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On the Spectral Radius and Sombor Energy of the Non-Commuting Graph for Dihedral Groups

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Abstract The non-commuting graph, denoted by Γ_G, is defined on a finite group G, with its vertices are elements of G excluding those in the center $Z(G)$ of G. In this graph, two distinct vertices are adjacent whenever they do not commute in G. The graph Γ_G can be associated with several matrices including the most basic matrix, which is the adjacency matrix, $A(\Gamma_G)$, and a matrix called Sombor matrix, denoted by $S(\Gamma_G)$. The entries of $S(\Gamma_G)$ are either the square root of the sum of the squares of degrees of two distinct adjacent vertices, or zero otherwise. Consequently, the adjacency and Sombor energies of Γ_G is the sum of the absolute eigenvalues of the adjacency and Sombor matrices of Γ_G, respectively, whereas the spectral radius of Γ_G is the maximum absolute eigenvalues. Throughout this paper, we find the spectral radius obtained from the spectrum of Γ_G and the Sombor energy of Γ_G for dihedral groups of order $2n$, D_{2n}, where $n \geq 3$. Moreover, there is an almost linear correlation between the Sombor energy and the adjacency energy of Γ_G for D_{2n} which is slightly different than reported earlier in previous literature.

Keywords: Sombor matrix, energy of graph, spectral radius, non-commuting graph, dihedral group.

Introduction

The concept of the non-commuting graph was first introduced in 1976 by Neumann [17]. However, Abdollahi *et al*. [1] started extensive research in exploring more on the properties of the non-commuting graph, which includes graph connectivity, regularity, and the relation of these graph properties with group properties.

The non-commuting graph, denoted by Γ_G, is defined on a finite group G, with its vertices are elements of $G\backslash Z(G)$, where $Z(G)$ is the center of G. This graph has to satisfy a condition in which $v_p, v_q \in G\backslash Z(G)$, and $v_p \neq v_q$, are joined by an edge whenever $v_p v_q \neq v_q v_p$. If there is an edge between v_p and v_q in Γ_G, then they are called adjacent. This adjacency property of Γ_G can be represented in a matrix named adjacency matrix, $A(\Gamma_G) = [a_{pq}]$ of size $n \times n$, in which $a_{pq} = 1$, if v_p and v_q are adjacent, and $a_{pq} = 0$ otherwise. Furthermore, the characteristic polynomial formula of $A(\Gamma_G)$, $P_{A(\Gamma_G)}(\lambda) = |\lambda I_n - A(\Gamma_G)|$, where I_n is the identity matrix of size $n \times n$. The eigenvalues of Γ_G are the roots of $P_{A(\Gamma_G)}(\lambda) = 0$, labeled as $\lambda_1, \lambda_2, \dots, \lambda_n$.

A non-commuting graph Γ_G is a simple graph that implies the adjacency matrix of Γ_G, $A(\Gamma_G)$, to be symmetric with its real eigenvalues that can be arranged as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The list of all $\lambda_1, \lambda_2, \dots, \lambda_n$, denoted by $Spec(\Gamma_G) = \{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$ is called the spectrum of Γ_G with $k_1, k_2, \dots, k_n$ are their respective multiplicities. The spectral radius of Γ_G is defined as $\rho(\Gamma_G) = max\{|\lambda| : \lambda \in Spec(\Gamma_G)\}$. Clearly, $\rho(\Gamma_G)$ is a non-negative real number and is the smallest disc radius that includes all the eigenvalues of Γ_G with the center at the origin of the complex plane [11]. There are a number of papers focusing on the spectral radius of other types of graphs, such as the spectral radius of power graphs on dihedral groups [5], the spectral radius of directed graphs [6] and the spectral radius of the interval-valued fuzzy graph [25] with regards to signless Laplacian matrix.

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Moreover, the concept of the spectrum of the adjacency matrix associated with finite graphs can further be extended to the study of graph energies. Gutman in 1978 is the first to discover the (ordinary) energy of a finite graph in order to estimate the energy of the electrons in a chemical molecule which is regarded as a graph. For Γ_G with n vertices, the adjacency energy is denoted by $E_A(\Gamma_G)$ and is defined in [9] as $E_A(\Gamma_G) = \sum_{i=1}^{n}|\lambda_i|$. The graphs on n vertices with an energy of more than $E_A(K_n)$ can be classified as hyperenergetic, or in other words, when $E(\Gamma_G) > 2(n-1)$ [13]. It should be noted as well that the (ordinary) adjacency energy is neither an odd number [3] nor the square root of an odd number [18].

In 2021, a new graph matrix definition was put forward by Gowtham & Swamy [8], named the Sombor matrix of Γ_G, denoted by $S(\Gamma_G) = [s_{pq}]$ whose (p,q) −th entry is equal to

$$s_{pq} = \begin{cases} \sqrt{d_{v_p}^2 + d_{v_q}^2}, & \text{if } v_p \neq v_q \text{ and they are adjacent} \\ 0, & \text{otherwise,} \end{cases}$$

where d_{v_i} as the degree of v_i. This concept is the algebraic perspective of the Sombor index in mathematical chemistry that was established by [10].

Recently, the study on Sombor indices and Sombor energy of a graph has become one of the topics of interest for many researchers. There are many ways in which the Sombor index and its energy can be related to, for instance, see [23]. Moreover, the bound of graph energy has been derived in terms of the Sombor index, see [14,26]. Besides, Liu *et al*. [15] generalized the Sombor matrix definition and named it as p −Sombor matrices. They discussed the spectral properties of the graphs corresponding to these matrices. Rather & Imran [20] provided the Sombor energy updates for the extremal graph. Later they corrected their previous result in [21], one of the upper bounds of Sombor's energy was invalid. In correlation studies for molecules containing hetero atoms and their total electron energy with a correlation, Gowtham & Swamy [8] stated that the Sombor energy values were highly correlated with their total electron energy with a correlation coefficient of 0.952. However, a different result was shown by [22] in which they presented a numerical approach for comparing the Sombor energy and the adjacency energy of graphs, and it is still an open problem for Mathematical proof.

Throughout this paper, we focus on the Sombor matrix of Γ_G for the non-abelian dihedral groups of order $2n$, $n \geq 3$, denoted by D_{2n}. The dihedral group has a presentation $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1}\rangle$ and its elements can be written as a^i and $a^i b$ [2]. The center of D_{2n}, $Z(D_{2n})$ is either equal to the set $\{e\}$, if n is odd or $\{e, a^{\frac{n}{2}}\}$ for even n. The centralizer of the element a^i in D_{2n} is $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$ and for the element $a^i b$ is either $C_{D_{2n}}(a^i b) = \{e, a^i b\}$, if n is odd or $C_{D_{2n}}(a^i b) = \{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\}$, if n is even.

Some recent results in the energy of commuting and non-commuting graphs for D_{2n}, for $n \geq 3$, denoted by D_{2n} have been reported in [16,24]. They worked on adjacency and degree exponent sum matrices. To further extend this study, here we discuss on the spectral radius and Sombor energy of Γ_G for dihedral groups, D_{2n}. The methodology consists of constructing the Sombor matrix of Γ_G, finding the eigenvalues and the spectrum of the respective matrix, analyzing $\rho(\Gamma_G)$, computing the Sombor energy, and thus observing the relationship between $\rho(\Gamma_G)$ and the Sombor energy of the respected Γ_G. We also investigate the hyperenergetic property of Γ_G and we compare the Sombor energy, $E_S(\Gamma_G)$ and the (ordinary) energy $E_A(\Gamma_G)$ as one of the cases to answer the claim in [22].

Preliminaries

We study the non-commuting graph for G, Γ_G, being the subset of the dihedral groups of order $2n$, D_{2n}, where G is either G_1, G_2 or $G_1 \cup G_2$. We define $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n})$ and $G_2 = \{a^i b : 1 \leq i \leq n\}$. The Sombor energy of Γ_G is given by

$$E_S(\Gamma_G) = \sum_{i=1}^{n}|\lambda_i|, \tag{1}$$

with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) of $S(\Gamma_G)$. The Sombor spectral radius of Γ_G is

$$\rho_S(\Gamma_G) = max\{|\lambda| : \lambda \in Spec(\Gamma_G)\}. \tag{2}$$

Now since Γ_G has $2n-1$ (odd n) and $2n-2$ (even n) vertices, the Γ_G associated with the Sombor matrix can be classified as a hyperenergetic graph if the Sombor energy satisfies the following conditions:

$$E_S(\Gamma_G) > \begin{cases} 4(n-1), & \text{for odd } n \\ 4(n-1)-2, & \text{for even } n. \end{cases}$$

Now we are moving to the properties for constructing the Sombor matrix. The following are some underlying results focusing on the degree of vertices of Γ_G and the isomorphism of Γ_G with some common types of graphs for $G = G_1 \cup G_2$.

Theorem 2.1. [12] Let Γ_G be the non-commuting graph for G, where $G = G_1 \cup G_2$. Then
1. the degree of a^i on Γ_G is $d_{a^i} = n$, and
2. the degree of $a^i b$ on Γ_G is $d_{a^i b} = \begin{cases} 2(n-1), & \text{if } n \text{ is odd} \\ 2(n-2), & \text{if } n \text{ is even.} \end{cases}$

Theorem 2.2. [12] Let Γ_G be the non-commuting graph for G.
1. If $G = G_1$, then $\Gamma_G \cong \overline{K}_m$, where $m = |G_1|$.
2. If $G = G_2$, then $\Gamma_G \cong \begin{cases} K_n, & \text{for odd } n \\ K_n - \frac{n}{2}K_2, & \text{for even } n, \end{cases}$

for a complete graph K_n on n vertices with $\overline{K}_n$ is the complement of K_n, and $\frac{n}{2}K_2$ denotes $\frac{n}{2}$ copies of K_2.

By those two theorems, we can construct the Sombor matrix of Γ_G and thus determine its characteristic polynomial. The following results are beneficial for simplifying the process of formulating the characteristic polynomial of Γ_G, $P_{S(\Gamma_G)}(\lambda)$, for $G = G_1 \cup G_2$.

Proposition 2.1. [19] If w, x, y and z are real numbers, and J_n is an $n \times n$ matrix whose all elements are equal to 1, then the determinant of the $(n_1 + n_2) \times (n_1 + n_2)$ matrix of the form

$$\begin{vmatrix} (\lambda + w)I_{n_1} - wJ_{n_1} & -yJ_{n_1 \times n_2} \\ -zJ_{n_2 \times n_1} & (\lambda + x)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified as the following expression

$$(\lambda + w)^{n_1 - 1}(\lambda + x)^{n_2 - 1}\big((\lambda - (n_1 - 1)w)(\lambda - (n_2 - 1)x) - n_1 n_2 yz\big),$$

where $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.

Theorem 2.3. [7] If a square matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be partitioned into four blocks, where $|A| \neq 0$, then

$$|M| = \begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix} = |A||D - CA^{-1}B|.$$

In order to compute the Sombor energy of Γ_G for $G = G_1 \cup G_2$, we need the spectrum of K_n as shown below.

Lemma 2.1. [4] If K_n is the complete graph on n vertices, then its adjacency matrix is $(J - I)_n$, and the spectrum of K_n is $\{(n-1)^1, (-1)^{n-1}\}$.

In order to determine the roots of $P_{S(\Gamma_G)}(\lambda) = 0$, elementary row and column operations on $P_{S(\Gamma_G)}(\lambda)$ need to be performed. Now suppose that R_i is the i-th row and R_i' is the new i-th row obtained from a row operation of $P_{S(\Gamma_G)}(\lambda)$. Also, let the i-th column as C_i and C_i' is the new i-th column obtained from a column operation of $P_{S(\Gamma_G)}(\lambda)$.

Moreover, to compare the Sombor energy and the adjacency energy of Γ_G for D_{2n}, we mention the previous result from [16] on the adjacency energy as follows:

Theorem 2.4. [16] Let Γ_G be the non-commuting graph on $G = G_1 \cup G_2$. Then the adjacency energy of Γ_G, $E_A(\Gamma_G)$, is
1. for odd n, $E_A(\Gamma_G) = (n-1) + \sqrt{5n^2 - 6n + 1}$, and
2. for even n, $E_A(\Gamma_G) = \begin{cases} 8, & \text{if } n = 4 \\ (n-2) + \sqrt{5n^2 - 12n + 4}, & \text{if } n > 4. \end{cases}$

Main Results

The following theorem gives the Sombor energy of Γ_G for both $G = G_1$ and $G = G_2$.

Theorem 3.1. Let Γ_G be the non-commuting graph on $G = D_{2n}$ and $E_S(\Gamma_G)$ be the Sombor energy of Γ_G.
1. If $G = G_1$, then $E_S(\Gamma_G) = 0$.
2. If $G = G_2$, then $E_S(\Gamma_G) = \begin{cases} 2\sqrt{2}(n-1)^2, & \text{if } n \text{ is odd} \\ 2\sqrt{2}(n-2)^2, & \text{if } n \text{ is even.} \end{cases}$

Proof.
1. Let $G = G_1$, then from Theorem 2.2 (1), $\Gamma_G \cong \overline{K}_m$ implies every vertex of Γ_G has a degree zero. For odd n, we the obtain $m = |G_1| = n - 1$, while for even n, $m = n - 2$ as the result of removing e and $a^{\frac{n}{2}}$ in

$Z(D_{2n})$. So, the Sombor matrix of Γ_G is $S(\Gamma_G) = [0]_{n-1}$ and $S(\Gamma_G) = [0]_{n-2}$ for odd and even n, respectively. Clearly, the only eigenvalue of $S(\Gamma_G)$ is zero. Thus, $E_S(\Gamma_G) = 0$.

2. When $G = G_2$ and n is odd, Theorem 2.2 (2) gives $\Gamma_G \cong K_n$, which means the degree of each vertex in $n-1$. Consequently, $S(\Gamma_G)$ is a matrix of size $n \times n$ whose (p,q)-th entry is $\sqrt{(n-1)^2 + (n-1)^2} = \sqrt{2}(n-1)$ for $v_p \neq v_q$ and they are adjacent, and zero otherwise, where the index of rows and columns are labeled by the vertices $\{b, ab, a^2b, \cdots, a^{n-1}b\}$, as follows

$$S(\Gamma_G) = \begin{matrix} & b & ab & \cdots & a^{n-1}b \\ b & 0 & \sqrt{2}(n-1) & \cdots & \sqrt{2}(n-1) \\ ab & \sqrt{2}(n-1) & 0 & \cdots & \sqrt{2}(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1}b & \sqrt{2}(n-1) & \sqrt{2}(n-1) & \cdots & 0 \end{matrix}$$
$$= \sqrt{2}(n-1)\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$
$$= \sqrt{2}(n-1) \cdot A(K_n).$$

Meanwhile, from Lemma 2.1, it follows that $Spec(K_n) = \{(n-1)^1, (-1)^{n-1}\}$. Then considering the adjacency energy of K_n that given by $2(n-1)$, we get the Sombor energy of Γ_G,

$$E_S(\Gamma_G) = \sqrt{2}(n-1) \cdot 2(n-1) = 2\sqrt{2}(n-1)^2.$$

For the second case, when n is even, as it is known from Theorem 2.2 (2), $\Gamma_G \cong K_n - \frac{n}{2}K_2$, which implies d_{a^ib} is $n-2$. Following the definition of the Sombor matrix of Γ_G, $S(\Gamma_G)$, we can construct $S(\Gamma_G)$ of size $n \times n$ whose (p,q)-th entry is $\sqrt{2}(n-2)$ for $v_p \neq v_q$ and they are adjacent and zero otherwise. In the same manner indexing rows and columns as in the odd n case, we obtain $S(\Gamma_G)$ as the following

$$S(\Gamma_G) = \begin{matrix} & b & ab & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \cdots & a^{n-1}b \\ b & 0 & \sqrt{2}(n-2) & \cdots & \sqrt{2}(n-2) & 0 & \sqrt{2}(n-2) & \cdots & \sqrt{2}(n-2) \\ ab & \sqrt{2}(n-2) & 0 & \cdots & \sqrt{2}(n-2) & \sqrt{2}(n-2) & 0 & \cdots & \sqrt{2}(n-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{\frac{n}{2}-1}b & \sqrt{2}(n-2) & \sqrt{2}(n-2) & \cdots & 0 & \sqrt{2}(n-2) & \sqrt{2}(n-2) & \cdots & 0 \\ a^{\frac{n}{2}}b & 0 & \sqrt{2}(n-2) & \cdots & \sqrt{2}(n-2) & 0 & \sqrt{2}(n-2) & \cdots & \sqrt{2}(n-2) \\ a^{\frac{n}{2}+1}b & \sqrt{2}(n-2) & 0 & \cdots & \sqrt{2}(n-2) & \sqrt{2}(n-2) & 0 & \cdots & \sqrt{2}(n-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1}b & \sqrt{2}(n-2) & \sqrt{2}(n-2) & \cdots & 0 & \sqrt{2}(n-2) & \sqrt{2}(n-2) & \cdots & 0 \end{matrix}.$$

In other words,

$$S(\Gamma_G) = \sqrt{2}(n-2)\begin{bmatrix} (J-I)_{\frac{n}{2}} & (J-I)_{\frac{n}{2}} \\ (J-I)_{\frac{n}{2}} & (J-I)_{\frac{n}{2}} \end{bmatrix}.$$

Here $J - I$ is a matrix with zero diagonal entries and non-diagonal entries being one. Then we get $P_{S(\Gamma_G)}(\lambda) = |\lambda I_n - S(\Gamma_G)|$, and it is equal to

$$P_{S(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda + \sqrt{2}(n-2))I_{\frac{n}{2}} - \sqrt{2}(n-2)J_{\frac{n}{2}} & -\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} \\ -\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} & (\lambda + \sqrt{2}(n-2))I_{\frac{n}{2}} - \sqrt{2}(n-2)J_{\frac{n}{2}} \end{vmatrix}. \quad (4)$$

Now, for $1 \leq i \leq \frac{n}{2}$, by the row operation $R'_{\frac{n}{2}+i} = R_{\frac{n}{2}+i} - R_i$, followed by column operation by replacing C_i with $C'_i = C_i + C_{\frac{n}{2}+i}$ on Equation (4), yield

$$P_{S(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda + 2\sqrt{2}(n-2))I_{\frac{n}{2}} - 2\sqrt{2}(n-2)J_{\frac{n}{2}} & -\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}.$$

According to Theorem 2.3, since $C = 0$, we then obtain $P_{S(\Gamma_G)}(\lambda)$ given in Equation (5)

$$P_{S(\Gamma_G)}(\lambda) = |A||D| = \left(\lambda + 2\sqrt{2}(n-2)\right)^{\frac{n}{2}-1}(\lambda - \sqrt{2}(n-2)^2)\lambda^{\frac{n}{2}}.$$

Therefore, the Sombor energy of Γ_G can be obtained as the summation of absolute roots of Equation (5),

$$E_S(\Gamma_G) = \left(\frac{n}{2} - 1\right)\left|-2\sqrt{2}(n-2)\right| + (1)\left|\sqrt{2}(n-2)^2\right| + \left(\frac{n}{2}\right)|0| = 2\sqrt{2}(n-2)^2.$$

□

We now formulate the characteristic polynomial of $S(\Gamma_G)$ and calculate the Sombor energy of Γ_G for $G = G_1 \cup G_2$.

Theorem 3.2. Let Γ_G be the non-commuting graph on G, where $G = G_1 \cup G_2$, then the characteristic polynomial of $S(\Gamma_G)$ is

1. for odd n,

$$P_{S(\Gamma_G)}(\lambda) = \lambda^{n-2}\left(\lambda + 2\sqrt{2}(n-1)\right)^{n-1}\left(\lambda^2 - 2\sqrt{2}(n-1)^2\lambda - n(n-1)(n^2+4(n-1)^2\right).$$

2. For even n,

$$P_{S(\Gamma_G)}(\lambda) = \lambda^{\frac{3(n-2)}{2}}\left(\lambda + 4\sqrt{2}(n-2)\right)^{\frac{n}{2}-1}\left(\lambda^2 - 2\sqrt{2}(n-2)^2\lambda - n(n-2)(n^2+4(n-2)^2\right).$$

Proof.

1. For the case of odd n, we know that $Z(D_{2n}) = \{e\}$ which implies that there are $2n-1$ vertices for Γ_G, where $G = G_1 \cup G_2$. We label the set G_1 as $\{a, a^2, \cdots, a^{n-1}\}$ and G_2 as $\{b, ab, a^2b, \cdots, a^{n-1}b\}$. From the fact that the centralizer of a^i in D_{2n} is $\{e, a, a^2, \cdots, a^{n-1}\}$, then the vertex a^i, for $1 \le i \le n-1$, is not adjacent to all vertices of G_1, however, it always has an edge with all members of G_2. While the centralizer of a^ib in D_{2n} is $\{e, a^ib\}$ implies that for $1 \le i \le n$, vertex a^ib is connected with all other elements of $G_1 \cup G_2$. Considering Theorem 2.1 we get $d_{a^i} = n$ and $d_{a^ib} = 2(n-1)$, for all for $1 \le i \le n$. Now the Sombor matrix for Γ_G, $S(\Gamma_G)$, is a $(2n-1)\times(2n-1)$ matrix

$$S(\Gamma_G) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array}\begin{array}{c} \begin{array}{cccc|cccc} a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & \sqrt{n^2+4(n-1)^2} & \sqrt{n^2+4(n-1)^2} & \cdots & \sqrt{n^2+4(n-1)^2} \\ 0 & 0 & \cdots & 0 & \sqrt{n^2+4(n-1)^2} & \sqrt{n^2+4(n-1)^2} & \cdots & \sqrt{n^2+4(n-1)^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{n^2+4(n-1)^2} & \sqrt{n^2+4(n-1)^2} & \cdots & \sqrt{n^2+4(n-1)^2} \\ \hline \sqrt{n^2+4(n-1)^2} & \sqrt{n^2+4(n-1)^2} & \cdots & \sqrt{n^2+4(n-1)^2} & 0 & 2\sqrt{2}(n-1) & \cdots & 2\sqrt{2}(n-1) \\ \sqrt{n^2+4(n-1)^2} & \sqrt{n^2+4(n-1)^2} & \cdots & \sqrt{n^2+4(n-1)^2} & 2\sqrt{2}(n-1) & 0 & \cdots & 2\sqrt{2}(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{n^2+4(n-1)^2} & \sqrt{n^2+4(n-1)^2} & \cdots & \sqrt{n^2+4(n-1)^2} & 2\sqrt{2}(n-1) & 2\sqrt{2}(n-1) & \cdots & 0 \end{array}\right] \end{array}.$$

Here the Sombor matrix of Γ_G can be obtained as the block matrix

$$S(\Gamma_G) = \begin{bmatrix} 0_{n-1} & \sqrt{n^2+4(n-1)^2}J_{(n-1)\times n} \\ \sqrt{n^2+4(n-1)^2}J_{n\times(n-1)} & 2\sqrt{2}(n-1)(J-I)_n \end{bmatrix},$$

and the determinant below is the characteristic polynomial for $S(\Gamma_G)$,

$$P_{S(\Gamma_G)}(\lambda) = \begin{vmatrix} \lambda I_{n-1} & -\sqrt{n^2+4(n-1)^2}J_{(n-1)\times n} \\ -\sqrt{n^2+4(n-1)^2}J_{n\times(n-1)} & \left(\lambda + 2\sqrt{2}(n-1)\right)I_n - 2\sqrt{2}(n-1)J_n \end{vmatrix}.$$

Repeated application of Proposition 2.1, with $w = 0$, $x = 2\sqrt{2}(n-1)$, $y = z = \sqrt{n^2+4(n-1)^2}$, $n_1 = n-1$, and $n_2 = n$, we get the required result.

2. Suppose now n is even. Since $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, Γ_G, where $G = G_1 \cup G_2$ has $2n-2$ vertices with $n-2$ vertices from a^i, for $1 \le i < \frac{n}{2}$, $\frac{n}{2} < i < n$, and n vertices from a^ib, for $1 \le i \le n$. We write the set G_1 as $\left\{a, a^2, \cdots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \cdots, a^{n-1}\right\}$ and G_2 $\{b, ab, a^2b, \cdots, a^{n-1}b\}$. Again, considering the centralizer of a^i in D_{2n}, then all the members of G_1 are only connected with the elements of G_2. Since the centralizer of a^ib is $\{e, a^{\frac{n}{2}}, a^ib, a^{\frac{n}{2}+i}b\}$, then two vertices a^ib and $a^{\frac{n}{2}+i}b$ are always disconnected in Γ_G. From Theorem 2.1, the fact that $d_{a^i} = n$ and $d_{a^ib} = 2(n-2)$, which implies $S(\Gamma_G)$ being the matrix of size $(2n-2)\times(2n-2)$ as follows,

$$
\begin{array}{c|ccc|ccc|ccc|}
 & a & \cdots & a^2 & b & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & \cdots & a^{n-1}b \\
a & 0 & \cdots & 0 & \sqrt{n^2+4(n-2)^2} & \cdots & \sqrt{n^2+4(n-2)^2} & \sqrt{n^2+4(n-1)^2} & \cdots & \sqrt{n^2+4(n-1)^2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^2 & 0 & \cdots & 0 & \sqrt{n^2+4(n-2)^2} & \cdots & \sqrt{n^2+4(n-2)^2} & \sqrt{n^2+4(n-1)^2} & \cdots & \sqrt{n^2+4(n-1)^2} \\
\hline
b & \sqrt{n^2+4(n-2)^2} & \cdots & \sqrt{n^2+4(n-2)^2} & 0 & \cdots & 2\sqrt{2}(n-2) & 0 & \cdots & 2\sqrt{2}(n-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{\frac{n}{2}-1}b & \sqrt{n^2+4(n-2)^2} & \cdots & \sqrt{n^2+4(n-2)^2} & 2\sqrt{2}(n-2) & \cdots & 0 & 2\sqrt{2}(n-2) & \cdots & 0 \\
\hline
a^{\frac{n}{2}}b & \sqrt{n^2+4(n-2)^2} & \cdots & \sqrt{n^2+4(n-2)^2} & 0 & \cdots & 2\sqrt{2}(n-2) & 0 & \cdots & 2\sqrt{2}(n-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a^{n-1}b & \sqrt{n^2+4(n-2)^2} & \cdots & \sqrt{n^2+4(n-2)^2} & 2\sqrt{2}(n-2) & \cdots & 0 & 2\sqrt{2}(n-2) & \cdots & 0
\end{array}.
$$

The Sombor matrix of Γ_G can be obtained as the block matrix:

$$
S(\Gamma_G)=\begin{bmatrix} 0_{n-2} & \sqrt{n^2+4(n-2)^2}J_{(n-2)\times\frac{n}{2}} & \sqrt{n^2+4(n-2)^2}J_{(n-2)\times\frac{n}{2}} \\ \sqrt{n^2+4(n-2)^2}J_{\frac{n}{2}\times(n-2)} & 2\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} & 2\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} \\ \sqrt{n^2+4(n-2)^2}J_{\frac{n}{2}\times(n-2)} & 2\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} & 2\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} \end{bmatrix},
$$

and the characteristic polynomial of $S(\Gamma_G)$ as follows

$$
P_{S(\Gamma_G)}(\lambda)=\begin{vmatrix} \lambda I_{n-2} & -2\sqrt{n^2+4(n-2)^2}J_{(n-2)\times\frac{n}{2}} & -\sqrt{n^2+4(n-2)^2}J_{(n-2)\times\frac{n}{2}} \\ -\sqrt{n^2+4(n-2)^2}J_{\frac{n}{2}\times(n-2)} & (\lambda+2\sqrt{2}(n-2))I_{\frac{n}{2}}-2\sqrt{2}(n-2)J_{\frac{n}{2}} & -2\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} \\ -\sqrt{n^2+4(n-2)^2}J_{\frac{n}{2}\times(n-2)} & -2\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} & (\lambda+2\sqrt{2}(n-2))I_{\frac{n}{2}}-2\sqrt{2}(n-2)J_{\frac{n}{2}} \end{vmatrix}. \tag{6}
$$

By applying the row operation $R'_{n-2+\frac{n}{2}+i}=R_{n-2+\frac{n}{2}+i}-R_{n-2+i}$, following by $C'_{n-2+i}=C_{n-2+i}+C_{n-2+\frac{n}{2}+i}$ on Equation (6) for $1\le i\le\frac{n}{2}$, we obtain

$$
P_{S(\Gamma_G)}(\lambda)=\begin{vmatrix} \lambda I_{n-2} & -2\sqrt{n^2+4(n-2)^2}J_{(n-2)\times\frac{n}{2}} & -\sqrt{n^2+4(n-2)^2}J_{(n-2)\times\frac{n}{2}} \\ -\sqrt{n^2+4(n-2)^2}J_{\frac{n}{2}\times(n-2)} & (\lambda+4\sqrt{2}(n-2))I_{\frac{n}{2}}-4\sqrt{2}(n-2)J_{\frac{n}{2}} & -2\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix}. \tag{7}
$$

Consequently, Equation (7) can be written as

$$
P_{S(\Gamma_G)}(\lambda)=\begin{vmatrix} A_{n-2+\frac{n}{2}} & B_{\left(n-2+\frac{n}{2}\right)\times\frac{n}{2}} \\ C_{\frac{n}{2}\times\left(n-2+\frac{n}{2}\right)} & D_{\frac{n}{2}} \end{vmatrix}, \tag{8}
$$

Where $A=\begin{vmatrix} \lambda I_{n-2} & -2\sqrt{n^2+4(n-2)^2}J_{(n-2)\times\frac{n}{2}} \\ -\sqrt{n^2+4(n-2)^2}J_{\frac{n}{2}\times(n-2)} & (\lambda+4\sqrt{2}(n-2))I_{\frac{n}{2}}-4\sqrt{2}(n-2)J_{\frac{n}{2}} \end{vmatrix}$,

$B=\begin{vmatrix} -\sqrt{n^2+4(n-2)^2}J_{(n-2)\times\frac{n}{2}} \\ -2\sqrt{2}(n-2)(J-I)_{\frac{n}{2}} \end{vmatrix}$, $C=\left|0_{\frac{n}{2}\times\left(n-2+\frac{n}{2}\right)}\right|$, and $D=\left|\lambda I_{\frac{n}{2}}\right|$. According to Theorem 2.3, since $C=0$, we then obtain Equation (8) as $P_{S(\Gamma_G)}(\lambda)=|A||D|$. By applying Proposition 2.1 to $|A|$, with $w=0$, $x=4\sqrt{2}(n-2)$, $y=2\sqrt{n^2+4(n-2)^2}$, $z=\sqrt{n^2+4(n-2)^2}$, $n_1=n-2$, $n_2=\frac{n}{2}$ and considering D is a diagonal matrix, we then get

$$
P_{S(\Gamma_G)}(\lambda)=(\lambda)^{\frac{3(n-2)}{2}}(\lambda+4\sqrt{2}(n-2))^{\frac{n}{2}-1}\left(\lambda^2-2\sqrt{2}(n-2)^2\lambda-n(n-2)(n^2+4(n-2)^2)\right).
$$

□

The following Theorems 3.3 and 3.4 give the spectrum, Sombor spectral radius, and Sombor energy of Γ_G for $G=G_1\cup G_2$. Then at the end of this paper, the relation between them is obtained.

Theorem 3.3. Let Γ_G be the non-commuting graph for G, where $G=G_1\cup G_2$, then the Sombor spectral radius for Γ_G is

1. $\rho_S(\Gamma_G)=\sqrt{2}(n-1)^2+\sqrt{2(n-1)^4+n(n-1)(n^2+4(n-1)^2)}$, for odd n, and

2. $\rho_S(\Gamma_G) = \sqrt{2}(n-2)^2 + \sqrt{2(n-2)^4 + n(n-2)(n^2+4(n-2)^2)}$, for even n.

Proof.
1. The result according to Theorem 3.2 (1) for odd n is the four eigenvalues obtained from $P_{S(\Gamma_G)}(\lambda)$. They are $\lambda_1 = 0$ of multiplicity $n-2$ and $\lambda_2 = -2\sqrt{2}(n-1)$ of multiplicity $n-1$. The other two eigenvalues are $\lambda_{3,4} = \sqrt{2}(n-1)^2 \pm \sqrt{2(n-1)^4 + n(n-1)(n^2+4(n-1)^2)}$ as the roots of the quadratic formula. Hence, the Sombor spectrum for Γ_G is as follows

$$Spec(\Gamma_G) = \left\{\left(\sqrt{2}(n-1)^2 + \sqrt{2(n-1)^4 + n(n-1)(n^2+4(n-1)^2)}\right)^1, 0^{n-2}, \left(\sqrt{2}(n-1)^2 - \sqrt{2(n-1)^4 + n(n-1)(n^2+4(n-1)^2)}\right)^1, \left(-2\sqrt{2}(n-1)\right)^{n-1}\right\}.$$

Now for $i = 1,2,3,4$, as mentioned in Equation (2), the maximum of $|\lambda_i|$ is the Sombor spectral radius of Γ_G,

$$\rho_S(\Gamma_G) = \sqrt{2}(n-1)^2 + \sqrt{2(n-1)^4 + n(n-1)(n^2+4(n-1)^2)}.$$

2. The eigenvalues of Γ_G for even n are given by the roots of $P_{S(\Gamma_G)}(\lambda) = 0$, which is obtained from Theorem 3.2 (2). The first eigenvalue is $\lambda_1 = 0$ of multiplicity $\frac{3(n-2)}{2}$, the second is $\lambda_2 = -4\sqrt{2}(n-2)$ of multiplicity $\frac{n}{2} - 1$, and the other two eigenvalues are $\lambda_{3,4} = \sqrt{2}(n-2)^2 \pm \sqrt{2(n-2)^4 + n(n-2)(n^2+4(n-2)^2)}$. So that the spectrum of Γ_G is

$$Spec(\Gamma_G) = \left\{\left(\sqrt{2}(n-2)^2 + \sqrt{2(n-2)^4 + n(n-2)(n^2+4(n-2)^2)}\right)^1, 0^{\frac{3(n-2)}{2}}, \left(\sqrt{2}(n-2)^2 - \sqrt{2(n-2)^4 + n(n-2)(n^2+4(n-2)^2)}\right)^1, \left(-2\sqrt{2}(n-2)\right)^{\frac{n}{2}-1}\right\}.$$

Taking the maximum absolute eigenvalues as stated in Equation (2), then we get the Sombor spectral radius of Γ_G,

$$\rho_S(\Gamma_G) = \sqrt{2}(n-2)^2 + \sqrt{2(n-2)^4 + n(n-2)(n^2+4(n-2)^2)}.$$

□

Theorem 3.4. Let Γ_G be the non-commuting graph for G, where $G = G_1 \cup G_2$, then the Sombor energy for Γ_G is
1. $E_S(\Gamma_G) = 2\sqrt{2}(n-1)^2 + 2\sqrt{2(n-1)^4 + n(n-1)(n^2+4(n-1)^2)}$, for odd n, and
2. $E_S(\Gamma_G) = 2\sqrt{2}(n-2)^2 + 2\sqrt{2(n-2)^4 + n(n-2)(n^2+4(n-2)^2)}$, for even n.

Proof.
1. By Equation (1), calculating the eigenvalues from $Spec(\Gamma_G)$ in the proving part of Theorem 3.3 (1), then the Sombor energy for Γ_G is given by

$$\begin{aligned} E_S(\Gamma_G) &= (n-2)|0| + (n-1)\left|-2\sqrt{2}(n-1)\right| + \left|\sqrt{2}(n-1)^2 \pm \sqrt{2(n-1)^4 + n(n-1)(n^2+4(n-1)^2)}\right| \\ &= 2\sqrt{2}(n-1)^2 + 2\sqrt{2(n-1)^4 + n(n-1)(n^2+4(n-1)^2)}. \end{aligned}$$

2. Using $Spec(\Gamma_G)$ given in Theorem 3.3 (2) for even n, we get the Sombor energy for Γ_G as follows

$$\begin{aligned} E_S(\Gamma_G) &= \left(\frac{3(n-2)}{2}\right)|0| + \left(\frac{n}{2} - 1\right)\left|-4\sqrt{2}(n-2)\right| + \left|\sqrt{2}(n-2)^2 \pm \sqrt{2(n-2)^4 + n(n-2)(n^2+4(n-2)^2)}\right| \\ &= 2\sqrt{2}(n-2)^2 + 2\sqrt{2(n-2)^4 + n(n-2)(n^2+4(n-2)^2)}. \end{aligned}$$

□

Discussion

In comparing the results of Theorem 3.3 and 3.4, we get the statement as follows.

Corollary 4.1. Let Γ_G be the non-commuting graph on $G = G_1 \cup G_2$, then $E_S(\Gamma_G) = 2 \cdot \rho_S(\Gamma_G)$.

As a result of Theorem 3.4, we obtain the classification of Sombor energy of Γ_G for D_{2n}.

Corollary 4.2. Let $G = G_1 \cup G_2 \subset D_{2n}$, Γ_G is hyperenergetic corresponding to Sombor matrix.

Moreover, according to the results presented in this paper, the energies in Theorem 3.4 deduce the following corollaries.

Corollary 4.3. Let Γ_G be the non-commuting graph on $G = G_1 \cup G_2$, then Sombor energy for Γ_G is never an odd integer.

The statement in Corollary 4.3 complies with well-known facts from (8) and (9) that the graph energy is neither an odd integer nor the square root of an odd integer. Moreover, as a comparison of the results from Theorem 2.4 and 3.4, we immediately have the following result.

Corollary 4.3. Let Γ_G be the non-commuting graph on $G = G_1 \cup G_2$, then $E_S(\Gamma_G) > E_A(\Gamma_G)$.

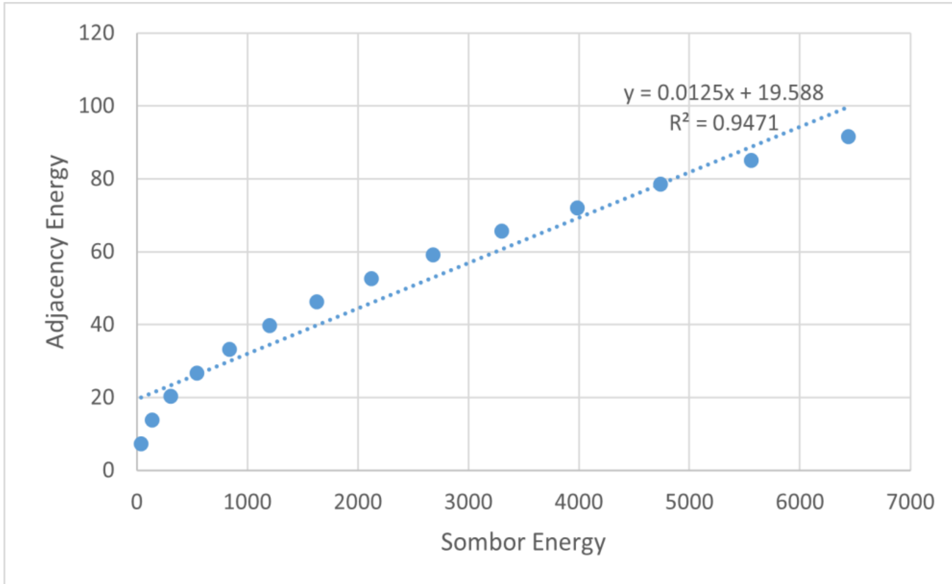

Figure 1. Correlation of $E_S(\Gamma_G)$ with $E_A(\Gamma_G)$ for odd n

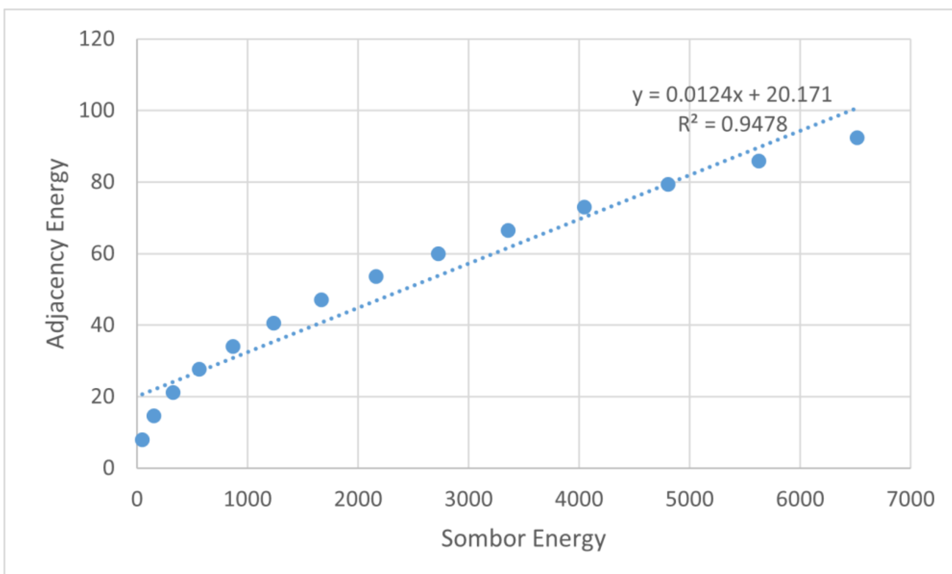

Figure 2. Correlation of $E_S(\Gamma_G)$ with $E_A(\Gamma_G)$ for even n

In Figures 1 and 2, the Sombor energy of Γ_G for D_{2n}, where $n \geq 3$ is always greater than the adjacency energy. Moreover, it can be seen that $E_S(\Gamma_G)$ has a significant correlation with $E_A(\Gamma_G)$, with a correlation coefficient of 0.9471 for odd n, and 0.9478 for even n. It is also clear that the Sombor energy of the non-commuting graph of dihedral group D_{2n} is minimum when $n = 3$ for odd n, or $n = 4$ for even n.

Conclusion

In this work, we provide the spectrum and spectral radius of Γ_G. We then presented the Sombor energy of Γ_G either for $G = G_1$, the set of rotation elements of D_{2n} removing members of $Z(D_{2n})$ or $G = G_2$, the set of reflection elements of D_{2n} or $G = G_1 \cup G_2$, the union of G_1 and G_2. We have shown that the Sombor energy of Γ_G is the multiple of two spectral radius of Γ_G and is always greater than its adjacency energy. Moreover, it is also observed that the correlation between $E_S(\Gamma_G)$ and $E_A(\Gamma_G)$ is almost linear.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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