\beta^{p,-}\right]^{\frac{1}{p}}}\leq\eta_\alpha^+$. Therefore

$$FP_p^L(S)_\alpha^- := \sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}} \leq \inf\{\eta_\alpha^+ : (9)\text{ holds}\} =: FP_p^L(S)_\alpha^+.$$

To establish that $[FP_p^L(S)^\sim]_\alpha$ fulfills the requirements of [9, Lemma 2.9]:

1. Let $0<\alpha_1\leq\alpha_2\leq1$. To show that $[FP_p^L(S)^\sim]_{\alpha_2}\subset[FP_p^L(S)^\sim]_{\alpha_1}$. Then

$$FP_p^L(S)_{\alpha_1}^- := \sup_{\beta<\alpha_1}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}$$

$$\leq \sup_{\beta<\alpha_2}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}} =: FP_p^L(S)_{\alpha_2}^-.$$

Since $0<\alpha_1\leq\alpha_2\leq1$ we obtain $\eta_{\alpha_2}^+\leq\eta_{\alpha_1}^+$ and then

$$FP_p^L(S)_{\alpha_2}^+ := \inf\{\eta_{\alpha_2}^+ : (9)\text{ holds}\} \leq \inf\{\eta_{\alpha_1}^+ : (9)\text{ holds}\} =: FP_p^L(S)_{\alpha_1}^+.$$

2. Let $(\alpha_k)_{k\in\mathbb{N}}$ be an increasing sequence in $(0,1]$ converging to α. To show that $[\lim_{k\to\infty} FP_p^L(S)_{\alpha_k}^-, \lim_{k\to\infty} FP_p^L(S)_{\alpha_k}^+] = [FP_p^L(S)_\alpha^-, FP_p^L(S)_\alpha^+]$. We have $\alpha_k\leq\alpha_{k+1}\leq\alpha$ and thus

$$\sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}} \leq \sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}. \tag{11}$$

Let $\epsilon>0$. Then there exist $\beta_0<\alpha$ such that

$$\sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}} - \epsilon < \sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_{\beta_0}^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_{\beta_0}^{p,-}\right]^{\frac{1}{p}}}.$$

Thus

$$\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_{\beta_0}^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_{\beta_0}^{p,-}\right]^{\frac{1}{p}}} \leq \sup_{\beta<\alpha_{k_0}}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}$$

$$\leq \sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^m \left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}.$$

Therefore

$$\sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}-\epsilon<\sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}.$$

As $\epsilon\longrightarrow 0$, we have

$$\sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}\leq\sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}. \tag{12}$$

From Inequalities (11) and (12), we obtain

$$\lim_{k\to\infty}\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}=\sup_k\sup_{\beta<\alpha_k}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}$$

$$=\sup_{\beta<\alpha}\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}.$$

$$\lim_{k\to\infty}\inf\{\eta_{\alpha_k}^{+}:(9)\text{ holds}\}=\inf_k\inf\{\eta_{\alpha_k}^{+}:(9)\text{ holds}\}=\inf\{\eta_{\alpha}^{+}:(9)\text{ holds}\}.$$

3. To show that $-\infty<FP_p^L(S)_\alpha^-\leq FP_p^L(S)_\alpha^+<\infty$, for all $\alpha\in(0,1]$. Since $0\leq\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}$ for all $x_j\neq y_j\in X$ and all $\beta\in(0,1]$. Then $0\leq FP_p^L(S)_\alpha^-$. Let $\eta\in\mathcal{F}$ such that for all $(x_j)_{j=1}^m$, $(y_j)_{j=1}^m$ in X and $m\in\mathbb{N}$, (9) holds. It follows that $\eta_\alpha^+<\infty$, for all $\alpha\in(0,1]$. Hence $FP_p^L(S)_\alpha^+<\infty$. Thus $FP_p^L(S)^\sim$ is a fuzzy real number.

Proposition 17 Let $1<p<\infty$. If $S\in F\mathfrak{P}_p^L(X,Y)$, then

$$\left[\sum_{j=1}^{m}\tilde{d}_Y(Sx_j,Sy_j)^p\right]^{\frac{1}{p}}\leq FP_p^L(S)^\sim\odot\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\sim^p\right]^{\frac{1}{p}}$$

for all $(x_j)_{j=1}^m$, $(y_j)_{j=1}^m$ in X and $m\in\mathbb{N}$.

Proof. Suppose that $(\beta_k)_{k\in\mathbb{N}}$ be an increasing sequence in $(0,1]$ converging to $\alpha\in(0,1]$. Since

$$\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}\leq\sup_{\substack{x_j,y_j\in X\\ x_j\neq y_j}}\frac{\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}}\leq FP_p^L(S)_\alpha^-.$$

Then $\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_{\beta_k}^{p,-}\right]^{\frac{1}{p}}\leq FP_p^L(S)_\alpha^-\cdot\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_{\beta_k}^{p,-}\right]^{\frac{1}{p}}$. Since $\beta_k\nearrow\alpha$, it follows from [9, Lemma 2.9 (b)] that

$$\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\alpha^{p,-}\right]^{\frac{1}{p}}=\lim_{k\to\infty}\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_{\beta_k}^{p,-}\right]^{\frac{1}{p}}\leq FP_p^L(S)_\alpha^-\cdot\lim_{k\to\infty}\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_{\beta_k}^{p,-}\right]^{\frac{1}{p}}$$

$$\leq FP_p^L(S)_\alpha^-\cdot\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\alpha^{p,-}\right]^{\frac{1}{p}}.$$

Then

$$\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\alpha^{p,-}\right]^{\frac{1}{p}}\leq FP_p^L(S)_\alpha^-\cdot\sup_{FLip(f)\leq\tilde{1}}\left[\sum_{j=1}^{m}\left|fx_j-fy_j\right|_\alpha^{p,-}\right]^{\frac{1}{p}}. \tag{13}$$

From our hypothesis we have

$$\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\alpha^{p,+}\right]^{\frac{1}{p}} \le \eta_\alpha^+ \cdot \sup_{FLip(f)\le\tilde{1}} \left[\sum_{j=1}^{m} \left|fx_j - fy_j\right|_\alpha^{p,+}\right]^{\frac{1}{p}}.$$

Then

$$\left[\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\alpha^{p,+}\right]^{\frac{1}{p}} \le \inf\{\eta_\alpha^+ : (9)\text{holds}\} \cdot \sup_{FLip(f)\le\tilde{1}} \left[\sum_{j=1}^{m} \left|fx_j - fy_j\right|_\alpha^{p,+}\right]^{\frac{1}{p}}. \tag{14}$$

From Inequalities (13) and (14), We meet the requirements.

Proposition 18 Let $1 < p < \infty$. If $S \in F\mathfrak{P}_p^L(X,Y)$, then $FP_p^L(S)^\sim \preceq \eta$, where η defined in (9).

Proof. From (10) we obtain

$$FP_p^L(S)_\alpha^- := \sup_{\beta<\alpha} \sup_{\substack{x_j,y_j\in X\\ x_j\ne y_j}} \frac{\left[\sum_{j=1}^m d_Y(Sx_j,Sy_j)_\beta^{p,-}\right]^{\frac{1}{p}}}{\sup_{FLip(f)\le\tilde{1}} \left[\sum_{j=1}^m \left|fx_j - fy_j\right|_\beta^{p,-}\right]^{\frac{1}{p}}} \le \eta_\alpha^-.$$

Since $FP_p^L(S)_\alpha^+ := \inf\{\eta_\alpha^+ : (9) \text{ holds}\} \le \eta_\alpha^+$, then $FP_p^L(S)_\alpha^+ \le \eta_\alpha^+$. Thus, we conclude that $FP_p^L(S)^\sim \preceq \eta$.

Proposition 19 If $1 < p < \infty$, then $\left[F\mathfrak{P}_p^L, FP_p^L(\cdot)^\sim\right]$ be a fuzzy Lipschitz ideal.

Proof. To prove that the conditions of Definition 14. First to show that Condition (I_0). Let $f \in \mathfrak{B}_{FX^\#}$ and $e \in F$ we get

$$\left[\sum_{j=1}^{m} \left\|f \boxdot e(x_j) - f \boxdot e(y_j)\right\|_\sim^p\right]^{\frac{1}{p}} = \left[\sum_{j=1}^{m} \left\|f(x_j)\cdot e - f(y_j)\cdot e\right\|_\sim^p\right]^{\frac{1}{p}} = \|e\|_\sim \odot \left[\sum_{j=1}^{m} \left|f(x_j) - f(y_j)\right|_\sim^p\right]^{\frac{1}{p}}$$

$$\preceq \|e\|_\sim \odot \sup_{FLip(f)\le\tilde{1}} \left[\sum_{j=1}^{m} \left|fx_j - fy_j\right|_\sim^p\right]^{\frac{1}{p}}. \tag{15}$$

with $FP_p^L(fe)^\sim \preceq \|e\|_\sim$. To prove that Condition (I_1). Let $S \in F\mathfrak{P}_p^L(X,Y)$ and $A \in FLip(Y,Z)$ we have

$$\left[\sum_{j=1}^{m} \tilde{d}_Z(ASx_j,ASy_j)^p\right]^{\frac{1}{p}} \preceq FLip(A) \odot \left[\sum_{j=1}^{m} \tilde{d}_Y(Sx_j,Sy_j)^p\right]^{\frac{1}{p}}$$

$$\preceq FLip(A) \odot FP_p^{L,\sim}(S) \odot \sup_{FLip(f)\le\tilde{1}} \left[\sum_{j=1}^{m} \left|fx_j - fy_j\right|_\sim^p\right]^{\frac{1}{p}}$$

with $FP_p^L(AS)^\sim \preceq FLip(A)^\sim \odot FP_p^L(S)^\sim$.

Theorem 20 Let $1 < p < \infty$. A fuzzy Lipschitz function $S \in F\mathfrak{P}_p^L(X,Y)$ if and only if there exist a fuzzy real number $\eta \in \mathcal{F}^+$ and a regular probability measure ν defined on $\mathfrak{B}_{FX^\#}$ such that

$$\tilde{d}_Y(Sx,Sy) \preceq \eta \odot \left(\int_{\mathfrak{B}_{FX^\#}} |fx - fy|_\sim^p d\nu(f)\right)^{\frac{1}{p}},$$

$\forall$ x and $y \in X$.

Proof. Let $x_1, \ldots, x_m$, $y_1, \ldots, y_m$ in X, $m \in \mathbb{N}$, and $\alpha \in (0,1]$,

$$\sum_{j=1}^{m} d_Y(Sx_j,Sy_j)_\alpha^{p,-} \le \eta_\alpha^{p,-} \cdot \sum_{j=1}^{m} \int_{\mathfrak{B}_{FX^\#}} \left|fx_j - fy_j\right|_\alpha^{p,-} d\nu(f)$$

$$\le \eta_\alpha^{p,-} \cdot \int_{\mathfrak{B}_{FX^\#}} \sum_{j=1}^{m} \left|fx_j - fy_j\right|_\alpha^{p,-} d\nu(f)$$

$$\le \eta_\alpha^{p,-} \cdot \sup_{FLip(f)\le\tilde{1}} \sum_{j=1}^{m} \left|fx_j - fy_j\right|_\alpha^{p,-}. \tag{16}$$

In the same way of Inequality (16), we have

$$\sum_{j=1}^{m} d_Y(Sx_j, Sy_j)_\alpha^{p,+} \leq \eta_\alpha^{p,+} \cdot \sup_{FLip(f)\leq\tilde{1}} \sum_{j=1}^{m} \left|fx_j - fy_j\right|_\alpha^{p,+}. \tag{17}$$

From Inequalities (16) and (17), we obtain $S \in F\mathfrak{P}_p^L(X, Y)$ with $FP_p^L(S)^\sim \preceq \eta$. Conversely, suppose that $S \in F\mathfrak{P}^L(X, Y)$. Each finite subset Ω of $X \times X$ we define a map ι_Ω on $\mathfrak{B}_{FX^\#}$ as follows

$$\iota_\Omega(f)^\sim := \sum_{(x,y)\in\Omega} \left(FP_p^L(S)^{p,\sim} \odot |f(x) - f(y)|_\sim^p \ominus \tilde{d}_Y(Sx, Sy)^p\right).$$

The fuzzy number $\iota_\Omega(f)^\sim$ is defined as follows $[\iota_\Omega(f)^\sim]_\alpha = [\iota_\Omega(f)_\alpha^-, \iota_\Omega(f)_\alpha^+]$, where

$$\iota_\Omega(f)_\alpha^- := \sum_{(x,y)\in\Omega} \left(FP_p^L(S)_\alpha^{p,-} \cdot |f(x) - f(y)|_\alpha^{p,-} - d_Y(Sx, Sy)_\alpha^{p,+}\right), \tag{18}$$

and

$$\iota_\Omega(f)_\alpha^+ := \sum_{(x,y)\in\Omega} \left(FP_p^L(S)_\alpha^{p,+} \cdot |f(x) - f(y)|_\alpha^{p,+} - d_Y(Sx, Sy)_\alpha^{p,-}\right). \tag{19}$$

Since for every $(x, y) \in \Omega$ the functions $R_{(x,y)}: \mathfrak{B}_{FX^\#} \longrightarrow \mathbb{R}$, $R_{(x,y)}(f) := |f(x) - f(y)|^p$, are continuous on $\mathfrak{B}_{FX^\#}$, obviously the maps $\iota_\Omega(\cdot)_\alpha^-$ and $\iota_\Omega(\cdot)_\alpha^+$ defined in (18) and (19) belong to $C(\mathfrak{B}_{FX^\#})$, respectively. Since $S \in F\mathfrak{P}_p^L(X, Y)$ and $\iota_\Omega(f)_\alpha^- \leq \iota_\Omega(f)_\alpha^+$ hence $\sup_{\|f\|^\sim\leq\tilde{1}} \iota_\Omega(f)_\alpha^- \geq 0$ and $\sup_{\|f\|^\sim\leq\tilde{1}} \iota_\Omega(f)_\alpha^+ \geq 0$. Note that $B_\alpha^- := \left\{\iota_{\Omega,\alpha}^-: \Omega \subset X \times X\right\}$ and $B_\alpha^+ := \left\{\iota_{\Omega,\alpha}^+: \Omega \subset X \times X\right\}$ be the convex subsets of $\mathfrak{B}_{FX^\#}$ for every $\alpha \in (0,1]$. Consider the open convex subset $F := \left\{\iota \in C(\mathfrak{B}_{FX^\#}): \sup_{FLip(f)\leq\tilde{1}} \iota(f) < 0\right\}$ of $C(\mathfrak{B}_{FX^\#})$. Since $F \cap \varrho_\alpha^- = \phi$ and $F \cap_\alpha^+ = \phi$ for every $\alpha \in (0,1]$, we have

$$\langle \nu, \iota \rangle < r_1 \leq \langle \nu, \iota\iota_{\Omega,\alpha}^- \rangle, \forall\ (\iota, \iota_{\Omega,\alpha}^-) \in F \times \varrho_\alpha^-, \tag{20}$$

and

$$\langle \nu, \iota \rangle < r_2 \leq \langle \nu, \iota_{\Omega,\alpha}^+ \rangle, \forall\ (\iota, \iota_{\Omega,\alpha}^+) \in F \times \varrho_\alpha^+. \tag{21}$$

Then

$$0 \leq \langle \nu, \iota_{\{x,y\},\alpha}^- \rangle = \int_{\mathfrak{B}_{FX^\#}} \left(FP_p^L(S)_\alpha^{p,-} \cdot |f(x) - f(y)|_\alpha^{p,-} - d_Y(Sx, Sy)_\alpha^{p,+}\right) d\nu(f), \forall\ x, y \in X.$$

Since $d_Y(Sx, Sy)_\alpha^{p,-} \leq d_Y(Sx, Sy)_\alpha^{p,+}, \forall\ \alpha \in (0,1]$ we obtain

$$d_Y(Sx, Sy)_\alpha^{p,-} \leq FP_p^L(S)_\alpha^{p,-} \cdot \int_{\mathfrak{B}_{FX^\#}} |f(x) - f(y)|_\alpha^{p,-} d\nu(f), \forall\ x, y \in X. \tag{22}$$

Also from (21) we have

$$0 \leq \langle \nu, \iota_{\{x,y\},\alpha}^+ \rangle = \int_{\mathfrak{B}_{FX^\#}} \left(FP_p^L(S)_\alpha^{p,+} \cdot |f(x) - f(y)|_\alpha^{p,+} - d_Y(Sx, Sy)_\alpha^{p,-}\right) d\nu(f), \forall\ x, y \in X.$$

Since $FP_p^L(S)_\alpha^{p,-} \leq FP_p^L(S)_\alpha^{p,+}, \forall\ \alpha \in (0,1]$ we obtain

$$d_Y(Sx, Sy)_\alpha^{p,+} \leq FP_p^L(S)_\alpha^{p,+} \cdot \int_{\mathfrak{B}_{FX^\#}} |f(x) - f(y)|_\alpha^{p,+} d\nu(f), \forall\ x, y \in X. \tag{23}$$

From Inequalities (22) and (23), we get

$$\tilde{d}_Y(Sx, Sy)^p \preceq FP_p^L(S)^{p,\sim} \odot \int_{\mathfrak{B}_{FX^\#}} |fx - fy|_\sim^p d\nu(f), \forall\ x, y \in X.$$

From Theorem 20 the following result is satisfied.

Proposition 21 If $p_1 \leq p_2$, then $\left[F\mathfrak{P}_{p_1}^L, FP_{p_1}^L(\cdot)^\sim\right] \subseteq \left[F\mathfrak{P}_{p_2}^L, FP_{p_2}^L(\cdot)^\sim\right]$.

5 Open Problems

1. Let S be a fuzzy bounded linear operator from X into Y and $(1 < p < \infty)$. Does the equality $FP_p^L(S)^\sim = FP_p(S)^\sim$ correct ?

2. Does the composition formula $FP_a^L(T \circ S)^\sim \preceq FP_b^L(T)^\sim \odot FP_c^L(S)^\sim$ correct for arbitrary absolutely fuzzy Lipschitz a-summing operators T, absolutely fuzzy Lipschitz b-summing operators S and $\frac{1}{a} \leq (\frac{1}{b} + \frac{1}{c}) \wedge 1$?

3. What is the dual of $F\mathfrak{P}_p^L(X, F)$, whenever F is a complete fuzzy normed space and X is finite fuzzy

pointed metric space ?

4. Find an algorithm to calculate the absolutely fuzzy Lipschitz p-summing norm of fuzzy Lipschitz maps between finite fuzzy pointed metric spaces exactly.

5. What findings about Lipschitz (crisp) ideals have analogs for fuzzy Lipschitz ideals ?

Conclusions

Although the systematic emergence of the theory of fuzzy functional analysis has begun in the past few years, we have begun to build a new theory of fuzzy Lipschitz ideals between pointed fuzzy metric spaces. The basic idea of the paper is to connecting both the fuzzy Lipschitz maps and the fuzzy Lipschitz theories, we study the class of absolutely fuzzy Lipschitz p-summing maps between arbitrary pointed fuzzy metric spaces. We pay attention to the fuzzyness of nonlinear domination theorem whose proof uses the abstract fuzzy version of the Pietsch domination theorem. The fruitful development of the theory of absolute summability for fuzzy Lipschitz operators will produce several generalizations to the nonlinear context. This is the case of fuzzy Lipschitz ideals, which will quickly attract the interest of many researchers trying to derive a parallel theory to the fuzzy Lipschitz one such as fuzzy Lipschitz p-integral operators, fuzzy Lipschitz p-nuclear operators, duality for fuzzy Lipschitz p-summing operators and two-fuzzy Lipschitz ideals.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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