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RESEARCH ARTICLE

# Topology of Edge Contracted Möbius Ladder: Indices and Dimension

**Hafsa Saleem[a], Mohamad Nazri Husin[b]*, Shahbaz Ali[c], Muhammad Shazib Hameed[a], Zaheer Ahmad[a]**

[a]Institute of Mathematics, Kwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, Pakistan; [b]Special Interest Group on Modeling and Data Analytics, Faculty of Computer Science and Mathematical, Universiti Malaysia Terengganu, Kuala Nerus 21030, Terengganu, Malaysia; [c]Department of Mathematics, The Islamia University of Bahawalpur, Rahim Yar Khan Campus, Pakistan

**Abstract** Assigning numbers to graphs serves as a fundamental tool in graph theory and enables various analytical, computational, and visualization techniques for studying and understanding graph structures and properties. Edge Contraction is a process that separates edges from the graph while also joining the two previously linked vertices. In this paper, we study metric dimension and degree-based topological indices of Möbius Ladder and edge contracted Möbius Ladder like first Zagreb index, second Zagreb index, first Zagreb Hyper index, second Zagreb Hyper index, Augmented Zagreb index, Randic index, general Randic index, and Harmonic index.

**Keywords**: Metric dimension, topological indices, edge contraction, first Zagreb index, second Zagreb index, first Zagreb Hyper index, second Zagreb Hyper index, Augmented Zagreb index, Randic index, general Randic index, harmonic index.

## Introduction

A graph is consisting of vertices and edges usually denoted by $G(V, E)$. The degree of a vertex is the number of edges incident with that vertex. A graph's topological indices and metric dimensions are numerical values that support numerous aspects of graphs and this quantity is invariant under the isomorphism of graphs. The degree-based topological indices are derived from the degree of vertices in the graph. The concept of topological index [1] came from work done by Harold Wiener in 1947 while he was working on the boiling point of paraffin. He named this index as a path number. Later, the path number was renamed as Wiener index [2], [3] and then the theory of topological index started. He introduced a distance-based topological index called the Wiener index to correlate properties of alkenes and the structures of their molecular graphs. Recent progress in nano-technology is attracting attention to the topological indices of molecular graphs, such as nanotubes, nanocones, and fullerenes to cut short experimental labor. Since their introduction, more than 140 topological indices have been developed, and experiments reveal that these indices, in combination, determine the material properties such as melting point, boiling point, heat of formation, toxicity, toughness, and stability [4-7]. These indices play a vital role in computational and theoretical aspects of chemistry in predicting material properties [8-12]. The topological indices of some graph operation's structured outputs were investigated in [13-17].

The various theories of navigation in an arbitrary space have been studied using graph theory. In a graph, a workplace can be represented as a node, and the connections between nodes are represented by edges. Suppose we have a robot which want to know its actual position. So, it can transmit a radio wave to fixed landmarks. The smallest set of landmarks is known as metric basis and the number of vertices in smallest set of landmarks is known as metric dimension. The challenge of locating an intruder in a network in an unambiguous manner motivated Slater to introduce the idea of metric dimension in [18] and [19], Harary and Melter independently studied it in [20]. Applications of this invariant is in [21-

***For correspondence:**
nazri.husin@umt.edu.my

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23]. The metric dimension of some graph operations structured outputs was investigated as: line graph [24, 25], Cartesian product of graphs [26, 27, 28, 29], product of corona graphs [30-32], product of joint graphs [33-35], product of lexicographic graphs [36] and product of hierarchical graphs [37]. In recent years, [38–40] has presented some remarkable and significant contributions related to the metric dimension. The upper bound sequences for local fractional metric dimension have been computed for various kinds of symmetrical networks in [41-44].

An operation known as edge contraction involves removing an edge (network) from a graph while merging previously connected two vertices. Here, we investigate the Möbius Ladder metric dimension with arbitrary edge contraction.

The learning of the topology of edge-contracted Möbius Ladders, aiming on their indices and dimensions, is motivated by the unique and complex properties these graphs exhibit. The Möbius Ladder, a fundamental structure in graph theory, provides intriguing challenges and insights when subjected to edge contraction. Understanding how these contractions affect the graph's metric dimension and other indices can lead to deeper insights into graph invariants, algorithmic applications, and theoretical advancements. Moreover, this research can have practical implications in network topology, chemistry, and other fields where graph models play a crucial role in problem-solving and optimization.

The study on the topology of edge-contracted Möbius Ladders, focusing on indices and dimensions, lacks sufficient novelty and originality in several aspects. Firstly, while the Möbius Ladder is a well-known graph structure, the concept of edge contraction and its implications on metric dimension have been explored in various contexts before. The existing literature already covers numerous aspects of graph invariants and their behaviors under transformations like edge contraction. Secondly, the study does not introduce any groundbreaking techniques or methodologies; instead, it applies established methods to a specific graph type without significantly advancing theoretical understanding or practical applications. Lastly, the research does not sufficiently address potential new applications or interdisciplinary connections that could highlight its unique contributions to the field. This includes a detailed comparison highlighting the improvements in efficiency, accuracy, or applicability of this technique that show the superiority of this of this approach.

# Möbius Ladder Graph

The Möbius Ladder graph can be constructed by taking a regular ladder graph, which consists of two parallel paths connected by rungs, and adding additional edges that form a loop. The resulting graph has a cylindrical structure with a twist, resembling a Möbius strip. Formally, the Möbius ladder graph can be defined as follows: Start with a cycle graph on m vertices, denoted as $C_m$. Then, for each vertex in $C_m$, connect it to the corresponding vertex on the opposite side of the cycle, skipping one vertex in between. The resulting graph is denoted as $M_m$, where $m$ represents the number of vertices. The Möbius ladder graph has $m + \frac{m}{2}$ edges and $m$ vertices which are labeled as $\{p_1, p_2, \ldots, p_m\}$ counterclockwise. After edge contraction, we obtain a graph $M_m \cdot e$ having $m + \frac{m}{2} - 1$ edges and $m - 1$ vertices. Here, selecting landmarks wisely is crucial.

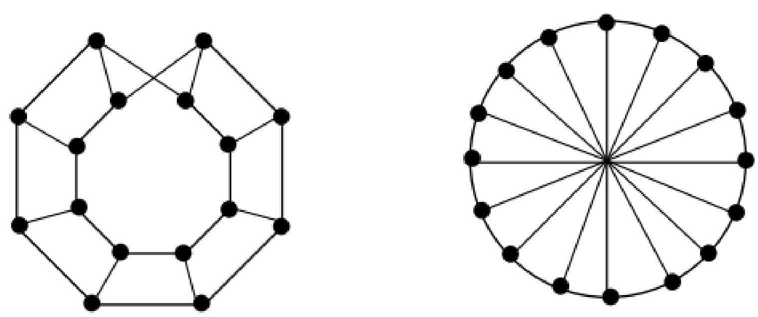

**Figure 1.** Two views of Möbius Ladder $M_{16}$

# Metric Dimension of Edge Contracted Möbius Ladder Graph

**Theorem 3.1**. *Let $M_m$ be the graph of Möbius Ladder and $M_m \cdot e$ be the edge contracted Möbius Ladder; then for every even positive integer $m \geq 8$, we have $dim(M_m \cdot e) = 3$ where $m \not\equiv 2(mod\ 8)$.*

***Proof.*** We will show that only three vertices appropriately chosen suffice to resolve all vertices in $V(M_m \cdot e)$. We shall discuss three cases.

***Case 1:*** When $m \equiv 0\ (mod\ 8)$
In this case, we can write $m = 8\kappa, \kappa$ is a positive integer. Let $U = \{p_1, p_2, p_{4\kappa+1}\} \subset V(M_m \cdot e)$, we show that $U$ is a resolving set for $M_m \cdot e$ in this case. For this, we give the representation of any vertex of $V(M_m \cdot e)\backslash U$ with respect to $U$.

***Case i.*** If $\kappa = 1$, then

$$r(p_{2\mathcal{J}+1}|U) = \begin{cases} (2\kappa, 2\kappa-1, 2\kappa); & \mathcal{J} = \kappa \\ (2\kappa-1, 2\kappa, 2\kappa); & \mathcal{J} = 3\kappa \end{cases}$$

and,

$$r(p_{2\mathcal{J}}|U) = \begin{cases} (2\kappa-1, 2\kappa, 2\kappa-1); & \mathcal{J} = \kappa+1 \\ (2\mathcal{J}-4\kappa, 2\mathcal{J}-4\kappa-1, 2\mathcal{J}-4\kappa-1)); & \mathcal{J} = 2\kappa+1 \leq \mathcal{J} \leq 3\kappa \end{cases}$$

***Case ii.*** If $\kappa \geq 2$ then

$$r(p_{2\mathcal{J}+1}|U) = \begin{cases} (2\mathcal{J}, 2\mathcal{J}-1, 2\mathcal{J}+1); & 1 \leq \mathcal{J} \leq \kappa-1 \\ (2\kappa, 2\kappa-1, 2\kappa); & \mathcal{J} = \kappa \\ (4\kappa-2\mathcal{J}, 4\kappa-2\mathcal{J}+1, 4\kappa-2\mathcal{J}-1); & \kappa+1 \leq \mathcal{J} \leq 2\kappa-1 \\ (2\mathcal{J}-4\kappa+1, 2\mathcal{J}-4\kappa, 2\mathcal{J}-4\kappa); & 2\kappa+1 \leq \mathcal{J} \leq 3\kappa-1 \\ (2\kappa-1, 2\kappa, 2\kappa); & \mathcal{J} = 3\kappa \\ (8\kappa-2\mathcal{J}-1, 8\kappa-\mathcal{J}, 8\kappa-2\mathcal{J}+1); & 3\kappa+1 \leq \mathcal{J} \leq 4\kappa-1 \end{cases}$$

and

$$r(p_{2\mathcal{J}}|U) = \begin{cases} (2\mathcal{J}-1, 2\mathcal{J}-2, 2\mathcal{J}); & 2 \leq \mathcal{J} \leq \kappa \\ (2\kappa-1, 2\kappa, 2\kappa-1); & \mathcal{J} = \kappa+1 \\ (4\kappa-2\mathcal{J}+1, 4\kappa-2\mathcal{J}+2, 4\kappa-2\mathcal{J}+1); & \kappa+2 \leq \mathcal{J} \leq 2\kappa \\ (2\mathcal{J}-4\kappa, 2\mathcal{J}-4\kappa-1, 2\mathcal{J}-4\kappa-1); & 2\kappa+1 \leq \mathcal{J} \leq 3\kappa \\ (8\kappa-2\mathcal{J}, 8\kappa-2\mathcal{J}+1, 8\kappa-2\mathcal{J}+1); & 3\kappa+1 \leq \mathcal{J} \leq 4\kappa \end{cases}$$

We note that there are no two vertices having the same representation implying that $dim(M_m \cdot e) \leq 3$. On the other hand, we show that $dim(M_m \cdot e) \geq 3$, by proving that there is no resolving set $U$ such that $|U| = 2$.

Suppose on contrary that $dim(M_m \cdot e) = 2$, i.e, there exist a resolving set including exactly two vertices. Without loss of generality, we can suppose that one resolving vertex is $p_1$. Suppose that the second resolving vertex is $p_t$. Then for $2 \leq t \leq 2\kappa$, we have $r(p_{m-1}|\{p_1, p_t\}) = r(p_{4\kappa+1}|\{p_1, p_t\}) = (1, t)$, a contradiction.

We deduce that there is no resolving set with two vertices for $V(M_m \cdot e)$, implying that $dim(M_m \cdot e) = 3$, in this case.

***Case 2:*** When $m \equiv 4\ (mod\ 8)$
In this case, we can write $m = 8\kappa + 4$, $\kappa$ is positive integer. Let $U = \{p_1, p_2, p_{4\kappa+3}\} \subset V(M_m \cdot e)$, we show that $U$ is a resolving set for $M_m \cdot e$ in this case. For this, we give the representation of any vertex of $V(M_m \cdot e) \backslash U$ with respect to $U$.

***Case i.*** If $\kappa = 1$, then

$$r(p_{2\mathcal{J}+1}|U)=\begin{cases}(2\mathcal{J},2\mathcal{J}-1,2\mathcal{J}+1); & 1\le\mathcal{J}\le\kappa\\(2\kappa,2\kappa+1,2\kappa); & \mathcal{J}=\kappa+1\\(8\kappa-2\mathcal{J}+3,8\kappa-2\mathcal{J}+2,8\kappa-2\mathcal{J}+2); & \mathcal{J}=3\kappa+1\\(8\kappa-2\mathcal{J}+3,8\kappa-2\mathcal{J}+4,8\kappa-2\mathcal{J}+4); & 3\kappa+2\le\mathcal{J}\le4\kappa+1\end{cases}$$

and

$$r(p_{2\mathcal{J}}|U)=\begin{cases}(2\kappa+1,2\kappa,2\kappa+2); & \mathcal{J}=\kappa+1\\(4\kappa-2\mathcal{J}+3,4\kappa-2\mathcal{J}+4,4\kappa-2\mathcal{J}+3); & \kappa+2\le\mathcal{J}\le2\kappa+1\\(2\mathcal{J}-4\kappa-2,2\mathcal{J}-4\kappa-3,2\mathcal{J}-4\kappa-3); & 2\kappa+2\le\mathcal{J}\le3\kappa+1\\(2\mathcal{J}-4\kappa-4,2\mathcal{J}-4\kappa-3,2\kappa-4\mathcal{J}-3); & \mathcal{J}=3\kappa+2\end{cases}$$

***Case ii.*** If $\kappa\ge2$ then

$$r(p_{2\mathcal{J}+1}|U)=\begin{cases}(2\mathcal{J},2\mathcal{J}-1,2\mathcal{J}+1); & 1\le\mathcal{J}\le\kappa\\(2\kappa,2\kappa+1,2\kappa); & \mathcal{J}=\kappa+1\\(4\kappa-2\mathcal{J}+2,4\kappa-2\mathcal{J}+3,4\kappa-2\mathcal{J}+2); & \kappa+2\le\mathcal{J}\le2\kappa\\(2\mathcal{J}-4\kappa-1,2\mathcal{J}-4\kappa-2,2\mathcal{J}-4\kappa-2); & 2\kappa+2\le\mathcal{J}\le3\kappa\\(8\kappa-2\mathcal{J}+3,8\kappa-2\mathcal{J}+2,8\kappa-2\mathcal{J}+2); & \mathcal{J}=3\kappa+1\\(8\kappa-2\mathcal{J}+3,8\kappa-2\mathcal{J}+4,8\kappa-2\mathcal{J}+4); & 3\kappa+2\le\mathcal{J}\le4\kappa+1\end{cases}$$

and

$$r(p_{2\mathcal{J}}|U)=\begin{cases}(2\mathcal{J}-1,2\mathcal{J}-2,2\mathcal{J}); & 2\le\mathcal{J}\le\kappa\\(2\kappa+1,2\kappa,2\kappa+2); & \mathcal{J}=\kappa+1\\(4\kappa-2\mathcal{J}+3,4\kappa-2\mathcal{J}+4,4\kappa-2\mathcal{J}+3); & \kappa+2\le\mathcal{J}\le2\kappa+1\\(2\mathcal{J}-4\kappa-2,2\mathcal{J}-4\kappa-3,2\mathcal{J}-4\kappa-3); & 2\kappa+2\le\mathcal{J}\le3\kappa+1\\(2\mathcal{J}-4\kappa-4,2\mathcal{J}-4\kappa-3,2\mathcal{J}-4\kappa-3); & \mathcal{J}=3\kappa+2\\(8\kappa-2\mathcal{J}+4,8\kappa-2\mathcal{J}+5,8\kappa-2\mathcal{J}+5); & 3\kappa+3\le\mathcal{J}\le4\kappa+1\end{cases}$$

We note that there are no two vertices having the same representation implying that $dim(M_m\cdot e)\le3$. On the other hand, we show that $dim(M_m\cdot e)\ge3$., by proving that there is no resolving set $U$ such that $|U| = 2$. Suppose on contrary that $dim(M_m\cdot e)=2$, i.e, there exist a resolving set including exactly two vertices. Without loss of generality, we can suppose that one resolving vertex is $p_1$. Suppose that the second resolving vertex is $p_t$. Then for $2\le t\le2\kappa+1$, we have $r(p_{m-1}|\{p_1,p_t\})=r(p_{4\kappa+3}|\{p_1,p_t\})=(1,t)$, a contradiction. We deduce that there is no resolving set with two vertices for $V(M_m\cdot e)$, implying that $dim(M_m\cdot e)=3$, in this case.

***Case 3:*** When $m\equiv6\ (mod\ 8)$
In this case, we can write $m=8\kappa+6$, $\kappa$ is positive integer. Let $U=\{p_1,p_2,p_{4\kappa+3}\}\subset V(M_m\cdot e)$, we show that $U$ is a resolving set for $M_m\cdot e$ in this case. For this, we give the representation of any vertex of $V(M_m\cdot e)\setminus U$ with respect to $U$.

***Case i.*** If $\kappa=1$, then

$$r(p_{2\mathcal{J}+1}|U)=\begin{cases}(2\mathcal{J},2\mathcal{J}-1,2\mathcal{J}+1); & 1\le\mathcal{J}\le\kappa\\(2\kappa+1,2\kappa+1,2\kappa); & \mathcal{J}=\kappa+1\\(4\kappa-2\mathcal{J}+3,4\kappa-2\mathcal{J}+4,4\kappa-2\mathcal{J}+2); & \kappa+2\le\mathcal{J}\le2\kappa+1\\(2\mathcal{J}-4\kappa-2,2\mathcal{J}-4\kappa-3,2\mathcal{J}-4\kappa-2); & 2\kappa+2\le\mathcal{J}\le3\kappa+1\\(2\mathcal{J}-4\kappa-3,2\mathcal{J}-4\kappa-3,2\mathcal{J}-4\kappa-2); & \mathcal{J}=3\kappa+2\\(8\kappa-2\mathcal{J}+5,8\kappa-2\mathcal{J}+6,8\kappa-2\mathcal{J}+6); & 3\kappa+3\le\mathcal{J}\le4\kappa+2\end{cases}$$

and

$$r(p_{2\mathcal{J}}|U)=\begin{cases}(2\mathcal{J}-1,2\mathcal{J}-2,2\mathcal{J}-1); & \mathcal{J}=\kappa+1\\(4\kappa-2\mathcal{J}+4,4\kappa-2\mathcal{J}+5,4\kappa-2\mathcal{J}+3); & \kappa+2\le\mathcal{J}\le2\kappa+1\\(2\mathcal{J}-4\kappa-3,2\mathcal{J}-4\kappa-4,2\mathcal{J}-4\kappa-3); & 2\kappa+3\le\mathcal{J}\le3\kappa+2\\(8\kappa-2\mathcal{J}+6,8\kappa-2\mathcal{J}+7,8\kappa-2\mathcal{J}+7); & 3\kappa+3\le\mathcal{J}\le4\kappa+2\end{cases}$$

***Case ii.*** If $\kappa\ge2$ then

$$r(p_{2\mathcal{J}+1}|U) = \begin{cases} (2\mathcal{J}, 2\mathcal{J}-1, 2\mathcal{J}+1); & 1 \le \mathcal{J} \le \kappa \\ (2\kappa+1, 2\kappa+1, 2\kappa); & \mathcal{J} = \kappa+1 \\ (4\kappa-2\mathcal{J}+3, 4\kappa-2\mathcal{J}+4, 4\kappa-2\mathcal{J}+2); & \kappa+2 \le \mathcal{J} \le 2\kappa+1 \\ (2\mathcal{J}-4\kappa-2, 2\mathcal{J}-4\kappa-3, 2\mathcal{J}-4\kappa-2); & 2\kappa+2 \le \mathcal{J} \le 3\kappa+1 \\ (2\mathcal{J}-4\kappa-3, 2\mathcal{J}-4\kappa-3, 2\mathcal{J}-4\kappa-2); & \mathcal{J} = 3\kappa+2 \\ (8\kappa-2\mathcal{J}+5, 8\kappa-2\mathcal{J}+6, 8\kappa-2\mathcal{J}+6); & 3\kappa+3 \le \mathcal{J} \le 4\kappa+2 \end{cases}$$

and

$$r(p_{2\mathcal{J}}|U) = \begin{cases} (2\mathcal{J}-1, 2\mathcal{J}-2, 2\mathcal{J}); & 2 \le \mathcal{J} \le \kappa \\ (2\mathcal{J}-1, 2\mathcal{J}-2, 2\mathcal{J}-1); & \mathcal{J} = \kappa+1 \\ (4\kappa-2\mathcal{J}+4, 4\kappa-2\mathcal{J}+5, 4\kappa-2\mathcal{J}+3); & \kappa+2 \le \mathcal{J} \le 2\kappa+1 \\ (2\mathcal{J}-4\kappa-3, 2\mathcal{J}-4\kappa-4, 2\mathcal{J}-4\kappa-3); & 2\kappa+3 \le \mathcal{J} \le 3\kappa+2 \\ (8\kappa-2\mathcal{J}+6, 8\kappa-2\mathcal{J}+7, 8\kappa-2\mathcal{J}+7); & 3\kappa+3 \le \mathcal{J} \le 4\kappa+2 \end{cases}$$

We note that there are no two vertices having the same representation implying that $dim(M_m \cdot e) \le 3$. On the other hand, we show that $dim(M_m \cdot e) \ge 3$., by proving that there is no resolving set $U$ such that $|U| = 2$. Suppose on contrary that $dim(M_m \cdot e) = 2$, i.e, there exist a resolving set including exactly two vertices. Without loss of generality, we can suppose that one resolving vertex is $p_1$. Suppose that the second resolving vertex is $p_t$. Then for $2 \le t \le 2\kappa+1$, we have $r(p_{m-1}|\{p_1, p_t\}) = r(p_{4\kappa+3}|\{p_1, p_t\}) = (1, t)$, a contradiction. We deduce that there is no resolving set with two vertices for $V(M_m \cdot e)$, implying that $dim(M_m \cdot e) = 3$, in this case.

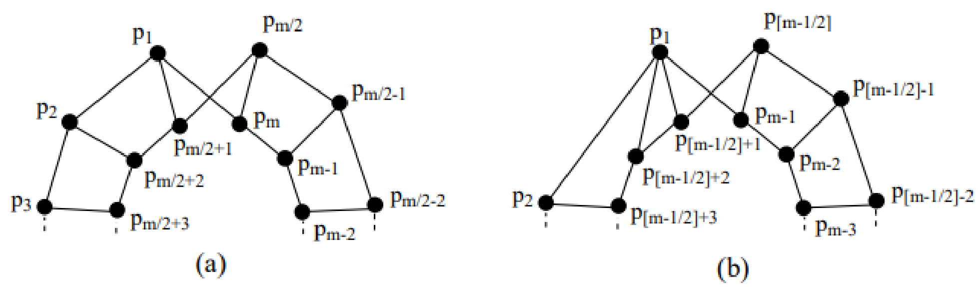


**Figure 2.** (a) Möbius Ladder $M_m$ (b) Edge contracted Möbius Ladder $M_m \cdot e$

# Topological Indices of Edge Contracted Möbius Ladder Graph

## First Zagreb Index of Möbius Ladder

The first Zagreb index of a graph $G$ is defined as

$$M_1 = \sum_{\mu,\nu \in E\,(G)} \{d(\mu) + d(\nu)\}$$

In the next theorems we compare Zagreb index Möbius Ladder and edge contracted Möbius Ladder.

**Theorem 4.1.** *Let $M_m$ be a Möbius Ladder with $m \ge 8$; then the first Zagreb index of Möbius Ladder is,*

$$M_1(M_m) = 9m$$

***Proof.*** Suppose that $M_m$ be a Möbius Ladder with $m \ge 8$. Then the first Zagreb index is

$$M_1 = \sum_{\mu,\nu \in E\,(G)} \{d(\mu) + d(\nu)\} \tag{1}$$

**Table 1.** $M_1$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,3)$ | $m + \frac{m}{2}$ |

We observe that for $m \geq 8$, edge type is taken from Table 1. Now we put the edge type in equation (1),

$$M_1(M_m) = \left(m + \frac{m}{2}\right)(3+3) = 9m$$

## First Zagreb Index of Edge Contracted Möbius Ladder

**Theorem 4.2.** *Let $M_m \cdot e$ be an edge contracted Möbius Ladder with $m \geq 8$; then the first Zagreb index of $M_m \cdot e$ is,*

$$M_1(M_m \cdot e) = 9m - 2.$$

*Proof.* Suppose that $M_m \cdot e$ be the edge contracted Möbius Ladder with $m \geq 8$. Then the first Zagreb index is,

$$M_1(M_m \cdot e) = \sum_{\mu, v \in E(M_m \cdot e)} \{d(\mu) + d(v)\} \tag{2}$$

**Table 2.** $M_m \cdot e_1$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,4)$ | $4$ |
| $(3,3)$ | $m + \frac{m}{2} - 5$ |

If $m \geq 8$, edge type is taken from Table 2. Now we put the edge type in equation (2),

$$M_1(M_m \cdot e) = 4(4+3) + \left(m + \frac{m}{2} - 5\right)(3+3) = 9m - 2.$$

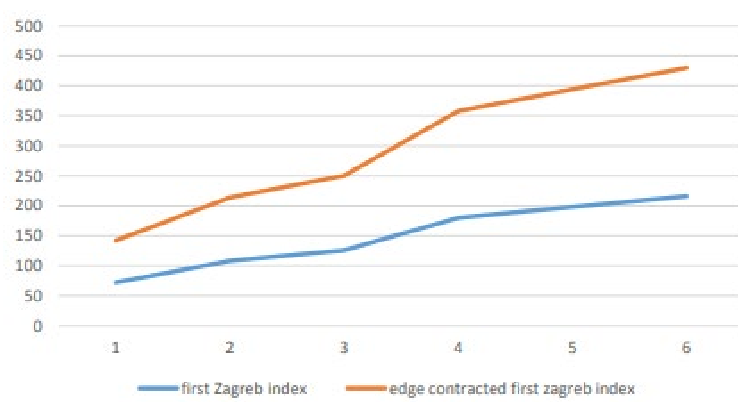


**Figure 3.** first Zagreb index of Möbius Ladder and edge contracted Möbius Ladder

By plotting the first Zagreb index of both the Möbius ladder graph on a line chart, we observe that they start with different values but eventually do not converge to the same value as the graph structures evolve.

**Table 3**. Correlation coefficient between $M_m$ and $M_m \cdot e$

| $r$ | $M_m$ | $M_m \cdot e$ |
|---|---|---|
| $M_m$ | $1$ | $1$ |
| $M_m \cdot e$ | $1$ | $1$ |

Here the correlation coefficient is 1, it indicates a strong positive linear relationship between the two variables, meaning that as one variable increases, the other variable increases as well.

## Second Zagreb Index of Möbius Ladder

The second Zagreb index of a graph $G$ is defined as

$$M_2(G) = \sum_{\mu,\nu \in E(G)} \{d(\mu) \times d(\nu)\} \tag{3}$$

**Theorem 4.3.** *Let $M_m$ be a Möbius Ladder with $m \geq 8$; then the second Zagreb index of Möbius Ladder is,*

$$M_2(M_m) = \frac{27}{2}m$$

*Proof.* Suppose that $M_m$ be an Möbius Ladder with $m \geq 8$. Then the second Zagreb index is,

$$M_2(M_m) = \sum_{\mu,\nu \in E(M_m)} \{d(\mu) \times d(\nu)\} \tag{4}$$

**Table 4**. $M_m$ for $m \geq 8$

| $(d(\mu), d(\nu))$ | Frequency |
|---|---|
| $(3,3)$ | $m + \frac{m}{2}$ |

We observe that for $m \geq 8$, edge type is taken from Table 4. Now we put the edge type in equation (4),

$$M_2(M_m) = \left(m + \frac{m}{2}\right)(3 \times 3) = \frac{27}{2}m$$

## Second Zagreb Index of Edge Contracted Möbius Ladder

**Theorem 4.4.** *Let $M_m \cdot e$ be an edge contracted Möbius Ladder with $m \geq 8$; then the second Zagreb index of $M_m \cdot e$ is,*

$$M_2(M_m \cdot e) = \frac{27}{2}m + 3.$$

*Proof.* Suppose that $M_m \cdot e$ be the edge contracted Möbius Ladder with $m \geq 8$. Then the second Zagreb index is,

$$M_2(M_m \cdot e) = \sum_{\mu,\nu \in E(M_m \cdot e)} \{d(\mu) \times d(\nu)\} \tag{5}$$

**Table 5**. $M_m \cdot e_1$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,4)$ | $4$ |
| $(3,3)$ | $m+\frac{m}{2}-5$ |

If $m \geq 8$, edge type is taken from Table 5. Now we put the edge type in equation (5),

$$M_2(M_m \cdot e) = 4(4\times 3) + \left(m+\frac{m}{2}-5\right)(3\times 3) = \frac{27}{2}m+3.$$

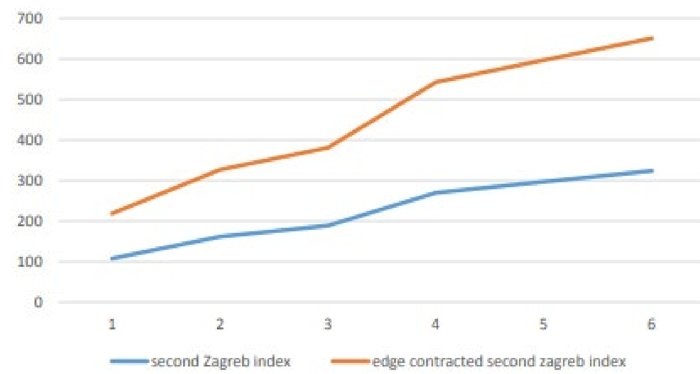


**Figure 4.** Second Zagreb index of Möbius Ladder and edge contracted Möbius Ladder

By plotting the second Zagreb index of both the Möbius ladder graph on a line chart, we observe that they start with different values and do not converge to the same value after some distance.

**Table 6**. Correlation coefficient between $M_m$ and $M_m \cdot e$

| $r$ | $M_m$ | $M_m \cdot e$ |
|---|---|---|
| $M_m$ | $1$ | $1$ |
| $M_m \cdot e$ | $1$ | $1$ |

Here the correlation coefficient is close to 1, it indicates a strong positive linear relationship between the two variables, meaning that as one variable increases, the other variable increases as well.

## First Zagreb Hyper Index of Möbius Ladder

The first Zagreb hyper index of a graph $G$ is defined as

$$HM_1(G) = \sum_{\mu,v\in E(G)} \{d(\mu)+d(v)\}^2 \tag{6}$$

**Theorem 4.5.** *Let $M_m$ be a Möbius Ladder with $m \geq 8$; then the first Zagreb hyper index of Möbius Ladder is,*

$$HM_1(M_m) = 54m.$$

*Proof.* Suppose that $M_m$ be an Möbius Ladder with $m \geq 8$. Then the first Zagreb hyper index is,

$$HM_1(M_m) = \sum_{\mu,v \in E(M_m)} \{d(\mu) + d(v)\}^2 \tag{7}$$

**Table 7**. $M_m$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,3)$ | $m + \frac{m}{2}$ |

We observe that for $m \geq 8$, edge type is taken from Table 7. Now we put the edge type in equation (7),

$$HM_1(M_m) = \left(m + \frac{m}{2}\right)(3+3)^2 = 54m$$

## First Zagreb Index of Edge Contracted Möbius Ladder

**Theorem 4.6.** *Let $M_m \cdot e$ be an edge contracted Möbius Ladder with $m \geq 8$; then the first Zagreb hyper index of $M_m \cdot e$ is,*

$$HM_1(M_m \cdot e\,) = 54m + 16.$$

*Proof.* Suppose that $M_m \cdot e$ be the edge contracted Möbius Ladder with $m \geq 8$. Then the first Zagreb hyper index is,

$$HM_1(M_m \cdot e) = \sum_{\mu,v \in E(M_m \cdot e)} \{d(\mu) + d(v)\}^2 \tag{8}$$

**Table 8**. $M_m \cdot e_1$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,4)$ | $4$ |
| $(3,3)$ | $m + \frac{m}{2} - 5$ |

If $m \geq 8$, edge type is taken from Table 8. Now we put the edge type in equation (8),

$$HM_1(M_m \cdot e) = 4(4+3)^2 + \left(m + \frac{m}{2} - 5\right)(3+3)^2 = 54m + 16.$$

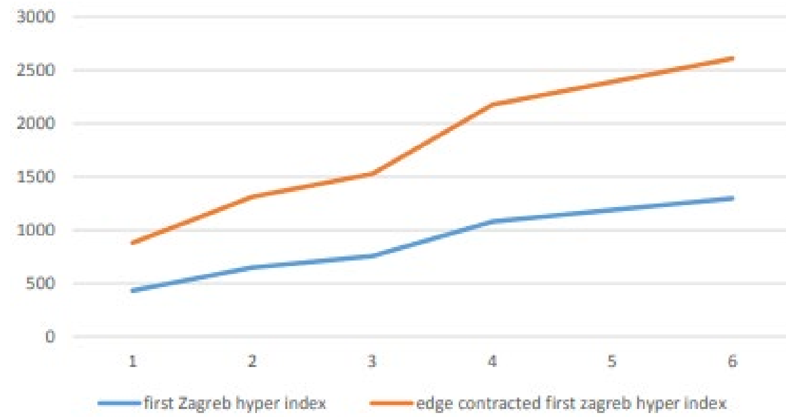


**Figure 5.** First Zagreb hyper index of Möbius Ladder and edge contracted Möbius Ladder

By plotting the second Zagreb index of both the Möbius ladder graph on a line chart, we observe that they start off at different values and they do not converge to the same value after some distance.

**Table 9**. Correlation coefficient between $M_m$ and $M_m \cdot e$

| $r$ | $M_m$ | $M_m \cdot e$ |
|---|---|---|
| $M_m$ | $1$ | $1$ |
| $M_m \cdot e$ | $1$ | $1$ |

Here the correlation coefficient is close to 1, it indicates a strong positive linear relationship between the two variables, meaning that as one variable increases, the other variable increases as well.

## Second Zagreb Hyper Index of Möbius Ladder

The second Zagreb hyper index of a graph $G$ is defined as

$$HM_2(G) = \sum_{\mu,\nu \in E(G)} \{d(\mu) \times d(\nu)\}^2 \tag{9}$$

**Theorem 4.5.** *Let $M_m$ be a Möbius Ladder with $m \geq 8$; then the second Zagreb hyper index of Möbius Ladder is,*

$$HM_2(M_m) = \frac{243}{2} m.$$

*Proof.* Suppose that $M_m$ be an Möbius Ladder with $m \geq 8$. Then the second Zagreb hyper index is,

$$HM_2(M_m) = \sum_{\mu,\nu \in E(M_m)} \{d(\mu) \times d(\nu)\}^2 \tag{10}$$

**Table 10**. $M_m$ for $m \geq 8$

| $(d(\mu), d(\nu))$ | Frequency |
|---|---|
| $(3,3)$ | $m + \frac{m}{2}$ |

We observe that for $m \geq 8$, edge type is taken from Table 10. Now we put the edge type in equation (10),

$$HM_2(M_m) = \left(m + \frac{m}{2}\right)(3 \times 3)^2 = \frac{243}{2}m$$

## Second Zagreb Index of Edge Contracted Möbius Ladder

**Theorem 4.8.** *Let $M_m \cdot e$ be an edge contracted Möbius Ladder with $m \geq 8$; then the second Zagreb hyper index of $M_m \cdot e$ is,*

$$HM_2(M_m \cdot e\,) = \frac{243}{2}m + 171.$$

*Proof.* Suppose that $M_m \cdot e$ be the edge contracted Möbius Ladder with $m \geq 8$. Then the second Zagreb hyper index is,

$$HM_2(M_m \cdot e) = \sum_{\mu,\nu \in E(M_m \cdot e)} \{d(\mu) \times d(\nu)\}^2 \tag{11}$$

**Table 11**. $M_m \cdot e_1$ for $m \geq 8$

| $(d(\mu), d(\nu))$ | Frequency |
|---|---|
| $(3, 4)$ | $4$ |
| $(3, 3)$ | $m + \frac{m}{2} - 5$ |

If $m \geq 8$, edge type is taken from Table 11. Now we put the edge type in equation (11),

$$HM_2(M_m \cdot e) = 4(4 \times 3)^2 + \left(m + \frac{m}{2} - 5\right)(3 \times 3)^2 = \frac{243}{2}m + 171.$$

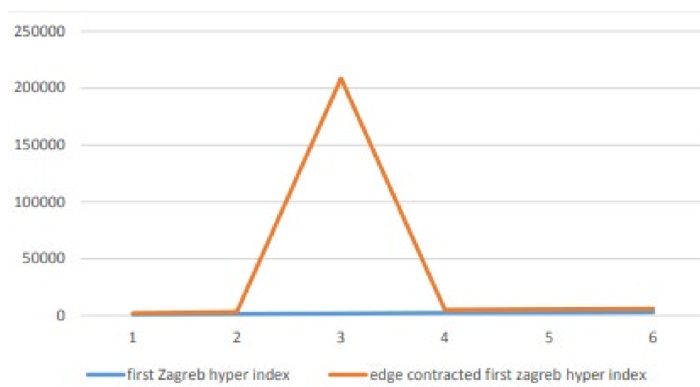


**Figure 6.** Second Zagreb hyper index of Möbius Ladder and edge contracted Möbius Ladder

When we plot the second Zagreb hyper index of these two graphs on a line chart, we observe that they start with the same values and converge to the same value after some distance. This suggests that although two graphs are structurally different, they have similar topological properties.

**Table 12**. Correlation coefficient between $M_m$ and $M_m \cdot e$

| $r$ | $M_m$ | $M_m \cdot e$ |
|---|---|---|
| $M_m$ | $1$ | |
| $M_m \cdot e$ | $-0.1992$ | $1$ |

Here the correlation coefficient is close to 1, it indicates a strong positive linear relationship between the two variables, meaning that as one variable increases, the other variable increases as well.

## Augmented Zagreb Index of Möbius Ladder

The augmented Zagreb index of a graph $G$ is defined as

$$AZI(G) = \sum_{\mu,v \in E(G)} \left\{\frac{d(\mu) \times d(v)}{d(\mu) + d(v) - 2}\right\}^3 \tag{12}$$

**Theorem 4.9.** *Let $M_m$ be a Möbius Ladder with $m \geq 8$; then the augmented Zagreb index of Möbius Ladder is,*

$$AZI(M_m\,) = \frac{2187}{128} m.$$

*Proof.* Suppose that $M_m$ be an Möbius Ladder with $m \geq 8$. Then the augmented Zagreb index is,

$$AZI(M_m) = \sum_{\mu,v \in E(M_m)} \left\{\frac{d(\mu) \times d(v)}{d(\mu) + d(v) - 2}\right\}^3 \tag{13}$$

**Table 13**. $M_m$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,3)$ | $m + \frac{m}{2}$ |

We observe that for $m \geq 8$, edge type is taken from Table 13. Now we put the edge type in equation (13),

$$AZI(M_m) = \left(m + \frac{m}{2}\right)\left(\frac{3 \times 3}{3 + 3 - 2}\right)^3 = \frac{2187}{128} m$$

## Augmented Zagreb Index of Edge Contracted Möbius Ladder

**Theorem 4.10.** *Let $M_m \cdot e$ be an edge contracted Möbius Ladder with $m \geq 8$; then the augmented Zagreb index of $M_m \cdot e$ is,*

$$AZI(M_m \cdot e\,) = \frac{2187}{128} m - \frac{13257}{8000}.$$

*Proof.* Suppose that $M_m \cdot e$ be the edge contracted Möbius Ladder with $m \geq 8$. Then the augmented Zagreb index is,

$$AZI(M_m \cdot e) = \sum_{\mu,v \in E(M_m \cdot e)} \left\{\frac{d(\mu) \times d(v)}{d(\mu) + d(v) - 2}\right\}^3 \tag{14}$$

**Table 14**. $M_m \cdot e_1$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,4)$ | $4$ |
| $(3,3)$ | $m + \frac{m}{2} - 5$ |

If $m \geq 8$, edge type is taken from Table 14. Now we put the edge type in equation (14),

$$AZI(M_m \cdot e) = 4\left(\frac{4 \times 3}{4+3-2}\right)^3 + \left(m + \frac{m}{2} - 5\right)\left(\frac{3 \times 3}{3+3-2}\right)^3 = \frac{2187}{128}m - \frac{13257}{8000}.$$

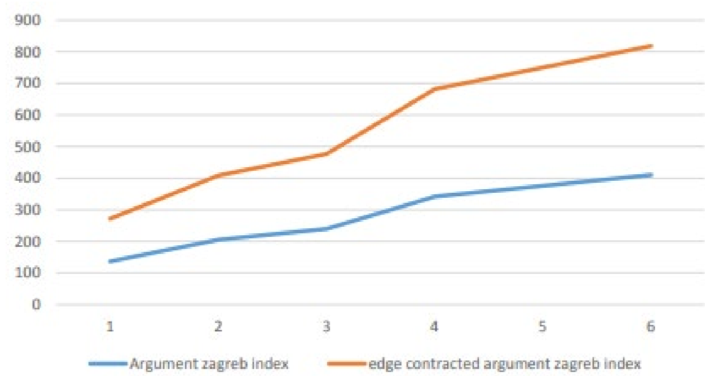


**Figure 7.** Augmented Zagreb index of Möbius Ladder and edge contracted Möbius Ladder

When we plot the augmented Zagreb index of these two graphs on a line chart, we observe that they start off at different values and do not converge to the same value after some distance.

**Table 15**. Correlation coefficient between $M_m$ and $M_m \cdot e$

| $r$ | $M_m$ | $M_m \cdot e$ |
|---|---|---|
| $M_m$ | $1$ | $1$ |
| $M_m \cdot e$ | $1$ | $1$ |

Here the correlation coefficient is close to 1, it indicates a strong positive linear relationship between the two variables, meaning that as one variable increases, the other variable increases as well.

## Randic Index of Möbius Ladder

The Randic index of a graph $G$ is defined as

$$R(G) = \sum_{\mu,v \in E(G)} \left\{\frac{1}{\sqrt{d(\mu)d(v)}}\right\} \qquad (15)$$

**Theorem 4.11.** *Let $M_m$ be a Möbius Ladder with $m \geq 8$; then the Randic index of Möbius Ladder is,*

$$R(M_m) = \frac{m}{2}.$$

*Proof.* Suppose that $M_m$ be an Möbius Ladder with $m \geq 8$. Then the Randic index is,

$$R(M_m) = \sum_{\mu,v \in E(M_m)} \left\{ \frac{1}{\sqrt{d(\mu)d(v)}} \right\} \tag{16}$$

**Table 16**. $M_m$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,3)$ | $m + \frac{m}{2}$ |

We observe that for $m \geq 8$, edge type is taken from Table 16. Now we put the edge type in equation (16),

$$R(M_m) = \left(m + \frac{m}{2}\right)\left(\frac{1}{\sqrt{3 \times 3}}\right) = \frac{m}{2}$$

## Randic Index of Edge Contracted Möbius Ladder

**Theorem 4.12.** *Let $M_m \cdot e$ be an edge contracted Möbius Ladder with $m \geq 8$; then the Randic index of $M_m \cdot e$ is,*

$$R(M_m \cdot e) = \frac{m}{2} - \frac{2 - 5\sqrt{3}}{\sqrt{3}}.$$

*Proof.* Suppose that $M_m \cdot e$ be the edge contracted Möbius Ladder with $m \geq 8$. Then the Randic index is,

$$R(M_m \cdot e) = \sum_{\mu,v \in E(M_m \cdot e)} \frac{1}{\sqrt{d(\mu)d(v)}} \tag{17}$$

**Table 17**. $M_m \cdot e_1$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,4)$ | $4$ |
| $(3,3)$ | $m + \frac{m}{2} - 5$ |

If $m \geq 8$, edge type is taken from Table 17. Now we put the edge type in equation (17),

$$R(M_m \cdot e) = 4\left(\frac{1}{\sqrt{4 \times 3}}\right) + \left(m + \frac{m}{2} - 5\right)\left(\frac{1}{\sqrt{3 \times 3}}\right) = \frac{m}{2} - \frac{2 - 5\sqrt{3}}{\sqrt{3}}.$$

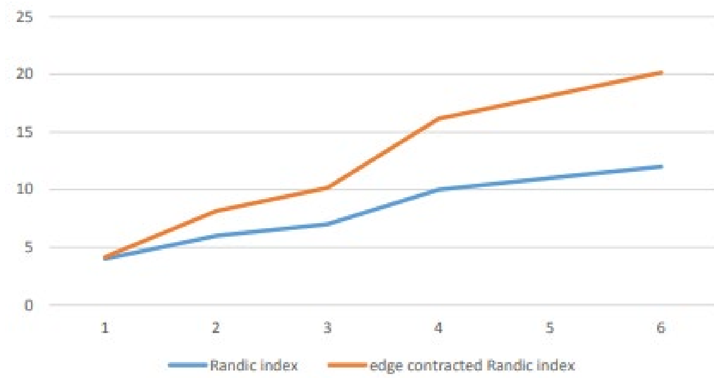


**Figure 8.** Randic index of Möbius Ladder and edge contracted Möbius Ladder

When we plot the Randic index of these two graphs on a line chart, we observe that they start with same values but converge to different after some distance.

**Table 18**. Correlation coefficient between $M_m$ and $M_m \cdot e$

| $r$ | $M_m$ | $M_m \cdot e$ |
|---|---|---|
| $M_m$ | $1$ | $1$ |
| $M_m \cdot e$ | $1$ | $1$ |

Here the correlation coefficient is close to 1, it indicates a strong positive linear relationship between the two variables, meaning that as one variable increases, the other variable increases as well.

## General Randic Index of Möbius Ladder

The general Randic index of a graph $G$ is defined as

$$R_\alpha(G) = \sum_{\mu,v \in E(G)} \{d(\mu)d(v)\}^\alpha \tag{18}$$

**Theorem 4.13.** *Let $M_m$ be a Möbius Ladder with $m \geq 8$; then the general Randic index of Möbius Ladder is,*

$$R_\alpha(M_m) = 3^{2\alpha+1} \cdot \frac{m}{2}.$$

*Proof.* Suppose that $M_m$ be an Möbius Ladder with $m \geq 8$. Then the general Randic index is,

$$R_\alpha(M_m) = \sum_{\mu,v \in E(M_m)} \{d(\mu)d(v)\}^\alpha \tag{19}$$

**Table 19**. $M_m$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,3)$ | $m + \frac{m}{2}$ |

We observe that for $m \geq 8$, edge type is taken from Table 19. Now we put the edge type in equation (19),

$$R_\alpha(M_m) = \left(m + \frac{m}{2}\right)(3 \times 3)^\alpha = 3^{2\alpha+1} \cdot \frac{m}{2}$$

## General Randic Index of Edge Contracted Möbius Ladder

**Theorem 4.14.** *Let $M_m \cdot e$ be an edge contracted Möbius Ladder with $m \geq 8$; then the general Randic index of $M_m \cdot e$ is,*

$$R_\alpha(M_m \cdot e) = 4^{\alpha+1} \cdot 3^\alpha + 3^{2\alpha+1} \cdot \frac{m}{2} - 5 \cdot 3^{2\alpha}.$$

*Proof.* Suppose that $M_m \cdot e$ be the edge contracted Möbius Ladder with $m \geq 8$. Then the general Randic index is,

$$R_\alpha (M_m \cdot e) = \sum_{\mu,\nu \in E(M_m \cdot e)} \{d(\mu)d(\nu)\}^\alpha \tag{20}$$

**Table 20**. $M_m \cdot e_1$ for $m \geq 8$

| $(d(\mu), d(\nu))$ | Frequency |
|---|---|
| $(3,4)$ | $4$ |
| $(3,3)$ | $m + \frac{m}{2} - 5$ |

If $m \geq 8$, edge type is taken from Table 20. Now we put the edge type in equation (20),

$$R_\alpha(M_m \cdot e) = 4(4 \times 3)^\alpha + \left(m + \frac{m}{2} - 5\right)(3 \times 3)^\alpha = 4^{\alpha+1} \cdot 3^\alpha + 3^{2\alpha+1} \cdot \frac{m}{2} - 5 \cdot 3^{2\alpha}.$$

## Harmonic Index of Möbius Ladder

The Harmonic index of a graph $G$ is defined as

$$H(G) = \sum_{\mu,\nu \in E(G)} \left\{\frac{2}{d(\mu) + d(\nu)}\right\} \tag{21}$$

**Theorem 4.15.** *Let $M_m$ be a Möbius Ladder with $m \geq 8$; then the Harmonic index of Möbius Ladder is,*

$$H(M_m) = \frac{m}{2}.$$

*Proof.* Suppose that $M_m$ be an Möbius Ladder with $m \geq 8$. Then the Harmonic index is,

$$H(M_m) = \sum_{\mu,\nu \in E(M_m)} \left\{\frac{2}{d(\mu) + d(\nu)}\right\} \tag{22}$$

**Table 21**. $M_m$ for $m \geq 8$

| $(d(\mu), d(\nu))$ | Frequency |
|---|---|
| $(3,3)$ | $m + \frac{m}{2}$ |

We observe that for $m \geq 8$, edge type is taken from Table 21. Now we put the edge type in equation (22),

$$H(M_m) = \left(m + \frac{m}{2}\right)\left(\frac{2}{3+3}\right) = \frac{m}{2}.$$

## Harmonic Index of Edge Contracted Möbius Ladder

**Theorem 4.16.** *Let $M_m \cdot e$ be an edge contracted Möbius Ladder with $m \geq 8$; then the Harmonic index of $M_m \cdot e$ is,*

$$H(M_m \cdot e) = \frac{m}{2} - \frac{11}{21}.$$

*Proof.* Suppose that $M_m \cdot e$ be the edge contracted Möbius Ladder with $m \geq 8$. Then the Harmonic index is,

$$H\,(M_m \cdot e) = \sum_{\mu,v \in E(M_m \cdot e)} \left\{\frac{2}{d(\mu) + d(v)}\right\} \tag{23}$$

**Table 22**. $M_m \cdot e_1$ for $m \geq 8$

| $(d(\mu), d(v))$ | Frequency |
|---|---|
| $(3,4)$ | $4$ |
| $(3,3)$ | $m + \frac{m}{2} - 5$ |

If $m \geq 8$, edge type is taken from Table 22. Now we put the edge type in equation (23),

$$H(M_m \cdot e) = 4\left(\frac{2}{4+3}\right) + \left(m + \frac{m}{2} - 5\right)\left(\frac{2}{3+3}\right) = \frac{m}{2} - \frac{11}{21}.$$

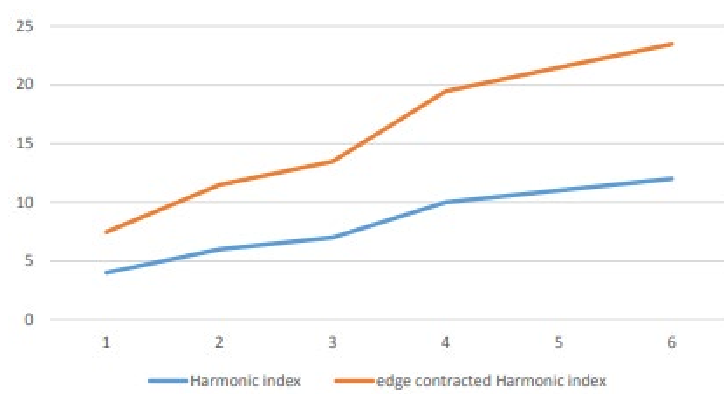


**Figure 9.** Harmonic index of Möbius Ladder and edge contracted Möbius Ladder

When we plot the Harmonic index of these two graphs on a line chart, we observe that they start off at different values but do not converge to the same values after some distance.

**Table 23**. Correlation coefficient between $M_m$ and $M_m \cdot e$

| $r$ | $M_m$ | $M_m \cdot e$ |
|---|---|---|
| $M_m$ | 1 | 1 |
| $M_m \cdot e$ | 1 | 1 |

Here the correlation coefficient is close to 1, it indicates a strong positive linear relationship between the two variables, meaning that as one variable increases, the other variable increases as well.

## Conclusions

The paper investigates the metric dimension and degree-based topological indices of the Möbius Ladder and edge-contracted Möbius Ladder. The purpose of this study is to analyze and understand how these indices are affected by the process of edge contraction in the context of the Möbius Ladder graph. Regular graphs have a metric dimension that is finite, as we can see and we have found that edge contraction does not affect the metric dimension for these families as it remains the same in many cases during this study. In the conclusions section, it is important to address the limitations of this study on the topology of edge-contracted Möbius Ladders, focusing on indices and dimensions. One limitation is that the study primarily relies on theoretical analysis and may benefit from empirical validation or practical applications to reinforce the findings. Additionally, the scope of the research is confined to specific types of edge contractions and particular graph indices, potentially overlooking other relevant transformations or invariants that could offer further insights. Future work should aim to explore a broader range of edge contraction scenarios and investigate their effects on different graph properties. Moreover, extending the study to other graph structures and examining real-world applications of these theoretical results could significantly enhance the relevance and impact of the research.

## Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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