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Fekete-Szegö Functional for Classes $X_q^n(\varphi)$ and $Y_q^n(\varphi)$

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Abstract Two new subclasses of analytic functions are proposed by applying *q*-differential operator which is denoted as $M_q^n f(z)$. Throughout this study, we acquired the initial coefficients a_2 and a_3 and the upper bound for the functional $|a_3 - \mu a_2^2|$ of the functions f in the classes $X_q^n(\varphi)$ and $Y_q^n(\varphi)$.

Keywords: Analytic function, Univalent function, *q*-differential operator, Fekete-Szegö functional, Subordination.

Introduction

The class for all analytic functions $f(z)$ within the open unit disk $\mathbb{U} = \{z: z \in \mathbb{C}, |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ is represented as A. According to Atassi [2], if $f(z)$ has a derivative at each point of R and if $f(z)$ is single valued, then a function $f(z)$ is known to be analytic within region R of the complex plane. Moreover, a function $f(z)$ is known to be analytic at a point z with the condition of z is an interior point of some region where $f(z)$ is analytic. Meanwhile, Kai [9] stated that for each $f \in A$, f has a Taylor series expansion written in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{j=2}^{\infty} a_j z^j, \qquad a_j \in \mathbb{C}, z \in \mathbb{U}. \tag{1.1}$$

The definition of subordination according to Jeyaraman & Suresh [6] is as if f and g are in A, the function f is said to be subordinate to g or (equivalently) g is said to be superordinate to f,

$$f \prec g \text{ in } \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U})$$

if a Schwarz function, $\omega(z)$, analytic in $\mathbb{U}$ with $|\omega(z)| < 1$ and $\omega(0) = 0$ for all $z \in \mathbb{U}$ is exist. For example,

$$f(z) = g(\omega(z)) \qquad (z \in \mathbb{U}).$$

In particular, several researchers have done the research about the coefficients, $|a_2|$ and $|a_3|$, and the upper bound for $|a_3 - \mu a_2^2|$ which is known as Fekete-Szegö functional of function f. For example, Alsoboh and Darus [1], Aouf and Orhan [3], Janteng *et al*. [4], Janteng and Halim [5] and Pinhong *et al.* [11].

Therefore, this study is going to introduce new subclasses of analytic functions and further determine the upper bound for the Fekete-Szegö functional of functions f for particular subclasses of analytic univalent functions which is defined by subordination and q-differential operator. Jackson [7] was the earliest researcher developed the q-integral and q-derivative more systematically.

However, Ramachandran *et al.* [12] stated that the *q*-derivative operator for function f as

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$$\mathcal{D}_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, 0 < q < 1 \\ f'(0), & z = 0 \end{cases}$$

for functions f which are differentiable at $z = 0$.

Then, Koekoek and Koekoek [8] further defined $D_q^n f$ as

$$D_q^n f = D_q(D_q^{n-1} f)$$

for $n = 1,2,3, \ldots$, where D_q^0 denotes the identity operator.

For the used of $D_q f(z)$, Seoudy and Aouf [13] introduced the subclasses $S_q^*(\alpha)$ and $C_q(\alpha)$ of the class A for $0 \leq \alpha < 1$ which are defined by

$$S_q^*(\alpha) = \left\{ f \in A : Re \frac{zD_q f(z)}{f(z)} > \alpha, z \in \mathbb{U} \right\},$$

$$C_q(\alpha) = \left\{ f \in A : Re \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)} > \alpha, z \in \mathbb{U} \right\}.$$

Selvaraj *et al.* [14] noted that

$$f \in C_q(\alpha) \Leftrightarrow zD_q f \in S_q^*(\alpha),$$

Alsoboh and Darus [1] proposed *q*-differential operator of a function f in the form of (1.1) and denoted by $M_q^n f(z)$ as

$$M_q^0 f(z) = f(z), \qquad M_q^1 f(z) = zD_q f(z) = z + \sum_{j=2}^{\infty} [j]_q a_j z^j$$

$$M_q^n f(z) = zD_q\left(M_q^{n-1} f(z)\right) = z + \sum_{j=2}^{\infty} [j]_q^n a_j z^j \tag{1.2}$$

where $[j]_q = \frac{1-q^j}{1-q}$ which was defined by Jackson [7].
By using the *q*-differential operator in (1.2) and the principle of subordination, we propose two new subclasses, $X_q^n(\varphi)$ and $Y_q^n(\varphi)$, of A.

Let P to be denoted as class of all functions φ that is analytic and univalent in $\mathbb{U}$.
The definitions of classes $X_q^n(\varphi)$ and $Y_q^n(\varphi)$ where $\varphi \in P$ are given respectively.

Definition 1.1 A function $f \in A$ is categorized in the class $X_q^n(\varphi)$ if the following subordination condition hold

$$D_q\left(M_q^n f(z)\right) \prec \varphi(z), \qquad \varphi \in P, n \in N, 0 < q < 1, z \in \mathbb{U}.$$

Definition 1.2 A function $f \in A$ is categorized in the class $Y_q^n(\varphi)$ if the following subordination condition hold

$$(1-\delta)\frac{zD_q(M_q^n f(z))}{M_q^n f(z)} + \delta\left(1 + \frac{qzD_q\left(D_q M_q^n f(z)\right)}{D_q(M_q^n f(z))}\right) \prec \varphi(z),$$

$\varphi \in P,\ n \in N,\ 0 < q < 1, 0 \leq \delta \leq 1$ and $z \in \mathbb{U}$.

Next, the lemma that is used to validate the main results in order to get the upper bound for the Fekete-Szegö functional for $f \in X_q^n(\varphi)$ and $f \in Y_q^n(\varphi)$ is as below.

Lemma 1.1 ([10]) If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{U}$ and γ is a complex number, then

$$|c_2 - \gamma c_1^2| \leq 2 \max\{1; |2\gamma - 1|\}.$$

The result is sharp for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2}\ \text{and}\ p(z)=\frac{1+z}{1-z}.$$

Main Results

Theorem 2.1 Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3\ldots$ with $(B_1\neq 0)$, and f is given by (1.1) be in the class $X_q^n(\varphi)$ and μ is a complex number, then

$$|a_3-\mu a_2^2|\leq \frac{B_1}{[3]_q^{n+1}}\max\left\{1;\left|\frac{B_2}{B_1}-\frac{[3]_q^{n+1}\mu B_1}{[2]_q^{2n+2}}\right|\right\}.$$

Proof. If $f\in X_q^n(\varphi)$, then Schwarz function $\omega(z)$ is exist with $\omega(0)=0$ and $|\omega(z)|<1$ in $\mathbb{U}$ such that

$$D_q\left(M_q^n f(z)\right)=\varphi(\omega(z)). \tag{2.1}$$

The function $p(z)$ is defined as

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+\cdots \tag{2.2}$$

We see that $Re(p(z))>0$ and $p(0)=1$ with $\omega(z)$ as Schwarz function. Let

$$g(z)=D_q\left(M_q^n f(z)\right)=1+d_1z+d_2z^2+\cdots \tag{2.3}$$

From equations (2.1), (2.2) and (2.3), we get that

$$g(z)=\varphi\left(\frac{p(z)-1}{p(z)+1}\right).$$

By equation (2.2), we solve $\omega(z)$ in terms of $p(z)$, we get that

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1z+p_2z^2+\cdots}{2+p_1z+p_2z^2+\cdots},$$

where

$$\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3+\frac{p_1^3}{4}-p_1p_2\right)z^3+\cdots\right). \tag{2.4}$$

From equations $\varphi(z)$ and (2.4), we get that

$$g(z)=\varphi\left(\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3+\frac{p_1^3}{4}-p_1p_2\right)z^3+\cdots\right)\right)$$

$$=1+B_1\left(\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right)\right)+B_2\left(\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right)\right)^2+\cdots$$

$$=1+\frac{1}{2}B_1p_1z+\left(\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2\right)z^2+\cdots. \tag{2.5}$$

From (2.3) and (2.5), we obtain

$$d_1=\frac{1}{2}B_1p_1,\quad \text{and}\quad d_2=\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2.$$

From (1.2), a computation shows that

$$M_q^n f(z)=z+[2]_q^n a_2z^2+[3]_q^n a_3z^3+\cdots. \tag{2.6}$$

According to the definition of $\mathcal{D}_q f$ stated by Ramachandran *et al.* [12], we obtain

$$D_q\left(M_q^n f(z)\right) = 1 + (q+1)[2]_q^n a_2 z + (q^2+q+1)[3]_q^n a_3 z^2 + \cdots \tag{2.7}$$

According to the definition of $[j]_q$ by Jackson [7], let $j = 0,1,2$ and 3, we obtain that

when $j = 0$,

$$[0]_q = \frac{1-q^0}{1-q} = 0$$

when $j = 1$,

$$[1]_q = \frac{1-q^1}{1-q} = 1$$

when $j = 2$,

$$[2]_q = \frac{1-q^2}{1-q} = 1 + q \tag{2.8}$$

when $j = 3$,

$$[3]_q = \frac{1-q^3}{1-q} = q^2 + q + 1 \tag{2.9}$$

Substitute (2.8) and (2.9) into (2.7), we obtain

$$D_q\left(M_q^n f(z)\right) = 1 + [2]_q^{n+1} a_2 z + [3]_q^{n+!} a_3 z^2 + \cdots \tag{2.10}$$

Then, compared (2.3) to (2.10), we obtain

$$d_1 = [2]_q^{n+1} a_2$$

and

$$d_2 = [3]_q^{n+1} a_3$$

or equivalently we have

$$d_1 = \frac{1}{2} B_1 p_1 = [2]_q^{n+1} a_2,$$
$$a_2 = \frac{B_1 p_1}{2[2]_q^{n+1}}$$

and

$$d_2 = \frac{1}{2} B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} B_2 p_1^2 = [3]_q^{n+1} a_3,$$
$$a_3 = \frac{B_1}{2[3]_q^{n+1}}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_2 p_1^2}{4[3]_q^{n+1}}.$$

Now,

$$a_3 - \mu a_2^2 = \frac{B_1}{2[3]_q^{n+1}}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_2 p_1^2}{4[3]_q^{n+1}} - \mu\left(\frac{B_1 p_1}{2[2]_q^{n+1}}\right)^2,$$
$$a_3 - \mu a_2^2 = \frac{B_1 p_2}{2[3]_q^{n+1}} - \frac{B_1 p_1^2}{4[3]_q^{n+1}} + \frac{B_2 p_1^2}{4[3]_q^{n+1}} - \frac{\mu B_1^2 p_1^2}{4[2]_q^{2n+2}},$$
$$a_3 - \mu a_2^2 = \frac{B_1}{2[3]_q^{n+1}}\left(p_2 - \frac{p_1^2}{2} + \frac{B_2 p_1^2}{2B_1} - \frac{[3]_q^{n+1}\mu B_1 p_1^2}{2[2]_q^{2n+2}}\right),$$
$$a_3 - \mu a_2^2 = \frac{B_1}{2[3]_q^{n+1}}\left(p_2 - p_1^2\left(\frac{1}{2} - \frac{B_2}{2B_1} + \frac{[3]_q^{n+1}\mu B_1}{2[2]_q^{2n+2}}\right)\right),$$

consider

$$\gamma = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{[3]_q^{n+1}\mu B_1}{[2]_q^{2n+2}}\right).$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{B_1}{2[3]_q^{n+1}}(p_2 - \gamma p_1^2).$$

By applying Lemma 1.1, it shows that

$$|a_3 - \mu a_2^2| \le \frac{B_1}{[3]_q^{n+1}} \max\left\{1; \left|\frac{B_2}{B_1} - \frac{[3]_q^{n+1}\mu B_1}{[2]_q^{2n+2}}\right|\right\}.$$

The proof of Theorem 2.1 is done.

Taking $n = 0$ into Theorem 2.1, we acquire the corollary below.

Corollary 2.1 ([4]) Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 \ldots$ with $(B_1 \neq 0)$, and f is given by (1.1) be in the class $X_q^0(\varphi)$ and μ is a complex number, then

$$|a_3 - \mu a_2^2| \le \frac{B_1}{[3]_q} \max\left\{1; \left|\frac{B_2}{B_1} - \frac{[3]_q \mu B_1}{[2]_q^2}\right|\right\}.$$

Now, we show the results for class $Y_q^n(\varphi)$.

Theorem 2.2 Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 \ldots$ with $(B_1 \neq 0)$, and f is given by (1.1) be in the class $Y_q^n(\varphi)$ and μ is a complex number, then

$$|a_3 - \mu a_2^2| \le \frac{B_1}{2\left((1-\delta)\left([3]_q^n\right)\left([3]_q - 1\right) + \delta q[3]^{n+2}\right)} \max\left\{1; \left| -\left(\frac{B_1}{\left((1-\delta)[2]^n\left([2]_q - 1\right) + \delta q\right)^2}\left((1 - \delta)[2]_q^{2n}\left([2]_q - 1\right) + \delta q[2]_q^{2n+3} - \mu\left((1-\delta)\left([3]_q^n\right)\left([3]_q - 1\right) + \delta q[3]^{n+2}\right)\right)\right) - \frac{B_2}{B_1}\right|\right\}.$$

Proof. If $f \in Y_q^n(\varphi)$, then Schwarz function $\omega(z)$ is exist with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$(1-\delta)\frac{zD_q M_q^n f(z)}{M_q^n f(z)} + \delta\left(1 + \frac{qzD_q\left(D_q M_q^n f(z)\right)}{D_q\left(M_q^n f(z)\right)}\right) = \varphi(\omega(z)). \quad (2.11)$$

We see that $Re\left(p(z)\right) > 0$ and $p(0) = 1$ with $\omega(z)$ as Schwarz function. Let

$$g(z) = (1-\delta)\frac{zD_q(M_q^n f(z))}{M_q^n f(z)} + \delta\left(1 + \frac{qzD_q\left(D_q M_q^n f(z)\right)}{D_q\left(M_q^n f(z)\right)}\right) = 1 + d_1 z + d_2 z^2 + \cdots \quad (2.12)$$

From equations (2.2), (2.11) and (2.12), we get

$$g(z) = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right).$$

By equation (2.2), we solve $\omega(z)$ in terms of $p(z)$, we get

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1 z + p_2 z^2 + \cdots}{2 + p_1 z + p_2 z^2 + \cdots},$$

Where

$$\frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2\right) z^3 + \cdots\right). \quad (2.13)$$

From equations $\varphi(z)$ and (2.13), we get

$$g(z) = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right)$$
$$= \varphi\left(\frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2\right) z^3 + \cdots\right)\right)$$
$$= 1 + B_1\left(\frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right)\right) + B_2\left(\frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right)\right)^2 + \cdots$$

$$= 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right)z^2 + \cdots. \tag{2.14}$$

From (2.2) and (2.14), we obtain

$$d_1 = \frac{1}{2}B_1 p_1,$$

and

$$d_2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2.$$

Case 1 for the equation $(1-\delta)\frac{zD_q M_q^n f(z)}{M_q^n f(z)}$, we substitute (2.6) and (2.10) into the equation and obtain

$$(1-\delta)\frac{zD_q(M_q^n f(z))}{M_q^n f(z)} = (1-\delta)\left(\frac{z\left(1 + [2]_q^{n+1}a_2 z + [3]_q^{n+1}a_3 z^2 + \cdots\right)}{z + [2]_q^n a_2 z^2 + [3]_q^n a_3 z^3 + \cdots}\right)$$

$$= (1-\delta)\left(1 + [2]^n\left([2]_q - 1\right)a_2 z + \left([3]_q^n\left([3]_q - 1\right)a_3 - [2]_q^{2n}\left([2]_q - 1\right)a_2^2\right)z^2 + \cdots\right) \tag{2.15}$$

Case 2 for the equation $\delta\left(1 + \frac{qzD_q\left(D_q M_q^n f(z)\right)}{D_q(M_q^n f(z))}\right)$, by Alsoboh and Darus [1],

$$\delta\left(1 + \frac{qzD_q\left(D_q M_q^n f(z)\right)}{D_q(M_q^n f(z))}\right)$$

$$= \delta[1 + qa_2[2]_q^{n+2}z + q\left(a_3[3]^{n+2} - a_2^2[2]_q^{2n+3}\right)z^2 + \cdots] \tag{2.16}$$

Therefore, a computation of (2.15) and (2.16) shows that

$$(1-\delta)\frac{zD_q(M_q^n f(z))}{M_q^n f(z)} + \delta\left(1 + \frac{qzD_q\left(D_q M_q^n f(z)\right)}{D_q(M_q^n f(z))}\right)$$

$$= (1-\delta)\left(1 + [2]^n\left([2]_q - 1\right)a_2 z + \left([3]_q^n\left([3]_q - 1\right)a_3 - [2]_q^{2n}\left([2]_q - 1\right)a_2^2\right)z^2 + \cdots\right) + \delta\left(1 + qa_2[2]_q^{n+2}z + q\left(a_3[3]^{n+2} - a_2^2[2]_q^{2n+3}\right)z^2 + \cdots\right)$$

$$= 1 + \left(\left((1-\delta)[2]^n\left(\left[[2]_q - 1\right]a_2\right)\right) + \delta qa_2[2]_q^{n+2}\right)z + \left((1-\delta)\left([3]_q^n\right)\left([3]_q - 1\right)a_3 - (1-\delta)[2]_q^{2n}\left([2]_q - 1\right)a_2^2 + \delta q\left(a_3[3]^{n+2} - a_2^2[2]_q^{2n+3}\right)\right)z^2 + \cdots \tag{2.17}$$

Then, compared (2.14) to (2.17), we get

$$d_1 = (1-\delta)[2]^n\left(\left[[2]_q - 1\right]a_2\right) + \delta qa_2[2]_q^{n+2},$$
$$d_1 = a_2\left((1-\delta)[2]^n\left([2]_q - 1\right) + \delta q[2]_q^{n+2}\right)$$

and

$$d_2 = (1-\delta)\left([3]_q^n\right)\left([3]_q - 1\right)a_3 - (1-\delta)[2]_q^{2n}\left([2]_q - 1\right)a_2^2 + \delta q\left(a_3[3]^{n+2} - a_2^2[2]_q^{2n+3}\right),$$
$$d_2 = a_3\left((1-\delta)\left([3]_q^n\right)\left([3]_q - 1\right) + \delta q[3]^{n+2}\right) - a_2^2\left((1-\delta)[2]_q^{2n}\left([2]_q - 1\right) + \delta q[2]_q^{2n+3}\right)$$

or equivalently we have

$$d_1 = \frac{1}{2}B_1 p_1 = a_2\left((1-\delta)[2]^n\left([2]_q - 1\right) + \delta q[2]_q^{n+2}\right),$$
$$a_2 = \frac{B_1 p_1}{2\left((1-\delta)[2]^n\left([2]_q - 1\right) + \delta q[2]_q^{n+2}\right)},$$

and

$$d_2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2$$
$$= a_3\left((1-\delta)\big([3]_q^n\big)\big([3]_q - 1\big) + \delta q[3]^{n+2}\right) - a_2^2\big((1-\delta)[2]_q^{2n}\big([2]_q - 1\big) + \delta q[2]_q^{2n+3}\big),$$

$$a_3\left((1-\delta)\big([3]_q^n\big)\big([3]_q - 1\big) + \delta q[3]^{n+2}\right)$$
$$= \left(\frac{B_1p_1}{2\left((1-\delta)[2]^n\big([2]_q - 1\big) + \big(\delta q[2]_q^{n+2}\big)\right)}\right)^2 \big((1-\delta)[2]_q^{2n}\big([2]_q - 1\big) + \delta q[2]_q^{2n+3}\big)$$
$$+ \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2,$$

$$a_3 = \frac{1}{\left((1-\delta)\big([3]_q^n\big)\big([3]_q - 1\big) + \delta q[3]^{n+2}\right)}\left(\left(\frac{B_1^2p_1^2\big((1-\delta)[2]_q^{2n}\big([2]_q - 1\big) + \delta q[2]_q^{2n+3}\big)}{4\left((1-\delta)[2]^n\big([2]_q - 1\big) + \delta q[2]_q^{n+2}\right)^2}\right)\right.$$
$$\left. + \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2\right).$$

Now,

$$a_3 - \mu a_2^2 = \frac{1}{(1-\delta)\big([3]_q^n\big)\big([3]_q - 1\big) + \delta q[3]^{n+2}}\left(\frac{B_1^2p_1^2\big((1-\delta)[2]_q^{2n}\big([2]_q - 1\big) + \delta q[2]_q^{2n+3}\big)}{4\left((1-\delta)[2]^n\big([2]_q - 1\big) + \delta q[2]_q^{n+2}\right)^2}\right.$$
$$\left. + \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2\right) - \mu\left(\frac{B_1p_1}{2\left((1-\delta)[2]^n\big([2]_q - 1\big) + \delta q[2]_q^{n+2}\right)}\right)^2,$$

$$a_3 - \mu a_2^2 = \frac{1}{(1-\delta)\big([3]_q^n\big)\big([3]_q - 1\big) + \delta q[3]^{n+2}}\left(\left(\frac{B_1p_1}{2\left((1-\delta)[2]^n\big([2]_q - 1\big) + \delta q[2]_q^{n+2}\right)}\right)^2 \Big((1\right.$$
$$- \delta)[2]_q^{2n}\big([2]_q - 1\big) + \delta q[2]_q^{2n+3} - \mu\left((1-\delta)\big([3]_q^n\big)\big([3]_q - 1\big) + \delta q[3]^{n+2}\right)\Big)$$
$$\left. + \left(\frac{B_1p_2}{2} - \frac{B_1p_1^2}{4}\right) + \frac{1}{4}B_2p_1^2\right),$$

$$a_3 - \mu a_2^2 = \frac{1}{(1-\delta)\big([3]_q^n\big)\big([3]_q - 1\big) + \delta q[3]^{n+2}}\left(p_2\left(\frac{B_1}{2}\right)\right.$$
$$+ p_1^2\left(\left(\frac{B_1}{2\left((1-\delta)[2]^n\big([2]_q - 1\big) + \delta q[2]_q^{n+2}\right)}\right)^2 \Big((1-\delta)[2]_q^{2n}\big([2]_q - 1\big) + \delta q[2]_q^{2n+3}\right.$$
$$\left.\left. - \mu\left((1-\delta)\big([3]_q^n\big)\big([3]_q - 1\big) + \delta q[3]^{n+2}\right)\Big) + \frac{B_2}{4} - \frac{B_1}{4}\right)\right),$$

$$a_3-\mu a_2^2=\frac{B_1}{2\left((1-\delta)\left([3]_q^n\right)\left([3]_q-1\right)+\delta q[3]^{n+2}\right)}\left(p_2\right.$$
$$-p_1^2\left(-\left(\frac{B_1}{2\left((1-\delta)[2]^n\left([2]_q-1\right)+\delta q[2]_q^{n+2}\right)^2}\left((1-\delta)[2]_q^{2n}\left([2]_q-1\right)+\delta q[2]_q^{2n+3}\right.\right.\right.$$
$$\left.\left.\left.\left.-\mu\left((1-\delta)\left([3]_q^n\right)\left([3]_q-1\right)+\delta q[3]^{n+2}\right)\right)\right)-\left(\frac{B_2}{2B_1}-\frac{1}{2}\right)\right)\right),$$

consider

$$\gamma=-\left(\frac{B_1}{2\left((1-\delta)[2]^n\left([2]_q-1\right)+\delta q[2]_q^{n+2}\right)^2}\left((1-\delta)[2]_q^{2n}\left([2]_q-1\right)+\delta q[2]_q^{2n+3}\right.\right.$$
$$\left.\left.-\mu\left((1-\delta)\left([3]_q^n\right)\left([3]_q-1\right)+\delta q[3]^{n+2}\right)\right)\right)-\left(\frac{B_2}{2B_1}-\frac{1}{2}\right).$$

Therefore,

$$a_3-\mu a_2^2=\frac{B_1}{2\left((1-\delta)\left([3]_q^n\right)\left([3]_q-1\right)+\delta q[3]^{n+2}\right)}\left(p_2-\gamma p_1^2\right).$$

By applying Lemma 1.1, it shows that

$$|a_3-\mu a_2^2|$$
$$\leq\frac{B_1}{2\left((1-\delta)\left([3]_q^n\right)\left([3]_q-1\right)+\delta q[3]^{n+2}\right)}\max\left\{1;\left|-\left(\frac{B_1}{\left((1-\delta)[2]^n\left([2]_q-1\right)+\delta q[2]_q^{n+2}\right)^2}\left((1\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\delta)[2]_q^{2n}\left([2]_q-1\right)+\delta q[2]_q^{2n+3}-\mu\left((1-\delta)\left([3]_q^n\right)\left([3]_q-1\right)+\delta q[3]^{n+2}\right)\right)\right)-\frac{B_2}{B_1}\right|\right\}.$$

The proof of Theorem 2.2 is done.

Taking $\delta=1$ into Theorem 2.2, we acquire the corollary below.

Corollary 2.2 ([1]) Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3\ldots$ with $(B_1\neq 0)$, and f is given by (1.1) be in the class $Y_q^n(\varphi)$ and μ is a complex number, then

$$|a_3-\mu a_2^2|\leq\frac{B_1}{2q[3]^{n+2}}\max\left\{1;\left|\frac{B_2}{B_1}+B_1\left(\frac{1}{[2]_q}-\mu\frac{[3]_q^{n+2}}{[2]_q^{2n+4}q}\right)\right|\right\}.$$

Conclusions

In conclusion, we acquired the initial coefficients a_2 and a_3 and the upper bound for the functional $|a_3-\mu a_2^2|$ of the functions f in the class $X_q^n(\varphi)$ and class $Y_q^n(\varphi)$.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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